HYDRODYNAMIC LIMIT FOR THE BAK-SNEPPEN BRANCHING DIFFUSIONS

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ABSTRACT. We prove a hydrodynamic limit for a system of N diffusions moving in an open domain $D \subseteq \mathbb{R}^d$ undergoing branching when one particle reaches a certain subset of the boundary. The particle at the boundary and another random neighbor are eliminated and replaced with two new particles created instantaneously at a random point with distribution $\gamma(dx)$ in D. The mechanism represents a hybrid between the Fleming-Viot branching and a mean-field version of the Bak-Sneppen fitness model where the absorbing boundary represents the *minimal* configuration, seen as biologically not viable. The limiting profile is the normalization of the solution of a heat equation with mass creation, which is studied using its representation via an auxiliary measure-valued supercritical process. Self-organized criticality is manifested by the emergence of a quasi-stationary distribution a formal limiting profile under equilibrium.

1. INTRODUCTION

Let $D \subseteq \mathbb{R}^d$ an open domain with piecewise smooth boundary ∂D and a diffusion on D generated by $(L, \mathcal{D}(L))$, where L is strongly elliptic with smooth coefficients up to the boundary and $\mathcal{D}(L) \subseteq C_b^1(\bar{D}) \cap C^2(D)$ is given by boundary conditions obtained by partitioning $\partial D = (\partial D)_r \cup (\partial D)_a$ in a relatively open part $(\partial D)_r$, the regular component, and a relatively closed part $(\partial D)_a$, the absorbing component. This is the *underlying, or driving diffusion*. To fix ideas, we assume reflecting boundary conditions on $(\partial D)_r$. While D may be unbounded, and the regular component may be taken empty, it will be assumed that the hitting time τ^D of the absorbing component will have an exponential moment, i.e. there

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exits $\beta_1 > 0$ such that

$$E_x[e^{\beta_1 \tau^D}] < +\infty\,, \qquad \forall x \in D\,. \tag{1.1}$$

Alternatively, we could consider a diffusion killed at rate $V(x)$ on D, where V is a non-negative Borel measurable function. In that case $(\partial D)_a$ may be empty. Since the interaction considered here is of mean-field type, the soft obstacle V is easier to handle at the level of the law of large numbers, more precisely the hydrodynamic limit, given in Theorem 1. Other aspects of the model are non-trivial in that case, i.e. related to quasi-stationarity, fluctuations, large deviations, and shall be pursued in other work.

We introduce the *Bak-Sneppen Branching Diffusion* (BDBD), a particle system with a fixed number of individuals N, with dynamics described as a hybrid between the Fleming-Viot (FV) particle system (e.g. [6, 10, 16, 7, 1]) and the Bak-Sneppen self-organizing fitness model from [2, 3], explaining the name.

In genome population interpretation, the particles undergo *mutation* represented by a diffusive term (Brownian), *selection*, represented by drift (in the probabilistic, not geneticists' sense) and *recombination*, represented by branching with redistribution at a random point $\sim \gamma(dx)$ where the new mass is born. Genetic recombination can be seen as a repair mechanism to damaged DNA. If artificial, it is under the effect of a *catalyst*, here seen as contact with the absorbing boundary.

The main result is a hydrodynamic limit stated in Theorem 1. The motivation of the model, and of this paper, is the the parallelism with the FV process. In a nutshell, as $N \to \infty$, the FV process converges to the normalized density of the dissipative heat equation (subcritical), whereas the BSBD converges to the normalized density of the accretive heat equation (6.1)-(1.21) introduced in [13, 20], referred to as *the heat equation with mass creation*, which is super-critical. This comparison is discussed in more detail in Subsection 1.4.

1.1. **Conditions on the underlying diffusion.** For a test function $\phi \in C^1(\bar{D}) \cap C^2(D)$, we shall denote

$$\phi \in (BC)_r \qquad \text{if} \qquad \nabla\phi(x)\cdot \mathbf{n} = 0\,, \quad x \in (\partial D)_r\,, \tag{1.2}$$

where $\mathbf{n}$ is the normal to the boundary $(\partial D)_r$, which will be prescribed for the underlying diffusion, respectively

$$\phi \in (BC)_a \quad (\in (BC)_{ac}) \quad \text{if } \phi(x) \text{ vanishes (is constant) on } x \in (\partial D)_a \,. \tag{1.3}$$

The boundary $(\partial D)_a$ can be assimilated to the cemetery state $\mathfrak{b}$ and a function $\phi \in (BC)_{ac}$ will take constant value $\phi(\mathfrak{b})$ on $(\partial D)_a$, and that constant will be zero if $\phi \in (BC)_a$.

It is assumed that the underlying diffusion defines a *strongly continuous Feller-Dynkin semigroup* in the sense of [17], Chapter III.6, that is, the transition probabilities satisfy

$$S_t^D\phi(x) = E_x[\phi(x_t)] = \int_D p^D(t,x,dy)\phi(y) \in C_b(D)\,, \quad \phi \in C_b(D) \ \text{(Feller property)} \tag{1.4}$$

and determine a C_0 (i.e. strongly continuous) semigroup on $C_0(D)$, the space continuous functions vanishing at infinity on D with the supremum norm.

The diffusion described solves the martingale problem $(L, \mathcal{D}(L))$ with

$$\mathcal{D}(L) = \{\phi \in C_b^1(\bar{D}) \cap C_b^2(D) \,|\, \phi \in (BC)_r \cap (BC)_a\}\,, \tag{1.5}$$

where C_b^j designates that all j derivatives are bounded. We mention that the set $\mathcal{D}(L)$ is a class of test functions larger than the domain of the generator and even more so $\mathcal{D}(L)$ includes $C_c^\infty(D)$, the set of infinitely differentiable functions with compact support.

When the boundary condition on $(\partial D)_a$ is replaced by $(BC)_{ac}$ we denote the set $\mathcal{D}_c(L)$,

$$\mathcal{D}_c(L) = \{\phi \in C^1(\bar{D}) \cap C^2(D) \,|\, \phi \in (BC)_r \cap (BC)_{ac}\}\,. \tag{1.6}$$

Additionally, we shall require that the heat kernel $p^D(t,x,y)$ be sufficiently smooth to have

$$S_t^D\phi(x)\,,\ f_D(t,x) \in C_b^{1,2}([t_0,\infty) \times \bar{D})\,, \qquad \phi \in C_b(\bar{D})\,, \quad t_0 > 0\,. \tag{1.7}$$

and for any $t_0 > 0$ there exists two positive constants c_- and c_+ such that

$$c_- \le p^D(t,x,y) \le c_+\,, \qquad t \ge t_0\,. \tag{1.8}$$

Remark.

1) Some of these conditions are redundant if D is bounded. Even for unbounded domains, in most applications, if the coefficients of L are C_b^∞ functions, then (1.7) and even stronger

smoothness properties would immediately hold, with precise bounds on the heat kernel when $t \to 0$ and $x, y \to \infty$.

2) Part of the interest is in examples with unbounded D, for example $d \geq 1$ the upper half-space $D = \{\mathbf{x} \in \mathbb{R}^d \,|\, x_d > 0\,,\, \mathbf{x} = (x_1, \ldots, x_d)\}$, $(\partial D)_a = \{x_d = 0\}$ and $(\partial D)_r = \emptyset\}$. It is sufficient to have a negative drift along the normal to the boundary to ensure (1.1). Ornstein-Uhlenbeck processes, BM with drift, allow explicit formulas for the heat kernel. In $d = 1$ we briefly discuss, right before (6.3), Brownian motion in $D = (0, 1)$, killed at $x = 0$ and reflected at $x = 1$.

The presence of a two boundary parts, reflecting and absorbing, is not essential, as mentioned before, but is motivated by applications, e.g. the case $d = 1$ cf. (6.3) and also [20], where the interest is to create the closest analogue to the classical Bak-Sneppen self-organizing fitness model [2, 3]. This is revisited in the next subsection, where we discuss the origin of the model.

In the same spirit, weaker regularity conditions on the coefficients of L as well as non-smooth (Lipschitz) domains are easy to consider, as long as (1.7)-(1.8) hold, for instance when L is in divergence form and heat kernel bounds are available (for instance, see [19]). However, this direction is less illustrative for the main idea, which is the representation (1.20) of the macroscopic profile of the system (1.13) based on a super-critical process (6.4) and its corresponding diffusion with mass creation (6.1) .

1.2. **The Bak-Sneppen branching diffusions (BSBD).** We start with $N \in \mathbb{N}$ particles moving independently according to $(L, \mathcal{D}(L))$ until the first, say of index i, hits $(\partial D)_a$. We then choose particle $j \neq i$, $1 \leq j \leq N$ uniformly, i.e. with probability $1/(N-1)$. The two particles *instantaneously jump* at the same random point chosen with distribution $\gamma \in M_1(D)$. It is important that

$$\gamma \text{ does not charge the boundary, i.e. } \gamma(D) = 1. \tag{1.9}$$

We emphasize that, as far as the empirical distribution is concerned, the construction is equivalent in distribution to the killing of the pair (i, j) with instantaneous birth at the new location of two new particles evolving independently from then on. This approach is consistent to the birth and death dynamics, but we prefer a finite system with simpler particle labelling for our purpose.

The number of particles N did not change and the independent motion of the system is restarted afresh until the next particle is killed, when the same branching mechanism redistributes the N particle system inside the domain D and continues with a new iteration.

The resulting process

$$(X_t^N(\omega))_{t\geq 0}\,, \quad X_t^N(\omega) = (X_t^{N,1}(\omega), \ldots, X_t^{N,N}(\omega)) \tag{1.10}$$

is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\omega \in \Omega$, where $\mathcal{F}_t$ satisfies the usual conditions, and shall be called the *Bak-Sneppen branching diffusion*, or BSBD - process. By construction, $(X_t^N(\omega))_{t\geq 0}$ is a jump-diffusion on the Skorokhod space $\mathbb{D}^N([0,\infty), D^N)$ of right continuous with left limits paths.

1.3. **Non-explosive behavior.** It is shown in [11], Subsection 4.2, that as long as the underlying diffusion $(L, \mathcal{D}(L))$ has a heat kernel (density) satisfying (i) a Doeblin recurrence condition - for instance, guaranteed by the lower bound of (1.8); (ii) has an a.s. positive lifetime τ^D with finite expectation, here stated in (1.1), and the redistribution/creation distribution γ, i.e. does not charge the boundary, here required as stated in (1.9), then the process is non-explosive and only one particle branches at a time, almost surely.

In the following we shall have the number of times particle i hits $(\partial D)_a$ up to time $t \geq 0$

$$A_t^{N,i}(\omega) = \int_0^t \mathbf{1}_{(\partial D)_a}(X_{s-}^{N,i})\, ds \tag{1.11}$$

and the average number of boundary hits

$$A_t^N(\omega) = \frac{1}{N-1}\sum_{i=1}^{N} A_t^{N,i}(\omega)\,, \tag{1.12}$$

where the normalization constant $(N-1)^{-1}$ is chosen for convenience and asymptotically consistent to the total number N of particles. These processes are naturally adapted to the filtration $\mathcal{F}_t$. We shall omit ω unless absolutely necessary.

1.4. **Comparison with the Bak-Sneppen and Fleming-Viot models.** In the celebrated Bak-Sneppen fitness model [2, 3], a system of N *fitness columns* on $D = (0,1)$, corresponding to N species, are re-sampled with distribution γ equal to $unif(0,1)$ at discrete time intervals. It is done by picking the minimal value, together with its two neighbors.

One difference is that we look at one neighbor only, so the number of individuals branching is $K = 2$; that is not significant qualitatively, noting that the proofs would remain almost identical for a fixed number of neighbors $K > 2$. It is the random choice of the "neighbor" that makes our current model mean-field, and as such, closable. Another important difference is, of course, that instead of a dynamic value of the "minimum", we trigger branching only by contact with the boundary $(\partial D)_a = \{0\}$, an absolute minimum value. Nonetheless, the most important feature of the Bak-Sneppen model remains present: *self-organizing criticality*, in the sense that the relaxation limit (as $t \to \infty$) of the macroscopic profile is equal to the quasi-stationary distribution of the supercritical branching system described in the Appendix, Subsection 6.3. It is exactly the normalized resolvent kernel of the underlying diffusion, calculated at a value $\alpha_* > 0$ (determined by K), the same critical value described in Theorem 4, which depends only on the number of neighbors re-sampled. This produces a particle representation of the resolvent of $(L, \mathcal{D}(L))$. These aspects are significant but not necessary for our proof. They are studied in detail in [13].

In the FV case [6, 10, 18, 16, 7, 1] the hydrodynamic limit is the normalization of the solution to heat equation with Dirichlet boundary conditions, which is dissipative, mass vanishing exponentially fast at rate $e^{\lambda_1 t}$, with $\lambda_1 < 0$. This is exactly the first eigenvalue for the Dirichlet Laplacian when $(L, \mathcal{D}(L))$ is BM killed at the boundary. In the BSBD case, if K is the number of individuals re-sampled (here $K = 2$), the hydrodynamic limit is the normalization of ν_t, a process accruing mass exponentially fast at rate $e^{\alpha_* t}$, where α_* and $\log E[K]$ have the same sign, implying super-criticality in our case cf. [13]. Essentially, we need a non-conservative process in either model.

While the dissipative case allows a representation with a single particle as in [10, 1], the mass creation can be modeled stochastically using a Markov semigroup only as a measure-valued process as in Theorem 3 in the Appendix. The Yaglom limit and quasi-stationarity are also discussed in both sub- and super- critical cases in [13].

1.5. **Main result.** We prove a Law of Large Numbers for the time-dependent empirical measure process

$$t \longrightarrow \mu_t^N(dy, \omega) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{N,i}(\omega)}(dy) \in \mathbf{D}([0,\infty), M_1(D)) \,, \tag{1.13}$$

or *hydrodynamic limit*, where $\mathbb{D}([0,\infty), M_1(D))$ denotes the Skorokhod space of probability measure - valued paths on D. To simplify notation, the random element ω will be omitted unless absolutely necessary. We shall write $\langle m, \psi \rangle$ for the integral of any bounded function

ψ against a finite measure $m(dx)$ on D. Define the time-space set of test functions $\phi(t,x)$,

(1.14)
$$\mathcal{D} = \{\phi \in C_b^{1,0}([0,\infty)\times \bar{D},\mathbb{R}) \,|\, \phi(t,\cdot) \in C_b^1(D\cup(\partial D)_r)\cap C_b(D\cup(\partial D)_a)\cap C_b^2(D)\}\,.$$

For sufficiently small $\delta > 0$, due to the smooth boundary of D, we denote

$$D_\delta = \{x \in D \,|\, dist(x,(\partial D)_a) > \delta\}\,, \quad \delta > 0\,. \tag{1.15}$$

We shall say that the process is regular near the absorbing boundary $(\partial D)_a$ if there exists $\delta > 0$ such that

(1.16) $(C0)$ The operator L has bounded coefficients on $D \setminus D_\delta$.

A *separation condition* (C1) between the three "boundaries" involved in the jump/branching mechanism: $(\partial D)_a$, $(\partial D)_r$, and $supp(\gamma)$, will also be needed.

Condition 1. *Let $q(dx) \in M_1(D)$. We shall say that the absorbing boundary $(\partial D)_a$ is* separated *from $(\partial D)_r$ and $q(dx)$ if there exists $d_a > 0$ such that*

$$(C1) \qquad dist((\partial D)_a, supp(q)\cup(\partial D)_r) \geq d_a\,, \tag{1.17}$$

where $supp(q)$ is the topological support of $q(dx)$.

Remarks.

1) Condition (C1) will be required for $q(dx) = \gamma(dx)$ in Theorem 1. It is formulated more generally, because it will be used several times, for other initial measures than $\gamma(dx)$ in the study of the tagged particle, an essential step in establishing *tightness*, e.g. in Lemma 2.

2) Condition $(C0)$ is used in (3.6). It is not restricting models with unbounded drift like the Ornstein-Uhlenbeck process in $d = 1$ (or along a certain direction in higher dimensions), provided the absorbing boundary is not extending to infinity, like in an exterior domain.

Definition 1. *A sequence of processes $(Y_\cdot^N)_{N>0}$ on a Polish space $(\mathbb{X},||\cdot||)$* converges in probability *to $(Y_\cdot)$, uniformly in finite time, if*

(i) For any $t \geq 0$, $(Y_t^N)_{N>0}$ is a tight family and

(ii) For any $T > 0$, the process $t \to (Y_t^N)_{t\geq 0}$ satisfies

$$\forall \epsilon > 0 \qquad \lim_{N\to\infty} P\Big(\sup_{t\in[0,T]} ||Y_t^N - Y_t|| > \epsilon\Big) = 0\,. \tag{1.18}$$

Definition 2. *The process* $(\mu^N_\cdot)_{N>0}$ converges weakly in probability *to* $(\mu_\cdot)$ *if for any test function* $\phi \in \mathcal{D}_c$*, the process* $t \to \langle \mu^N_t, \phi(t,\cdot)\rangle_{t\geq 0}$ *converges in probability to* $t \to \langle \mu_t, \phi(t,\cdot)\rangle_{t\geq 0}$ *in the sense of the Definition 1.*

Remark. If D is bounded, condition (i) and the boundedness of the test functions are redundant. Condition (ii) can be stated equivalently using only smooth functions with compact support depending only on the space variable.

Assume $\mu_0(dx)$ is a non-random measure in $M_1(D)$ and the initial condition

$$\mu^N_0 \text{ converges weakly in probability to } \mu_0 \text{ as } N \to \infty\ . \tag{1.19}$$

Remark.

1) Theorems 3 and 4 from [13], are presented in the Appendix, because they are an essential part of understanding the hydrodynamic limit $\mu_\cdot$ from Theorem 1, and for keeping this paper self-contained.

2) It is important to point out that the Appendix has the unique role of explaining the solution of the pde satisfied by the hydrodynamic limit (1.22). In spite of its probabilistic representation (6.5), there is no overlap between the proofs in [13] and the results on the BSBD particle process studied in the present paper, beyond the existence, uniqueness and smoothness of the solution to the pde.

3) Conditions (1.4), (1.5), (1.7), (1.8) from Subsection 1.1 are sufficient for Theorems 3 and 4.

Let ν_t be the unique solution to the heat equation with particle creation (6.1)-(1.21) and initial value $\nu_0 = \mu_0$. Write $n_t = \langle \nu_t, 1\rangle$ for its total mass and set the notations

$$\mu_t = \frac{\nu_t}{\langle \nu_t, 1\rangle}\,, \qquad \ln n_t = A_t = \int_0^t a_s ds\,. \tag{1.20}$$

For ϕ with $\phi(t,\cdot) \in \mathcal{D}_c$, $t > 0$, define the condition

$$\phi(t,\cdot) \in \mathcal{D}_c(L) \quad \text{and} \quad 2\langle \gamma, \phi(t,\cdot)\rangle - \phi(t,\mathfrak{b}) = 0\,. \tag{1.21}$$

Theorem 1. *Assuming (C0) and (C1) for the measure* $\gamma(dx)$*, together with the initial condition (1.19), the empirical measure process (1.13) converges weakly in probability, as* $N \to \infty$*, to the deterministic trajectory* $\mu_\cdot \in \mathbf{C}([0,\infty), M_1(D))$*, and* $A^N_\cdot$ *converges in probability to* $A_\cdot$ *from (1.20). The trajectory* $\mu_\cdot$ *is unique, having all the regularity properties* *inherited from Theorem 3, being absolutely continuous for $t > 0$ with density $\rho(t,x)$ and continuous for $t \geq 0$ in the topology of convergence in distribution. For any $\phi \in \mathcal{D}_c$ with the boundary condition (1.21) and $t \geq 0$, it satisfies*

$$\langle \mu_t, \phi(t,\cdot)\rangle = \langle \mu_0, \phi(0,\cdot)\rangle + \int_0^t \langle \mu_s, \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot) - a_s\phi(s,\cdot)\rangle \, ds \,. \tag{1.22}$$

Remark. When the asymptotic profile is a function, i.e.$\mu_t(dy) = \rho(t,y)dy$, it satisfies the forward equation $\partial_t\rho_t + L^*\rho - a_s\rho = 0$, where L^* is the formal adjoint of L. See in example (6.3) how the concrete conjugate boundary conditions look like when the redistribution measure is a delta function.

2. The Martingale Characterization of the BSBD - process and Ito's formula

Let $\mathcal{C}^N = C^{1,2}((0,\infty)\times D^N, \mathbb{R}) \cap C^{0,1}([0,\infty)\times \bar{D}^N, \mathbb{R})$ be the class of N - dimensional time-space test functions $F(t,x)$ *continuous up to the boundary* and, by analogy to (1.6)

$$\mathcal{D}_c^N = \{F \mid F \in \mathcal{C}^N \,, F(t,\cdot)_{|x_i} \in (BC)_r \cap (BC)_{ac} \,, 1 \leq i \leq N \,,\, t > 0\} \,, \tag{2.1}$$

where $F_{|x_i}$ is the marginal function when all but component x_i are fixed and the boundary conditions are described in the paragraph containing the definition (1.5).

Denote $L^{\otimes N}$ the direct sum of the one variable operator L, and by F^{ij} (defined precisely below) the configuration under F after redistribution of the particle i.

This is obtained as particle i has reached ∂D, has chosen particle $j \neq i$ uniformly, and both are created anew at the same random point with distribution $\gamma(dx)$. Using the vector notation $X = (x^1, x^2, \ldots, x^N)$,

$$L^{\otimes N} F(s,X) = \sum_{i=1}^N L_{x_i} F(s,\ldots,x_i,\ldots) \tag{2.2}$$

$$F^{ij}(s,X) = 2\int_D\int_D \mathbf{1}(x_i = x_j) F(s,\ldots,x_i,\ldots,x_j\ldots)\gamma(dx_i)\gamma(dx_j) \,, \tag{2.3}$$

where the identical entries are on position i and j.

Let $A_t^{N,i}$ be the number of hits of particle i to the absorbing boundary $(\partial D)_a$ from (1.11). Notice that $X_{t-}^{N,i} = \mathfrak{b}$ if and only if the counting process $A_t^{N,i}$ has a discontinuity, with probability one.

The joint set of interacting processes $(X_t^{N,i}, A_t^{N,i})_{t\geq 0}$, for $1 \leq i \leq N$, was defined constructively in Section 1, based on the strong Markov property, the fact that there are no simultaneous boundary hits. For a similar construction in more detail, more details, see [10].

We also denote by $\mathcal{M}_t^F$, for each $F \in \mathcal{C}^N$ the processes

$$\mathcal{M}_t^F = F(t, X_t^N) - F(0, X_0^N) - \int_0^t L^{\otimes N} F(s, X_s^N)\, ds \tag{2.4}$$

$$- \sum_{i=1}^N \int_0^t \Big(\frac{1}{N-1} \sum_{j\neq i} F^{i,j}(s, X_{s-}^N) - F(s, X_{s-}^N) \Big)\, dA_s^{N,i}\,. \tag{2.5}$$

Set $F(t,x) = \frac{1}{N}\sum_{i=1}^N \phi(t, x_i)$, for $\phi(t,\cdot) \in \mathcal{D}_c(L)$ Then, the expressions (2.4) (which will be shown to be martingales) read

$$\mathcal{M}_t^\phi = \langle \mu_t^N, \phi(t,\cdot)\rangle - \langle \mu_0^N, \phi(0,\cdot)\rangle - \int_0^t \langle \mu_s^N, L\phi(s,\cdot)\, ds \tag{2.6}$$

$$- \int_0^t \Big[(2\langle \gamma, \phi(s,\cdot)\rangle - \phi(s, \mathfrak{b})) - \langle \mu_{s-}^N, \phi(s,\cdot)\rangle \Big] - \frac{2}{N} \langle \gamma, \phi(s,\cdot)\rangle\, dA_s^N\,. \tag{2.7}$$

Proposition 1. *The processes $(\mathcal{M}_t^F)$ are $\mathcal{F}_t$ - martingales with continuous and jump components $\mathcal{M}_t^F = \mathcal{M}_t^{F,c} + \mathcal{M}_t^{F,J}$, such that $\mathcal{N}_t^{F,c}$, respectively $\mathcal{N}_t^{F,J}$ are also $\mathcal{F}_t$ - martingales, where*

$$\mathcal{N}_t^{F,c} = (\mathcal{M}_t^{F,c})^2 - \sum_{i=1}^N \int_0^t (L_{x_i} F^2 - 2\langle F, L_{x_i} F\rangle)(s, X_s^N) ds \tag{2.8}$$

$$\mathcal{N}_t^{F,J} = (\mathcal{M}_t^{F,J})^2 - \sum_{i=1}^N \int_0^t \frac{1}{N-1} \sum_{j\neq i} (F^{i,j}(s, X_{s-}^N) - F(s, X_{s-}^N))^2\, dA_s^{N,i}\,. \tag{2.9}$$

Moreover, theres exists a constant $C(\gamma)$, independent of t and N but dependent on the initial limiting profile μ_0, such that, for all $t \geq 0$ and $N \in \mathbb{Z}_+$,

$$E[\sum_{i=1}^N A_t^{N,i}] \leq C(\gamma) N t\,. \tag{2.10}$$

Remark. As *Step 1* below shows it, it is not hard to see that the processes in the statement are local martingales. In fact, all the processes in Proposition 1 are proper martingales, which is equivalent to showing that $E[A_t^{N,i}] < \infty$ for all components $1 \leq i \leq N$ and $t \geq 0$.

Proof. Step 1. The process (X_t^N) is non-explosive, as shown in Subsection 4.2 in [11]. We then know that $\lim_{t\to\infty} A_t^{N,i} = +\infty$ a.s., which implies, due to the boundedness of all integrand terms in the martingales, that setting T_m, $m \geq 1$ the first hitting time of the positive integer m by the sum $\sum_{i=1}^N A_t^{N,i}$, the processes (2.4), (2.8), (2.9) are local martingales by setting $t \to t \wedge T_m$, in other words with localization sequence T_m.

Step 2. We prove the processes are martingales. Set $F(t,X) = \frac{1}{N}\sum_{i=1}^N \phi(x_i)$ for a function $\phi \in (BC)_a$, $0 \leq \phi \leq 1$ with $c_\gamma = 2\langle \gamma, \phi\rangle - 1 > 0$. Such a function exists since γ has integral one and ϕ can be taken as a smooth function approximating the indicator function of a compact set in D. In that case, the integrand of the dA_t^N term in (2.6) is greater or equal to c_γ, so we obtain, almost surely,

$$c_\gamma A^N_{t\wedge T_m} \leq -\mathcal{M}^F_{t\wedge T_m} + F(t\wedge T_m, X^N_{t\wedge T_m}) - F(0, X^N_0) - \int_0^{t\wedge T_m} L^{\otimes N} F(s, X_s^N)\, ds\,. \tag{2.11}$$

Taking the expected value, we see that there exists a constant $C(\gamma)$, independent of t and N because it is simply a uniform bound on the function ϕ and its derivatives, such that $E[A^N_{t\wedge T_m}] \leq C(\gamma) t$. Since $\lim_{m\to\infty} T_m = +\infty$ a.s. we obtain by dominated convergence the same bound for $E[A_t^N]$, proving the proposition. □

3. Tightness

We start with two lemmas. One shows that the number of particles near the absorbing boundary remains small, uniformly in N, provided that it was small at time zero. The other one shows that even though the particles are not independent, the duration between visits to the absorbing boundary cannot be very short, provided the starting point is, in some sense, distributed away from the boundary, like is the case with the point with distribution $\gamma(dx)$. Since the time of return is controllable, uniformly in N, there cannot be too many boundary visits in a short time interval.

Given a small $\delta > 0$, let $U_t^N(\delta)$ be the number of particles within distance δ from the absorbing boundary at time t, $\nu^N \Rightarrow \nu$ denote the convergence in distribution for a sequence $(\nu^N) \in M_1(D)$.

Lemma 1. *Assume μ_0^N converges weakly in probability to $\mu_0 \in M_1(D)$ as $N \to \infty$. Let $T > 0$ and $0 < t_0 < T$. Then, there exists a constant $C(t_0, T)$, depending on t_0, T but* *independent of N, such that, for any $g \in C(\bar{D})$,*

$$\limsup_{N\to\infty} \sup_{t\in[t_0,T]} E[\langle \mu_t^N, g\rangle] \le C(t_0,T) \int_D g(y)dy \,. \tag{3.1}$$

As a consequence, we have the convergence, uniformly in time,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{t\in[t_0,T]} E[\frac{U_t^N(\delta)}{N}] = 0 \,. \tag{3.2}$$

Remark. In fact, $C(t_0,T) \le c(t_0)e^{\lambda_* T}$ (cf. [13]), where $c(t_0)$ depends only on t_0 and λ_* is the growth rate of the total mass of the auxiliary process (ζ_t) in Theorem 3.

Proof. Let g be a smooth function and $v(t,x) = E_x[\langle \zeta_t, g\rangle]$ as in the representation stated in Theorem 3. For a fixed $t > 0$, the function $\bar{v}(s,x) = v(t-s,x)$, $s \in [0,t]$, satisfies the backward equation $\partial_s \bar{v} + L\bar{v} = 0$ with terminal condition $\bar{v}(t,x) = g(x)$ and the boundary conditions $(BC)_r$ from (1.5), together with $2\langle\gamma, \bar{v}(s,\cdot)\rangle = \bar{v}(s,\mathfrak{b})$ from (1.21).

Setting $\bar{v} \to \phi$ in (2.6) we obtain that $s \to \langle \mu_s^N, \bar{v}(s,\cdot)\rangle$ is a super-martingale. The expected values at $s=0$ and $s=t$ give the inequality

$$E[\langle \mu_t^N, g\rangle] \le E[\langle \mu_0^N, v(t,\cdot)\rangle] = E[\int_D v(t,x)\mu_0^N(dx)] = E[E_{\mu_0^N}[\langle \zeta_t, g\rangle]] \tag{3.3}$$

First, we recall that for $t \in [t_0,T]$,

$$\begin{aligned} E_{\mu_0^N}[\langle \zeta_t, g\rangle] &= \int_D \int_D v^x(t,y) g(y) \mu_0^N(dx) \\ &\le \Big(\sup_{t\in[t_0,T],x,y\in D} |v^x(t,y)| \Big) \int_D g(y)dy \le C(t_0,T) \int_D g(y)dy \,, \end{aligned} \tag{3.4}$$

where we used (6.2). Taking the supremum over $t \in [t_0,T]$ in both (3.3) and then (3.4) we obtain (3.1).

The last assertion follows from taking a smooth approximation of the indicator function of the complement of the compact set $\bar{D}_{2\delta}$, as in (1.15), which is well defined since D has smooth boundary. □

The underlying process with generator L is a *one-particle process* and its hitting time of the absorbing boundary τ^D is introduced in (1.1). In the following, we shall need the same hitting time τ_X^D for the *tagged particle process* $(X_t^{N,i})_{t\ge 0}$, i.e. the process with fixed label $1 \le i \le N$. For simplicity, we suppress the index i, since in Lemma 2 and in its applications the label will never change, and it would be redundant.

Lemma 2. *Assume conditions $(C0)$ and $(C1)$ are satisfied for a probability measure $q(dx)$. Let $1 \le i \le N$ be a fixed index of one of the particles. We assume the N - component vector process X_t^N starts at a finite stopping time τ from a configuration with marginal distribution of particle i equal to $q(dx) \in M_1(D)$. Then there exists a constant $c(q)$, dependent only on $q(dx)$ only, and a fortiori independent of N, such that, for any $\eta > 0$*

$$P_{X_\tau^N}(\tau_X^D \le \tau + \eta) \le c(q)\eta\,. \tag{3.5}$$

Remarks. 1) Inequality (3.5) is valid pointwise, holding simply due to the distribution $\gamma(dx)$ of the tagged particle.

2) This lemma will be applied twice, once for $\tau = 0$ and q equal to the distribution of $X_0^{N,i}$, in order to prove tigtness for the tagged particle, and another time with τ a time when $X_{\tau-}^{N,i} \in (\partial D)_a$ and $q = \gamma$. In the second case it will be essential that $q(dx)$, and consequently $c(q)$, do not depend on τ, N or the index i.

3) Lemma 2 is the only place where the condition that $supp(\gamma)$ (the topological support of the redistribution measure) is at a positive distance from the absorbing boundary.

Proof. We construct a coupling between two processes, one without jumps, and then use a small ball estimate based on Doob's maximal inequality.

Step 1. Let $\psi \in C^2(\bar{D}, \mathbb{R})$ be a test function with the properties

1) $0 \le \psi(x) \le 1$,

2) $\psi(x) = 1$ on $supp(\gamma)$ and $\psi(x) = 0$ if and only if $x \in (\partial D)_a$,

3) There exists $0 < \delta < \frac{d_a}{2} \wedge 1$, such that $\psi(x) = dist(x, (\partial D)_a)$ on $D \setminus D_\delta$.

4) $\psi \in (BC)_r$

Define $y_t = \psi(X_t^{N,i})$, $t \ge \tau$. Notice that by construction, at any τ', a jump time of $X_t^{N,i}$, y_t jumps $y_{\tau'} - y_{\tau'-} \ge 0$, a non-negative jump. This is because the values on the support of γ, where it jumps, are guaranteed to equal the maximum value of ψ over the full set $\bar{D}$. We notice that $(y_t) \in [0,1]$ is a semi-martingale, adapted to $(\mathcal{F}_{t\wedge\tau})$, driven by the *full process* (X_t^N), not just the particle i, due to the jumps it undergoes at times when $X_t^{N,i}$ is chosen randomly by another particle hitting the absorbing boundary, in addition to its own jumps triggered by hitting the absorbing boundary. This process will be coupled with a new process denoted $(z_t)_{t\ge\tau}$, with the same initial value, driven by the same equations between jumps, only *with all jumps suppressed.* Then

$$0 \leq z_t \leq y_t \leq 1 \qquad a.s.$$

and $(z_t)_{t\geq\tau}$ is an Ito process $dz_t = \alpha_t dt + \beta_t dw_t$, with coefficients given by

$$dz_t = L\psi(X_t^{N,i})dt + (\nabla\psi)(X_t^{N,i}) \cdot [\sigma(s, X_t^{N,i})dw_t] \,, \quad z_0 = y_0 = \psi(X_\tau^{N,i}) \,,$$

if the underlying diffusion is given by $L\phi = \sum b_k \partial_k \phi + \frac{1}{2}\sum(\sigma^*\sigma)_{kl}\partial_{kl}\phi$ and B_t is the d - dimensional Brownian motion used in the construction of (X_t^N). We can see that the times to hit zero are ordered a.s. for the three processes $\tau_z^0 \leq \tau_y^0 \leq \tau_X^D$, where τ_X^D is the hitting time of the absorbing boundary by the process $X_t^{N,i}$.

Here is where condition $(C0)$ is used. Let $\alpha_0 \geq 0$ and $\beta_0 \geq 0$ be bounds for the coefficients

$$\alpha_0 = \sup_{x\in D} |L\psi(x)| \,, \quad \beta_0^2 = \sup_{x\in D} ||\sigma^*\sigma|| \, ||D^2\psi(x)|| \tag{3.6}$$

where the norms are the sum of the maximum of all elements of a matrix/vector, depending on ψ and its derivatives, and L.

It remains to evaluate, for an initial value $X_\tau^{N,i}$ as prescribed in the lemma, the sequence of upper bounds

$$P(\tau_X^D \leq \tau + \eta | X_\tau^{N,i}) \leq P(\tau_z^0 \leq \tau + \eta | X_\tau^{N,i}) \leq P(\inf_{t\in[\tau,\tau+\eta]} z_t \leq 1 - d_a | z_\tau = 1)$$

$$\leq P(\sup_{t\in[\tau,\tau+\eta]} |z_t - 1| \geq d_a | z_\tau = 1)$$

$$\leq P(\sup_{t\in[\tau,\tau+\eta]} |\int_\tau^t \beta_s dw_s| \geq d_a - \alpha_0\eta) \leq \left(\frac{\beta_0}{d_a - \alpha_0\eta}\right)^2 \eta \leq \frac{4\beta_0^2}{d_a^2}\eta$$

as soon as $0 < \eta < \frac{\beta_0}{2\alpha 0}$. Taking $c(q) = \frac{2\alpha_0}{\beta 0} \vee \frac{4\beta_0^2}{d_a^2}$ we conclude the proof. □

We move on to prove the tightness for both the empirical measure and the number of boundary hits. Additionally, we shall prove that for each fixed index i, $A_t^{N,i}$ is tight.

Naturally $A_t^{N,i}(\omega)$, $1 \leq i \leq N$ and their average $A_t^N(\omega)$, $\omega \in \Omega$, are random variables for all $t \in [0, \infty)$ and we omitted the sample space element ω to simplify notation. If (3.7)-(3.8) are satisfied, then a limit point $(A_t(\omega))_{t\geq 0}$ is a stochastic process with almost surely continuous paths. We can also verify that in this particular case, it is non-decreasing.

Proposition 2. *Assume $\mu_0^N \Rightarrow \mu_0$ and $\mu_0 \in M_1(D)$. Then, for any arbitrary but fixed $T > 0$,*

$$\limsup_{N\to\infty} E[A_T^N] < +\infty \tag{3.7}$$

$$\lim_{\eta\to 0} \limsup_{N\to\infty} \sup_{t\in[0,T]} P(A_{t+\eta}^N - A_t^N > \epsilon) = 0\,. \tag{3.8}$$

Remarks. 1) Evaluating (3.8) is based on the argument from line (3.10), which is a form of Wald's theorem for non-iid random variables $(\tau_X^D)_i,\ i \geq 1$, the waiting times between visits to the absorbing boundary. Independence is replaced by the condition in Lemma 2 and the strong Markov property.

2) Condition (3.8) is stronger than Aldous's criterion. It says cf. [14] that $(A_\cdot^N)$ is C - tight in the Skorokhod space, i.e. tight and that any limit point is continuous in time. Alternatively, if tightness is shown in the Skorokhod space, we recall that the *maximum jump size* $J_T(\omega(\cdot))$ of a path in D is a continuous functional in the *Skorokhod* J_1 - norm (not the same as the notation used below for the first jump). Since the jumps of $A_\cdot^N$ are at most of size $1/N$, it follows that a limit point $A_\cdot$ is continuous. This approach would prove immediately that $\mu_\cdot(dx)$ is also continuous in time.

Proof. Let $t \in [0,T]$, $\eta > 0$ and $J_1 < J_2 < \ldots$ be the ordered jump times after t. Then

$$A_{t+\eta}^{N,i} - A_t^{N,i} = [1 + m^\gamma(J_1, t+\eta)]\mathbf{1}_{\{J_1 \leq t+\eta\}}\,, \tag{3.9}$$

with $m^\gamma(s,t)$ denoting the number of episodes when $X_\cdot^i$ travels from the redistribution point with distribution γ to the absorbing boundary, observed in the time interval $(s,t]$, $0 \leq s \leq t$. Recall that τ^D is the hitting time of the boundary at $x = 0$ by the underlying diffusion process. Applying the Markov property, we can start at the vector configuration X_t^N.

$$E[A_{t+\eta}^{N,i} - A_t^{N,i}] = \sum_{k=1}^{\infty} P(A_{t+\eta}^{N,i} - A_t^{N,i} \geq k)$$
$$\leq E[P_{X_t^N}(\tau_X^D \leq \eta)] + \sum_{k=1}^{\infty} E[P_{X_{J_1}^N}(m^\gamma(J_1, t+\eta) \geq k)]\,.$$

Notice that $\{X_{J_1}^{N,i} \sim \gamma\}$ has probability one. We condition on this event in order to emphasize the label i that undergoes a jump. The general term of the infinite sum can be bounded

$$P_{X^N_{J_1}}(m^c(J_1,t+\eta)\geq k\,|\,X^{N,i}_{J_1}\sim\gamma)\leq P_{X^N_{J_1}}((\tau^D_X)_1+\dots(\tau^D_X)_k\leq\eta\,|\,X^{N,i}_{J_1}\sim\gamma) \tag{3.10}$$

$$\leq P_{X^N_{J_1}}(\max_{1\leq l\leq k}(\tau^D_X)_l\leq\eta\,|\,X^{N,i}_{J_1}\sim\gamma)\leq P_{X^N_{J_1}}((\tau^D_X)_k\leq\eta\,|\,\mathcal{A}_{k-1})P_{X^N_{J_1}}(\mathcal{A}_{k-1})\,.$$

where $\mathcal{A}_{k-1}=\{\max_{1\leq l\leq k-1}(\tau^D_X)_l\leq\eta\}$. In our count, $J_2-J_1=(\tau^D_X)_1$, ending with the k-th episode between jumps $J_{k+1}-J_k=(\tau^D_X)_k$. Taking the expectation under the initial condition X^N_t and using the strong Markov property recursively, we get the further bound

$$E_{X^N_t}[\Pi^k_{l=1}P_{X^N_{J_l}}((\tau^D_X)_l\leq\eta)]\leq[c(\gamma)\eta]^k\,, \tag{3.11}$$

This is due to the fact that $X^{N,i}_{J_l}$, $l\geq 1$ starts with distribution γ, which allows using Lemma 2 recursively. Summarizing (3.10)-(3.11) we see that *independently of the configuration* X^N_t,

$$P_{X^N_t}(m^c(J_1,t+\eta)\geq k)\leq[c(\gamma)\eta]^k\,,\qquad k\geq 1\,. \tag{3.12}$$

We obtained

$$E[A^{N,i}_{t+\eta}-A^{N,i}_t]\leq E[P_{X^i_t}(\tau^D\leq\eta)]+\frac{c(\gamma)\eta}{1-c(\gamma)\eta}\,. \tag{3.13}$$

After summation and division by $N-1$,

$$E[A^N_{t+\eta}-A^N_t]\leq\frac{1}{N-1}\sum_{i=1}^N E[P_{X^i_t}(\tau^D\leq\eta)]+(\frac{N}{N-1})\frac{c(\gamma)\eta}{1-c(\gamma)\eta} \tag{3.14}$$

To prove (3.7) we pick $\eta=[2c(\gamma)]^{-1}$. Then we put back to back at most $[\frac{T}{\eta}]+1$ intervals of length η to see that

$$E[A^N_T]\leq 2(\frac{N}{N-1})([\frac{T}{\eta}]+1)\leq 8c(\gamma)T\,. \tag{3.15}$$

We now turn to (3.8). Let $\delta > 0$ be an arbitrary number not exceeding $d_a/2$. Working on the first term

$$\begin{aligned}
(3.16) \qquad & \frac{1}{N}\sum_{i=1}^{N} E[P_{X_t^i}(\tau^D \leq \eta)] \\
& \leq \left[\sup_{dist(X_t^{N,i},(\partial D)_a)\geq \frac{\delta}{2}} P_{X_t^N}(\tau_X^D \leq \eta)\right] E[1-\frac{U_t(\frac{\delta}{2})}{N}] + E[\frac{U_t(\frac{\delta}{2})}{N}] \\
& \leq c(\delta)\eta + E[\frac{U_t(\frac{\delta}{2})}{N}]\,,
\end{aligned}$$

where $c(\delta)$ refers to the constant corresponding to an initial value away from the absorbing boundary at least by δ.

To finalize the proof, we turn to (3.8). Let $0 < \eta_0 < \eta$, momentarily fixed. We split the interval $[0, T]$, to calculate

$$\begin{aligned}
(3.17) \qquad & \sup_{t\in[0,\eta_0]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] \leq E[A_{2\eta_0}^{N,i} - A_0^{N,i}] = E[A_{2\eta_0}^{N,i}] \\
& \leq \frac{1}{N-1}\sum_{i=1}^{N} E[P_{X_0^i}(\tau^D \leq 2\eta_0)] + (\frac{N}{N-1})\frac{c(\gamma)(2\eta_0)}{1-c(\gamma)(2\eta_0)}
\end{aligned}$$

and

(3.18)

$$\sup_{t\in[\eta_0,T]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] \leq \sup_{t\in[\eta_0,T]}\left(\frac{1}{N-1}\sum_{i=1}^{N} E[P_{X_t^i}(\tau^D \leq \eta)]\right) + (\frac{N}{N-1})\frac{c(\gamma)\eta}{1-c(\gamma)\eta}$$

The first term on the right-hand side of these inequalities is reduced to a bound on the number of particles within $\delta > 0$, for (3.17), respectively $\delta' > 0$ for (3.18), as we did in (3.16). Taking $\eta c(\gamma) < \frac{1}{2}$ and $N \geq 2$, we obtain

$$\begin{aligned}
(3.19) \qquad \sup_{t\in[0,T]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] & \leq \sup_{t\in[0,\eta_0]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] + \sup_{t\in[\eta_0,T]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] \\
& \leq [4c(\gamma) + 2c(\delta')](2\eta_0) + 2E[\frac{U_0(\frac{\delta'}{2})}{N}]\,, \\
& + [4c(\gamma) + 2c(\delta)]\eta + 2\sup_{t\in[\eta_0,T]} E[\frac{U_t(\frac{\delta}{2})}{N}]\,.
\end{aligned}$$

Lemma 1 (3.2) concludes the proof, by having the limits over $N \to \infty$, $\eta \to 0$, $\delta \to 0$, $\eta_0 \to 0$, and finally $\delta' \to 0$, in this order. □

In fact, we can prove more than (3.7).

Proposition 3. *For any $T > 0$, $\beta > 0$*

$$M(\beta, T) = \limsup_{N\to\infty} E[e^{\beta A_T^N}] < \infty\,. \tag{3.20}$$

Proof. From Hölder's inequality we see that it is sufficient to prove the exponential bound for each tagged particle, where $i \le N$ is fixed, i.e.

$$M_i(\beta, T) = \limsup_{N\to\infty} E[e^{\beta A_T^{N,i}}] < \infty\,. \tag{3.21}$$

Let $\eta > 0$ be such that $\eta < (c(\gamma)e^{\beta})^{-1}$. Assume, for a moment, that there exists a number $\bar{M}(\beta, \eta) > 0$, independent of N, such that for any $t \ge 0$, *independently of* X_t^N,

$$E_{X_t^N}[e^{\beta A_\eta^{N,i}}] \le \bar{M}(\beta, \eta)\,. \tag{3.22}$$

The uniformity in the initial condition is inherited from (3.12), which, in its turn, comes from Lemma 2.

The Markov property shows that

$$E[e^{\beta A_T^{N,i}}] = E[E[e^{\beta(A_T^{N,i} - A_{T-\eta}^{N,i})} \mid \mathcal{F}_{T-\eta}] e^{\beta A_{T-\eta}^{N,i}}]$$
$$= E[E_{X_{T-\eta}^N}[e^{\beta A_\eta^{N,i}}] e^{\beta A_{T-\eta}^{N,i}}] \le \bar{M}(\beta, \eta)^{[\frac{T}{\eta}]+1} < \infty\,,$$

an upper bound independent of N, proving that $M_i(\beta, T) < \infty$. It remains to show (3.22). Recall that (3.12) holds uniformly in the initial state X_t^N. Since

$$P_{X_t^N}\left(A_\eta^{N,i} > \frac{\ln s}{\beta}\right) \le (c(\gamma)\eta)^{[\frac{\ln s}{\beta}]} \le (c(\gamma)\eta)^{\frac{\ln s}{\beta} - 1} \le (c(\gamma)\eta)^{-1} s^{\beta^{-1}\ln(c(\gamma)\eta)}$$

that

$$E_{X_t^N}[e^{\beta A_\eta^{N,i}}] = \int_1^\infty P_{X_t^N}\left(A_\eta^{N,i} > \frac{\ln s}{\beta}\right) ds \le (c(\gamma)\eta)^{-1} \int_1^\infty s^{-\beta^{-1}\ln(\frac{1}{c(\gamma)\eta})}\, ds < +\infty\,,$$

due to the choice of η. □

Theorem 2. *Under the same conditions of Theorem 1, the pair $(\mu_\cdot^N, A_\cdot^N)_{N>1}$ is C - tight on $D([0,\infty), M_1(D) \times \mathbb{R}_+)$, i.e. is tight and the limit is continuous in time.*

Proof. We can apply (2.6) for $\phi \in \mathcal{D}_c$ for two times t, t' in $[0, T]$ with $0 < t' - t < \eta$. There exist constants $K(c, \phi)$, $K(J, \phi)$, independent of t, N such that the squares of the martingales are bounded by $N^{-1}K(c, \phi)T$ for the continuous part and $N^{-1}K(J, \phi)A_T^N$ for the jump part. In similar fashion, the integrands of dt and dA_t^N parts are bounded by $K(c,\phi)\eta$, respectively $K(J,\phi)(A_{t'}^N - A_t^N)$. Due to Proposition 2, part (ii) of Definition 1 is satisfied. To obtain (i) we turn to (3.1) for g a smooth approximation of the indicator function of the complement of a compact set in D. The bound we need to prove is pointwise in t, due to the rcll property and the compactness of $[0,T]$; in that sense, less than (3.1) is needed, more precisely (3.3) is sufficient. All measures are concentrated, within $\epsilon > 0$ error, on a compact set, if the same is true at time $t = 0$. This is true simply because μ_0 charges D and not the boundary. The C - tightness is true because the criterion we used (i), (ii) in Proposition 2 implies C - tightness. □

4. Identification of the limit as the solution to the weak heat equation

Proposition 4. *The pair $(\nu_\cdot^N, n_\cdot^N)$, obtained by the transformation*

$$\nu_t^N = e^{A_t^N}\mu_t^N\,, \quad n_t^N = e^{A_t^N}$$

is C - tight and has hydrodynamic limit, in the sense of Definition 2, componentwise, the solution $\nu_\cdot$ to (6.1)-(1.21), respectively its total mass $n_\cdot$ from Theorem 4.

Proof. We write Ito's formula for semi-martingales [14]. Tightness follows from the tightness of the pair $(\mu_\cdot^N, A_\cdot^N)$ (Theorem 2); the exponential will also remain bounded in expectation due to the fact that all possible integrands in (2.6), including in the quadratic variations of the martingales, are dominated by constant multiples of $\exp A_T^N$ or $A_T^N \exp A_T^N$, both bounded above by $\exp 2A_T^N$, guaranteed by Proposition 3. Denote a generic element of $\mathbb{D}([0,\infty), M_F(D))$ by σ. Given any ϕ satisfying (1.21), using the notation $\sigma_s \in M_F(D)$ for the value at time $s \in [0,\infty)$, define the functional $\Phi : \mathbb{D}([0,\infty), M_F(D)) \to \mathbb{R}$

$$\Phi(\sigma_\cdot) := \sup_{t\in[0,T]} \Big| \langle \sigma_t, \phi(t,\cdot)\rangle - \langle \sigma_0, \phi(0,\cdot)\rangle - \int_0^t \langle \sigma_s, \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot)\rangle\, ds \Big|\,. \tag{4.1}$$

The exponential bounds obtained above will show that Φ is bounded and continuous, practically following steps 2-4 of Proposition 2 in [10].

Assuming that, the same bounds on the integrands, together with Doob's maximal inequality applied to the martingale part will show that

$$\lim_{N\to\infty} E[\Phi(\nu_\cdot^N)] = 0\,. \tag{4.2}$$

Let $(\nu_\cdot, n_\cdot)$ be a limit point of the tight pair of transformed processes. Since $(\nu_\cdot^N, n_\cdot^N) \Rightarrow (\nu_\cdot, n_\cdot)$ and Φ is continuous and bounded, we obtained that

$$E[\Phi(\nu_\cdot)] = 0 \quad \text{and then} \quad \Phi(\nu_\cdot) = 0 \quad a.s. \tag{4.3}$$

It is sufficient to remark that, being C - tight, the limit is continuous in time. It follows that we can pick a set of measure zero, common to all $t \in [0, T]$, and as a consequence, common for all $t \in [0, \infty)$ by choosing $T = r$, $r \in \mathbb{N}$, so that $\Phi(\nu_\cdot) = 0$ on its complement. We proved that $\nu_\cdot$ solves (6.1)-(1.21). By uniqueness, we are done with the claim on ν_t^N. When D is bounded, it is sufficient to integrate against the constant 1. A variation of the argument with approximations of indicator functions of a sequence of nested compacts will prove the same if D is unbounded. We see that since $n_t^N = \langle \nu_t^N, 1\rangle = \exp(A_t^N)$, then $n_\cdot^N \Rightarrow n_\cdot$. Finally, since the convergence is uniform in t over $[0, T]$, and the limit is a delta function (i.e. delta concentrated at the unique deterministic solution), we have that convergence in distribution implies convergence in probability. □

4.1. **Proof of Theorem 1.** At this point we have to reverse the transformation from Proposition 4. We notice that trivially both $n_t^N \geq 1$ and $n_t \geq 1$ and such have a lower bound away from zero. This allows to derive that (i), (ii) of Definition 1, in particular (1.18) is satisfied for $Y_t^N = \langle \mu_t^N, \phi\rangle$ and its limit $Y_t = \mu_t, \phi\rangle$ as soon as it is satisfied for $\langle \nu_t^N, \phi\rangle$, with its limit $\langle n_t, \phi\rangle$. The same is true for $Y_t^N = A^N = \ln n_t^N$, with its limit $\ln n_t = A_t$, since $x \to \ln x$ is uniformly Lipschitz on $x \in [1, \infty)$. Theorem 4 gives the explicit partial differential equation (1.22). □

5. Sketch of the tagged particle limit

The material proved in Sections 3 and 4 allows to develop the scaling limit of the tagged particle. We do not prove the result here, leaving it to an upcoming paper. However, we formally identify the limit in Subsection 5.2. The technical steps are outlined in a result we obtained in [12].

Fix the particle tag i and consider $N \geq i$ or simply take $i = 1$. We are interested in proving

$$X_\cdot^N \Rightarrow X. \tag{5.1}$$

and identifying the limit X_t as a stochastic process indexed by $t \geq 0$. These results require both convergence in distribution of $\mu_0^N \Rightarrow \mu_0$ and $X_0^{N,1} \Rightarrow X_0^1$.

5.1. **Tightness.** To prove the tightness *of each individual particle's number of visits to the absorbing boundary* $(A_t^{N,i})$, which is well defined for $N \geq i$, but of course is not continuous, even in the limit, we turn to the tightness criterion for processes in the Skorokhod space.

Proposition 5. *Let $i \in \mathbb{N}$ fixed and assume $X_0^{N,i} \Rightarrow X_0^i$ with $P(X_0^i \in dx) \in M_1(D)$. Then, $(A_t^{N,i})$ is tight, verifying, for any $T > 0$*

$$\sup_{N \geq 1} E[A_T^{N,i}] < +\infty \tag{5.2}$$

$$\lim_{\eta \to 0} \limsup_{N \to \infty} \sup_{t \in [0,T]} P(w'_{A^{N,i}}(\eta) > \epsilon) = 0\,. \tag{5.3}$$

Proof. Being a counting process, it follows that the only way the modulus of continuity $w'_{A^{N,i}}(\eta)$ in the Skorokhod J_1 - topology would exceed $\epsilon > 0$ is that it is at least one. More precisely, the union of the hitting times and the initial $t = 0$ must contain at least two elements within distance η. Otherwise, we can always optimize the partition of mesh η as to include that times and in that case $w'_{A^{N,i}}(\eta) = 0$. Then, either

(i) *there are at least two hits to the boundary* in $[t, t + \eta]$, or

(ii) *there is exactly one hit, but within η from $t = 0$.*

In case (i), the particle is redistributed, meaning that $\{\tau_X^D \leq \eta\}$ is a sub-event, fitting the exact conditions of Lemma 2 with $q(dx) = \gamma(dx)$.

In case (ii) $\{\tau_X^D \leq \eta\}$ is a sub-event as well. We split

$$P(\tau_X^D \leq \eta) = E[P_{X_0^N}(\tau_X^D \leq \eta\,,\, dist(X_0^{N,i}, (\partial D)_a) \geq \delta)]$$

$$+E[P_{X_0^N}(\tau_X^D \leq \eta\,,\, dist(X_0^{N,i}, (\partial D)_a) \geq \delta)] \leq c(\delta)\eta + P(dist(X_0^{N,i}, (\partial D)_a) < \delta)\,,$$

because the first term in the upper bound fits the exact conditions of Lemma 2 as, for example, it was applied in (3.16), while the second term will be vanishing due to the continuity theorem and the assumption that the initial point converges in distribution to a value that does not charge $(\partial D)_a$. □

Using Propositions 2 and 5 we write the differential equation for the test function corresponding to the tagged particle, i.e. of the form $F(X) = \phi(X_1)$, $\phi \in C^2(\bar{D})$. All integrands are bounded, and the integrators in time are either the Lebesgue measure dt or one of the counting measures A_t^N or $A_t^{N,1}$. It follows that $(X^{N,1})_{N>1}$ is tight. Moreover, it satisfies the following martingale problem, defining a Markov process which is time inhomogeneous.

We know from Theorem 2 and Theorems 3 and 4 in the Appendix that $A_\cdot^N$ converges in probability to the deterministic, continuous, increasing function $A_\cdot$, defining an absolutely continuous measure $dA_t = a_t dt$. This measure induces a non-homogeneous Poisson measure $\alpha(t)$ with jumps at times $A^{-1}(\theta)$, where θ are the jumps of a Poisson process of intensity one. By construction, this process can be independent from a countable sequence of mutually independent diffusions $(L, \mathcal{D}(L))$, which will serve as building blocks between jumps.

5.2. **The law of the tagged particle.** The tagged particle process $(X_t^1)_{t\geq 0}$ starts at X_0^1. It moves according to the diffusion $(L, \mathcal{D}(L))$ until the minimum of either the first arrival in $\alpha(t)$ or the first hitting time of the absorbing boundary. At such times, it instantaneously redistributes to a random point with distribution $\gamma(dx)$ and continue until the next jump time, dictated by the minimum described above. The process is well defined because no two jumps are simultaneous, and visits to the boundary are sufficiently far apart due to the tightness argument on $A_\cdot^{N,1} \Rightarrow A_\cdot^1$. The limit $A_\cdot^1$ is the number of visits to the absorbing boundary by the tagged particle $(X_t^1)_{t\geq 0}$, but its total average number of jumps is $\alpha(t) + A^1(t)$.

6. Appendix

Theorem 1, the main result of the paper, uses a partial differential equations result proven in detail in [13]. To keep this paper self-contained, we reproduce the two theorems we need. It is straightforward to verify that the conditions on the underlying process $(L, \mathcal{D}(L))$ stated in Subsection 1.1 imply the conditions required in [13] to proveTheorems 3 and 4.

6.1. **The heat equation with mass creation.** We shall say that $\nu_t(dy) \in C([0,\infty), M_1(D))$ is the weak solution to the heat equation for $(L, \mathcal{D}(L))$ with mass creation at γ and initial value ν_0 if

$$\langle \nu_t, \phi(t,\cdot)\rangle = \langle \nu_0, \phi(0,\cdot)\rangle + \int_0^t \langle \nu_s, \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot)\rangle\, ds \tag{6.1}$$

for any test function $\phi \in \mathcal{D}$ satisfying the boundary conditions (1.21) from Theorem 1 in Subsection 1.1.

Notice that the strong version of the equation satisfied by m is the *forward* equation, i.e. the time homogeneous heat equation for L^*, the formal adjoint of L, with boundary conditions obtained from (1.21).

Theorem 3 (from [13]). *For any $\nu_0 \in M_1$, equation (6.1) has a unique weak solution $\nu_\cdot \in C([0,\infty), M_F(D))$, where time continuity is defined in the topology of finite measures. This is a* strong solution *for $t > 0$ in the sense that $\nu_t(dy) = v(t,y)dy$, $t > 0$ with $v \in C^{1,2}((0,\infty)\times D)$. The solution admits the representation $\langle \nu_t, \phi\rangle = E_{\nu_0}[\langle \zeta_t, \phi\rangle]$, $t \geq 0$, for any $\phi \in \mathcal{D}$. Here $(\zeta_t)_{t\geq 0}$ is the auxiliary measure-valued process $(\zeta_t)_{t\geq 0}$ defined in Section 6.3.*

Theorem 4 (from [13]). *When $\nu_0 = \delta_x$, the solution is denoted $v^x(t,y)$ for $t > 0$, and for any $0 < t_0 < T$ there exists a constant $C(t_0,T) > 0$ such that*

$$\sup_{t\in[t_0,T],x,y\in D} v^x(t,y) = C(t_0,T) < \infty\,. \tag{6.2}$$

If, in addition, $\nu_0 = v_0(y)dy$, $v_0 \in C(\bar{D})$, then $v \in C([0,\infty)\times\bar{D}) \cap C^{1,2}((0,\infty)\times D)$. The total mass $n_t = \langle \nu_t, 1\rangle$ is positive, strictly increasing and there exists a constant $c(\eta_0)$ depending only on the initial value and $\lambda_* > 0$ such that $0 < n_t < c(\nu_0)e^{\lambda_* t}$, for any $t \geq 0$.

We conclude with a concrete example.

6.2. **Case $d = 1$.** Let $D = (0,1)$, $(\partial D)_r = \{1\}$, $(\partial D)_a = \{0\}$, $\gamma = \delta_c$, $c \in (0,1)$ and $L = \frac{1}{2}\frac{d^2}{dy^2}$ with $\nu_0(dx) = v_0(x)dx$. Then $L = L^*$, $\nu_t(dy) = v(t,y)dy$ with $v(0+,\cdot) = v_0(\cdot)$ and v has continuous time derivative. In addition, one can verify directly that for any $t > 0$, v is smooth in $(0,c)\cup(c,1)$ and satisfies the boundary conditions

$$\begin{aligned} v(t,c-) = v(t,c+)\,, \quad v'(t,1) = 0\,, \quad v(t,0) = 0 \\ (v'(t,c+) - v'(t,c-)) + 2v'(0) = 0\,. \end{aligned} \tag{6.3}$$

This case is studied in [20] with some additional considerations on the quasi-invariant measure.

6.3. **The auxiliary processes Z_t and ζ_t.** In this section, we outline the construction of a particle system Z_t having a random total number of particles N_t, which is a counting process as a result of branching. In that sense, our dynamics, including the conservative process (X_t^N) given in (1.10), is intimately related to super-critical behavior. See the comments in

Subsection 1.4. This states that the expected value of the empirical measure, seen as finite measure-valued random trajectory, is the solution to (6.1)-(1.21). The formal construction, definition, and proof of the regularity properties of this process, as well as related questions to its evolution semigroup, are done in [13].

At $t = 0$, a single particle is placed at a random point with distribution $m_0(dx) \in M_1(D)$. The particle, starts moving according to $(L, \mathcal{D}(L))$, until it reaches $(\partial D)_a$, when it dies. Instantaneously, two particles are born at the same random point in D chosen with distribution γ. All particles start afresh and continue an independent motion in D until the first one dies and the branching is repeated. We note that particles depend on each other only through ancestry, and not through their motion.

We shall make the convention that a particle hitting the absorbing boundary jumps, instead of being killed upon contact, which makes particle labelling easier. Then each particle has a Markovian motion once it is born, namely the *Brownian motion with rebirth* introduced in [8], also studied in [9, 4, 5]. Under (1.1), the particle system is well defined, having a constant number of particles between branchings. The branching times for a strictly increasing sequence, since they never coincide; all with probability one. We assume it is defined on a filtered probability space, and built constructively, up to the limit of the strictly increasing sequence of branching times, denoted by τ^*, a stopping time in $[0, +\infty]$.

The model can be easily generalized to have a random number K of offspring created at the recombination point, including a smaller number than one, leading to the possibility of dissipation of mass (e.g. K may be Poisson distributed), but we shall only consider a number of exactly $K = 2$ for our purpose of representing the solution of (6.1)-(1.21).

The first particle is denoted Z_t^1, the second Z_t^2, and so on. Let the number of particles at time t be denoted N_t, which, only in this special case, coincides with the number of branchings - a feature that while convenient, is not essential to the construction.

In principle, τ^* could be finite with positive probability, in which case the system is said *explosive*. In [13] it is shown that this is not the case. Moreover, N_t has exponential moments up to a critical value $\lambda_* > 0$, depending only on $(L, \mathcal{D}(L)))$ and the distribution of the number of offspring K. The constant is the same as the one given in Theorem 4, from [13]. Denote the empirical measure

$$\zeta_t = \sum_{i=1}^{N_t} \delta_{Z_t}. \tag{6.4}$$

This is a finite measure-valued Markov process, i.e. living on $\mathbb{D}([0,\infty), M_F(D))$.

Based on the estimate on N_t from [13], we define the expected value $\nu_t^x(dx)$ of the empirical measure of the process $(Z_t)_{t\geq 0}$ starting with one particle at x. Technically, we should denote this initial point by the non-random delta measure δ_x, for consistency with the measure valued setup. We can see that $x \to \nu_t^x(dx)$ is continuous in the topology of weak convergence and then the second integral in (6.5) is well defined.

For a bounded test function ϕ and a probability measure $\nu_0(dx) = v_0(x)dx \in M_1(D)$, we put

$$\langle \nu_t^x, \phi(t,\cdot)\rangle := E_x[\sum_{j=1}^{N_t} \phi(t, Z_t^j)], \qquad \nu_t^{v_0} = \int_D v_0(x)\nu_t^x dx\,. \tag{6.5}$$

It is part of the statement of Theorem 3 that the function $\nu_t^{v_0}$ is the stochastic representation of the unique weak solution ν_t of the heat equation with particle creation at $\gamma(dx)$, i.e. they are equal, hence satisfies (6.1)-(1.21).

The solution has the regularity properties of Theorem 3. Moreover, Theorem 4 shows that if $n_t := \langle \nu_t, 1\rangle$, then $n_t > 0$, $t \geq 0$, is differentiable with continuous derivative. Putting $\ln n_t = \int_0^t a_s ds$, then, for any test function $\phi \in \mathcal{D}$ with boundary conditions (1.21), the normalized solution $\mu_t = \nu_t/n_t$ satisfies (1.22). Relation (1.22) is obtained by an elementary calculation with $u_t = v_t/n_t$.

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