PATH COLLAPSE FOR MULTIDIMENSIONAL BROWNIAN MOTION WITH REBIRTH†

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Abstract. In a bounded open region of the d dimensional Euclidean space we consider a Brownian motion which is reborn at a fixed interior point as soon as it reaches the boundary. It was shown that in dimension one coupled paths starting at different points but driven by the same Brownian motion either collapse with probability one or never meet. In higher dimensions, for convex or polyhedral regions the paths with positive probability of collapse differ at start by a vector from a set of codimension one. The problem can be interpreted in terms of the long term mixing properties of the payoff of a portfolio of knock-out barrier options in derivatives markets.

1. Introduction

We denote by $\mathcal{R}$ a bounded open region in $\mathbb{R}^d$ with a piecewise smooth boundary of class C^2 satisfying the exterior cone condition and consider a fixed point in the interior, which will be identified as the origin without loss of generality. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{w_0(t,\omega)\}_{t\geq 0}$ a Brownian motion starting at the origin, adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on $\mathcal{F}$. The d-dimensional family of Brownian paths indexed by the starting points $x \in \mathbb{R}^d$, denoted by $w_x(\cdot,\omega) = x + w_0(\cdot,\omega)$ generates a family of coupled processes $\{z_x(t,\omega)\}_{t\geq 0}$ on the Skorohod space $\mathbf{D}([0,\infty), \mathcal{R})$, also indexed by the starting point $x \in \mathcal{R}$ according to the following mechanism. A Brownian particle is killed upon reaching the boundary and simultaneously a new particle is born at the origin which will perform a Brownian motion until it reaches the boundary in its turn. The process is repeated indefinitely (with

Date: October 6, 2003.

1991 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 60J50, 91B24.

Key words and phrases. Absorbing Brownian motion, harmonic measure, collapsing paths.

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probability equal to one, the procedure cannot end in finite time). The rigorous definition of the process is given in [4]. This paper completes and generalizes in higher dimensions the results in $d = 1$ of [2] and [3], where $\{z_x(t, \omega)\}_{t \geq 0}$ is referred to as *Brownian motion with rebirth.*

The goal of this paper is to answer the question whether paths corresponding to the same realization of the process, but starting at distinct initial points, will meet in finite time, or *collapse*. If we had $\mathcal{R} = \mathbb{R}^d$, the motion would be Brownian motion and the paths would simply be translates of each other with no possibility of collapse. The interesting feature is that for the *rebirth* process we can isolate a (possibly empty) set called the *grid* $G(\mathcal{R})$ of $\mathcal{R}$ such that if the coupled paths start at x and y then the probability of collapse is positive only if $k = x - y$ belongs to $G(\mathcal{R})$. Theorem 1, the main result of this paper, shows that in many relevant cases, more specifically when $\mathcal{R}$ is either convex or a polyhedron, we can describe the grid explicitly, being equal to the union of countably many $d-1$ dimensional hypersurfaces of positive surface area (thus having volume zero), each parallel to flat regions of the facets of $\mathcal{R}$, formally defined in (2.9).

Let the translation of a set $\mathcal{A} \subset \mathbb{R}^d$ by a vector $b \in \mathbb{R}^d$ be denoted by $\mathcal{A} + b$ where $\mathcal{A} + b = \{a + b : a \in \mathcal{A}\}$. We shall denote by $\lambda(\cdot)$ be the Lebesgue measure on smooth $d-1$ dimensional surfaces in $\mathbb{R}^d$. In dimension $d > 1$, the key observation leading to the notion of grid is that at least one of the two sets $(\partial\mathcal{R} \pm k) \cap \partial\mathcal{R}$ must have positive surface area in order to see the two paths collapse *in one step*, a consequence of Proposition 1. One would expect to construct inductively a skeleton of the grid $G'(\mathcal{R})$ even for arbitrary domains by setting

$$G'_0(\mathcal{R}) = \{k \in \mathbb{R}^d \,:\, \lambda((\partial\mathcal{R} \pm k) \cap \partial\mathcal{R}) > 0\}$$
$$G'_{n+1}(\mathcal{R}) = \{k \in \mathcal{R} \,:\, \lambda((\partial\mathcal{R} \pm k) \cap G'_n(\mathcal{R})) > 0\}$$

with $G'(\mathcal{R}) = \cup_{n \geq 0} G'_n(\mathcal{R})$. Due to the geometry of the region $\mathcal{R}$, this set is rather hard to describe. The present work focuses on the case (ii) of the discussion below, which permits a rigorous description of $G'(\mathcal{R}) \subseteq G(\mathcal{R})$ given in equation (3.2). In this particular case, the grid $G(\mathcal{R})$ is defined in (2.9) geometrically and not by induction, and the notion of skeleton $G'(\mathcal{R})$ will not be pursued in the rest of the paper. We can summarize the results concerning path collapse as follows.

(i) in $d = 1$ the grid $G(\mathcal{R})$ is the subset contained in $\mathcal{R}$ of the additive subgroup generated by the endpoints of the interval $\mathcal{R}$ and the paths with $x - y$ in the grid collapse with probability one. This case is presented in [2] and [3].

(ii) in $d > 1$, if $\mathcal{R}$ is convex or an arbitrary polyhedron, then only paths with $x - y$ in a subset of the grid have positive probability of collapsing, and this probability is strictly less than one. Example 1 shows that the grid may be nonempty yet there exist starting points with difference in the grid which have probability zero of collapsing. On the other hand, Corollary 1 illustrates a case in which if $x - y$ belongs to the grid, then the probability of collapse is always positive.

(iii) if $d > 1$ and the region $\mathcal{R}$ is not convex and has nonflat subsets of the boundary of positive surface area (is not a polyhedron), it is expected that the grid will have codimension higher than one. We do not pursue this case at this time, but would like to remark the geometric character of the problem.

The path collapse for Brownian motion with rebirth can be used to study the long-term behavior of the *double knock-out barrier options* in derivative markets, a special case of *lookback options*, characterized by the property that the payoff depends not on the value at a given time but on the path taken by the underlying asset process $\{S(t)\}_{t\geq 0}$ (for reference, see [1]). Usually $S(t)$ is modeled as a geometric Brownian motion, which, if $r(t) = \log S(t)$, is equivalent to

$$dr(t) = r_0 dt + \sqrt{a}\, dw(t)\,, \tag{1.1}$$

where $w(t)$ is a standard Brownian motion, $\sqrt{a}$ is the volatility and r_0 is the adjusted return rate of the market. The value of the double knock-out barrier options is driven by the market dynamics according to (1.1) until it hits the boundary or, in other words, one of the barriers, when its payoff is instantaneously reset to a fixed value (rebate). By a standard change of measure, we can reduce the problem to the analysis of the Brownian motion with rebirth. One can interpret the results as a characterization of the conditions under which the value of a portfolio with more than one instrument (higher dimensional case) remains positive on the long run.

The present analysis can be extended to diffusion process under general regularity properties (strong ellipticity, smooth coefficients, piecewise smooth boundary of the domain and exterior cone condition).

2. **Notations and Results**

Let $\mathcal{A}$ be an open region in $\mathbb{R}^d$ and $x \in \mathcal{A}$. We shall use the notation

$$T_x(\mathcal{A}) = \inf\{t > 0 \,:\, w_x(t,\omega) \notin \mathcal{A}\}\,, \tag{2.1}$$

the *exit time* from the region $\mathcal{A}$ for the Brownian motion starting at x. Occasionally we shall suppress either x or the set $\mathcal{A}$ if they are unambiguously defined in a particular context. In general, if $\mathcal{A}$ is a bounded open set in $\mathbb{R}^d$ and $p_{abs}(t,x,y)$ denotes the *absorbing Brownian kernel*, then

$$\int_{\mathcal{A}} p_{abs}(t,x,y)dy = P\Big(w_x(t,\omega) \in \mathcal{A}\,,\; t < T_x(\mathcal{A})\Big)\,. \tag{2.2}$$

Assume that $\mathcal{A}$ is a connected open and bounded set in $\mathbb{R}^d$ with piecewise smooth boundary satisfying the exterior cone condition. We denote by U a subset of $\partial\mathcal{A}$. If $\{w_k(t,\omega)\}_{t\geq 0}$ is the d-dimensional Brownian motion starting at $k \in \mathcal{A}$ and $T_k(\mathcal{A})$ will denote the first hitting time of the boundary, then

$$P\Big(w_k(T_k(\mathcal{A}),\omega) \in U\Big) = u(k,\mathcal{A},U) \tag{2.3}$$

where $u(k,\mathcal{A},U)$ is the harmonic measure centered at $k \in \mathcal{A}$ of the boundary $\partial\mathcal{A}$. One can prove directly or refer to more general results from [5] in order to show that under the present conditions on the set $\mathcal{A}$ the harmonic measure $u(k,\mathcal{A},du)$ and the Lebesgue surface area measure $\lambda(du)$ on the boundary $\partial\mathcal{A}$ are mutually absolutely continuous.

We shall call a *facet* of $\mathcal{R}$ any maximal smooth component of the boundary $\partial\mathcal{R}$.

Let $\{\tau_n^x\}_{n\geq 0}$ be the ordered sequence of stopping times when the particle driven by Brownian motion with rebirth starting at x reaches the boundary (in fact, the discontinuity points of the path). The reader is referred to [4] and also [2] for the rigorous construction of the process. With the notation $N_x(t) = \sum_{n=0}^{\infty} \mathbf{1}_{\{\tau_n^x \leq t\}}$, one can write inductively for $t \leq \tau_n^x$

$$z_x(t,\omega) = w_x(t,\omega) - \int_0^t z_x(s-,\omega)dN_x(s,\omega)\,, \tag{2.4}$$

such that $z_x(t,\omega) = 0$ for all $t = \tau_n^x$.

We are interested in the probability of collapse of two coupled paths $z_x(t,\omega)$ and $z_y(t,\omega)$, starting at x and y in $\mathcal{R}$. The coupling is understood in the sense that both processes are driven by the same realization $\omega \in \Omega$ of the standard Brownian motion $w_0(t,\omega)$ on $(\Omega,\mathcal{F},P)$. For $x \neq y$, the union of increasing sequences of a.s. finite hitting times of the boundary

$\{\tau_n^x\}$ and $\{\tau_n^y\}$ corresponding to x and y from $\mathcal{R}$ can be non-ambiguously rearranged in increasing order as long as the boundary is not hit by the two paths simultaneously and simply denoted by $\{\tau_n\}$. Let $T_c = \inf\{t > 0 : z_x(t,\omega) = z_y(t,\omega)\}$ be the time of collapse, which could possibly assume the value $T_c = \infty$. Since the paths are translates of each other (parallel) between boundary hits by either of them, collapse may only occur at one of the times $\{\tau_n\}$. At each such hitting time, one of the paths will fall back to zero, while the other one will be in $\mathcal{R}$. If the boundary is hit by the two paths in the same time, the two paths will collapse.

These considerations allow us to define a spatially inhomogeneous Markov chain $\{Y_n(\omega)\}_{n\geq 0}$ such that $Y_0(\omega) = 0$ if $\tau_0 = T_c$ and $Y_0(\omega) = z_r(\tau_0,\omega)$, where r is either x or y in such a way that $z_r(\tau_0,\omega)$ is the point which is *not* situated at zero at time τ_0 as long as $\tau_0 < T_c$. Inductively, $Y_n(\omega) = 0$ if $\tau_n \geq T_c$ and $Y_n(\omega) = z_r(\tau_n,\omega)$, where r is either x or y in such a way that $z_r(\tau_n,\omega)$ is the point which is *not* situated at zero at time τ_n as long as $\tau_n < T_c$.

Proposition 1. *Assume $d > 1$. The chain $\{Y_n(\omega)\}$ has a time homogeneous transition probability, defined for any $m \in \mathbb{Z}_+$, $k \in \mathcal{R}$ and any Borel set $A \in \mathcal{B}(\mathcal{R})$,*

$$
\begin{aligned}
P(Y_{m+1} \in A \,|\, Y_m = k) &= u(0, (\mathcal{R}-k)\cap\mathcal{R}, (A-k)\cap\partial\mathcal{R}) \\
&+ u(0, (\mathcal{R}-k)\cap\mathcal{R}, A\cap(\partial\mathcal{R}-k)) \\
&+ u(0, (\mathcal{R}-k)\cap\mathcal{R}, (\partial\mathcal{R}-k)\cap\partial\mathcal{R})\cdot \mathbf{1}_{\{0\}}(A)\,.
\end{aligned}
\tag{2.5}
$$

Remark 1. The transition probability is supported on the sets $\partial\mathcal{R} \pm k$ and $\{0\}$. Since the harmonic measure is absolutely continuous with respect to the Lebesgue surface area of the boundary, we do not need to write $A \setminus \{0\}$ in the first two lines of (2.5). Furthermore, the origin is an absorbing state.

Remark 2. Also note that $(\partial\mathcal{R}-k)\cap\mathcal{R}$, $(\mathcal{R}-k)\cap\partial\mathcal{R}$, $(\partial\mathcal{R}-k)\cap\partial\mathcal{R}$ is a partition of the boundary of $(\mathcal{R}-k)\cap\mathcal{R}$. Since $A \subseteq \mathcal{R}$, the exit sets on the first two lines of (2.5) can be written $(A-k)\cap\partial\mathcal{R} = (A-k)\cap(\mathcal{R}-k)\cap\partial\mathcal{R}$ and $A\cap(\partial\mathcal{R}-k) = A\cap(\partial\mathcal{R}-k)\cap\mathcal{R}$, respectively.

Proof. We shall refer to the event that the particle which was situated at the origin at time τ_m hits the boundary strictly before the particle that was at k hits the boundary as $\{0$ hits the boundary before k does$\}$ and analogously, the event that the particle which was situated at k at time τ_m hits the boundary strictly before the particle that was at

0 hits the boundary as $\{k$ hits the boundary before 0 does$\}$. When the two particles hit the boundary at the same time, we get $Y_{m+1} = 0$. In view of the remark following the statement of the proposition, the three distinct cases are represented by the events of the boundary hits taking place at points belonging to each of the sets of the partition.

The transition probability $P(Y_{m+1} \in A \,|\, Y_m = k)$ can be decomposed into

$$
\begin{aligned}
&P(Y_{m+1} \in A \,|\, Y_m = k) = \\
&P(Y_{m+1} \in A\,,\, \{0 \text{ hits the boundary before } k \text{ does}\} \,|\, Y_m = k) + \\
&P(Y_{m+1} \in A\,,\, \{k \text{ hits the boundary before } 0 \text{ does}\} \,|\, Y_m = k) + \\
&P(Y_{m+1} \in A\,,\, \{0 \text{ and } k \text{ hit the boundary simultaneously}\} | \, Y_m = k) \qquad .
\end{aligned}
$$

We write

$$
\begin{gathered}
P(Y_{m+1} \in A\,,\, \{0 \text{ hits the boundary before } k \text{ does}\} \,|\, Y_m = k) = \\
P(w_k(T_k((\mathcal{R} + k) \cap \mathcal{R}, \omega) \in (\partial\mathcal{R} + k) \cap \mathcal{R} \cap A) = \\
u(k, (\mathcal{R} + k) \cap \mathcal{R}, (\partial\mathcal{R} + k) \cap \mathcal{R} \cap A) = \\
u(0, (\mathcal{R} - k) \cap \mathcal{R}, (A - k) \cap (\mathcal{R} - k) \cap \partial\mathcal{R})\,.
\end{gathered} \tag{2.6}
$$

The last equality in (2.6) is due to the translation invariance of the Wiener measure. The exit set $(\partial\mathcal{R} + k) \cap \mathcal{R} \cap A$ is a subset of $(\partial\mathcal{R} + k) \cap \mathcal{R}$. In this case, it is true pathwise that the particle originally situated at 0 must hit the boundary first, otherwise the particle originally situated at k would have reached $\partial\mathcal{R}$ before. Finally, the set $A \subseteq \mathcal{R}$ allows us to simplify (2.6) to the first line of (2.5). Analogously

$$
\begin{gathered}
P(Y_{m+1} \in A\,,\, \{k \text{ hits the boundary before } 0 \text{ does}\} \,|\, Y_m = k) = \\
P(w_0(T_0((\mathcal{R} - k) \cap \mathcal{R}, \omega) \in (\partial\mathcal{R} - k) \cap \mathcal{R} \cap A) = \\
u(0, (\mathcal{R} - k) \cap \mathcal{R}, (\partial\mathcal{R} - k) \cap \mathcal{R} \cap A)\,.
\end{gathered} \tag{2.7}
$$

Lastly

$$
\begin{gathered}
P(Y_{m+1} \in A\,,\, \{0 \text{ and } k \text{ hit the boundary simultaneously}\} | \, Y_m = k) = \\
P(w_0(T_0((\mathcal{R} - k) \cap \mathcal{R}, \omega) \in (\partial\mathcal{R} - k) \cap \partial\mathcal{R}) \cdot \mathbf{1}_{\{0\}}(A) = \\
u(0, (\mathcal{R} - k) \cap \mathcal{R}, (\partial\mathcal{R} - k) \cap \partial\mathcal{R}) \cdot \mathbf{1}_{\{0\}}(A)\,.
\end{gathered} \tag{2.8}
$$

We conclude with the remark that in dimension $d > 1$ the event that $Y_{m+1} = 0$ due to the fact that one particle is at the origin at the exit time of the other one has zero probability, since it is equal to the harmonic measure of a single point from $\partial\mathcal{R}$. $\square$

From now on we will assume that our domain $\mathcal{R}$ is either convex or a polyhedron. We shall define a set $G(\mathcal{R})$, called the set of *grid* points of $\mathcal{R}$, in the following manner. By *planar* or *flat* set we shall understand a subset of a hyperplane in $\mathbb{R}^d$ with positive surface measure. If $\mathcal{R}$ has at least one facet with a planar subset we shall denote by $\mathbf{n}_1, \ldots, \mathbf{n}_{d(\mathcal{R})}$, all the possible normals to these planar subsets, without repetitions (we do not distinguish between $\mathbf{n}$ and $-\mathbf{n}$). For each j from 1 to $d(\mathcal{R})$, we denote by $\mathbf{F}^j$ the set $F_1^j, \ldots, F_{m(j)}^j$ of all flat subsets of $\partial\mathcal{R}$ with normal $\mathbf{n}_j$. Let $A^j = \{a_k^j\}_{1\leq k\leq m(j)}$ be the set of intersection points of the hyperplanes containing F_k^j with the line with direction $\mathbf{n}_j$ passing through the origin. We also denote by Z^j the additive subgroup of $\mathbb{R}$ generated by the family of numbers $\mathbf{n}_j \cdot A^j$.

Definition 1. *Let $\{\mathbf{n}_j\}_{1\leq j\leq d(\mathcal{R})}$ be the set of all the possible normal vectors to the flat subsets $\mathbf{F}^j$ of the boundary $\partial\mathcal{R}$ of the region $\mathcal{R}$ which is assumed to be either convex with piecewise smooth boundary or a polyhedron. The set $G(\mathcal{R})$ of grid points of $\mathcal{R}$ is defined as*

$$G(\mathcal{R}) = \left\{k \in \mathbb{R}^d \,:\, \exists\, j \quad \mathbf{n}_j \cdot k \in Z^j\right\}, \tag{2.9}$$

and we define the set $N(\mathcal{R})$ of all nodes of the grid $G(\mathcal{R})$, namely

$$N(\mathcal{R}) = \left\{k \in \mathbb{R}^d \,:\, \exists\, j' \neq j'' \quad \mathbf{n}_{j'} \cdot k \in Z^{j'} \text{ and } \mathbf{n}_{j''} \cdot k \in Z^{j''}\right\}. \tag{2.10}$$

Remark. The *nodes* defined at (2.10) may not be points as the name suggests, but unions of surfaces of codimension 2. Also we observe that the set $G(\mathcal{R})$ of the *grid points* is an empty set if the boundary $\partial\mathcal{R}$ has no points of zero curvature. More generally, if for any vector k, $\partial\mathcal{R} \cap (\partial\mathcal{R} + k)$ has Lebesgue surface measure zero, then there exists no grid.
We are ready for the main result of the paper.

Theorem 1. *Let $\mathcal{R}$ be a region in $\mathbb{R}^d$ with piecewise smooth boundary satisfying the exterior cone condition. If $\mathcal{R}$ is either convex or an arbitrary polyhedron and x and y are two points in $\mathcal{R}$, then the probability that the two paths $\{z_x(t,\omega)\}_{t\geq 0}$ and $\{z_y(t,\omega)\}_{t\geq 0}$ collapse is strictly less than one. If $x - y \in \mathcal{R} \setminus G(\mathcal{R})$, the probability to collapse is zero.*

Remark 1. This is a *necessary* condition theorem. In $d > 1$ there exists no pair of points x and y which ensure collapse a.s. while the positive probability of collapse is still valid only for a subset of volume zero - the grid $G(\mathcal{R})$. It even may turn out that the geometry of the set will prohibit the collapse in spite of the relative position of the initial points being on the grid. One such case is presented in Example 1. However, in some special cases we can make precise the sufficient condition on the set of $x - y$ which permits the paths to collapse, as seen in Corollary 1.

Remark 2. The set of *grid* points $G(\mathcal{R})$ corresponds in $d = 1$ to the family of linear combinations with integer coefficients of the endpoints of the interval (a, b), denoted by $Z_{a,b}$. In that case it is known from Theorem 1 in [3] that all such points will collapse almost surely to the origin.

Corollary 1. *If $\mathcal{R}$ is a d dimensional parallelepiped, then for all $x - y \in G(\mathcal{R})$, the paths collapse with positive probability strictly less than one.*

We shall prove right away the corollary.

Proof. The key observation is that any $k = y - x$ with $x, y \in \mathcal{R}$ can be translated along a Brownian path contained in the region all the way to the boundary. Since at least one coordinate of the vector k is in the grid, it follows that without loss of generality the hyperplane it is parallel to can be assumed horizontal ($x_d = 0$). It is a consequence of the $d = 1$ result stated in Theorem 1 in [3] that, conditional on the event that we hit a number of times in a row the opposite parallel horizontal facets of the parallelepiped (which can be done with positive probability) we shall reach a state k' with $k'_d = 0$ almost surely. With positive probability we can hit one of the horizontal boundaries once again, when both endpoints of k must collapse. $\square$

Example 1. *If $d > 1$ and $\mathcal{R}$ is the d-dimensional sphere containing the origin, then the probability that any two coupled paths starting at different points collapse in finite time is zero. Also, if $\mathcal{R}$ is the intersection of the d-dimensional sphere containing the origin with a half-space determined by a hyperplane perpendicular on a diameter at a distance strictly larger than half the radius, then there exist points x, y in $\mathcal{R}$ with the property that $x - y \in G(\mathcal{R})$ and $z_x(\cdot, \omega)$, $z_y(\cdot, \omega)$ collapse with probability zero.*

Proof. This is a consequence of Proposition 4. There is no grid for the sphere, a convex region. In the case of the sectioned sphere, the grid is represented by parallel planar regions parallel to the planar section - among them one through the origin. If the origin is situated very close to the section it is easy to pick two points $x, y \in \mathcal{R}$ such that $k = y - x$ is parallel to the grid yet $\|k\|$ exceeds the largest of the diameters of the grid. Collapse in one step becomes impossible and the probability to hit a non-grid point at exit is one. $\square$

3. **Proof of Theorem 1.**

Let $\mathbf{n}$ be an arbitrary unit vector in $\mathbb{R}^d$. We construct a subdomain $S_0(\mathbf{n}) \subseteq \mathcal{R}$, called a *solid radial domain of direction* $\mathbf{n}$ in the following manner. Since the origin is an interior point of $\mathcal{R}$ there exists a sufficiently small $r > 0$ such that a cube $C(0)$ of side length r will be contained in $\mathcal{R}$ and one pair of facets of $C(0)$ are perpendicular to $\mathbf{n}$. Informally, the cube can be extended in the direction $+\mathbf{n}$ by removing the face with projection $+r$ onto $\mathbf{n}$. The infinite half-cone intersected with $\partial\mathcal{R}$ is $S_0(\mathbf{n})$. Formally, let $H(\mathbf{n})$ be the hyperplane with normal $\mathbf{n}$ passing through the origin and $\Pi_{H(\mathbf{n})}$ the projection operator onto the hyperplane. Let $C_{d-1}(r)$ be a $d-1$ dimensional cube of side $2r$ in $H(\mathbf{n})$ centered at 0. We define

$$S_0(\mathbf{n}) = \{x \in \mathcal{R} \, : \, \Pi_{H(\mathbf{n})}(x) \in C_{d-1}(r) \, , \quad x \cdot \mathbf{n} \geq -r\} \, . \tag{3.1}$$

Proposition 2. *Let S be the intersection of a convex region of $\mathbb{R}^d$ with piecewise smooth boundary, containing the origin as an interior point with the region $\mathcal{R}$, having a nonempty intersection with $\partial\mathcal{R}$ of positive surface area. For any given $k \in \mathcal{R}$ consider the Markov chain $\{Y_n\}_{n\geq 0}$ from (2.5) derived from the Brownian motion with rebirth starting from k. Then the probability for the Markov chain Y_n to reach S in finite time is positive.*

Proof. Algorithm to transfer k to S with positive probability.
For an arbitrary unit vector $\mathbf{n} \in \mathbb{R}^d$ we construct a solid radial domain $S_0(\mathbf{n})$ with a sufficiently small $r > 0$ such that $S_0(\mathbf{n}) \subseteq S$. We recall that $S_0(\mathbf{n})$ is "oriented" in the sense that $\{x \in S_0(\mathbf{n}) \, : \, x \cdot \mathbf{n} \leq 0\}$ is included in $\mathcal{R}$.

Let $S_0(\mathbf{n}) + k$ be the translation of $S_0(\mathbf{n})$ by the vector $k \in \mathbb{R}^d$. We are in a position to move the Brownian path inside $S_0(\mathbf{n})$ at the endpoint containing O and inside $S_0(\mathbf{n}) + k$ at the endpoint containing k such that, with positive probability, we either hit the boundary $\partial\mathcal{R}$ first with the endpoint which was situated at k originally, in which case we are done, or we first hit the boundary $\partial\mathcal{R}$ with the endpoint which was situated at O originally, in which case we have moved a strictly positive distance in the direction $\mathbf{n}$. To make this more precise, the distance in the sense of projection onto the oriented vector $+\mathbf{n}$ of the difference between the new position and the position at start has a lower bound. This is the minimum of the projections onto $\mathbf{n}$ of $S_0(\mathbf{n}) \cap \partial\mathcal{R}$. For sufficiently small r, by continuity of the boundary surface, we can ensure that the lower bound is $|\mathbf{a}|/2$ where $\mathbf{a}$ is the intersection point of the straight line going through the origin in $+\mathbf{n}$ direction with the boundary $\partial\mathcal{R}$. By repeating the procedure, since the region $\mathcal{R}$ is finite, we shall hit the boundary with the endpoint which was originally at k with positive probability. This will bring k inside S with positive probability in a finite number of steps. □

Proposition 3. *Let $\mathbf{n}'$ be an arbitrary unit vector in $\mathbb{R}^d$. Then, there exist $r > 0$ and a unit vector $\mathbf{n} \in \mathbb{R}^d$ such that the solid radial domain $S_0(\mathbf{n})$ intersects the boundary over a smooth connected subset F which is not normal to $\mathbf{n}'$ in the sense that $\lambda\Big(H \cap F\Big) = 0$ for any hyperplane H with normal parallel to $\mathbf{n}'$, where $\lambda(\cdot)$ is the Lebesgue surface measure.*

Proof. For $(r, x) \in (0, \infty) \times \partial\mathcal{R}$ we define the solid radial domain of direction $\mathbf{n} = x/\|x\|$ of size r. For sufficiently small $r > 0$ this is a proper radial domain in the sense that it will intersect the boundary of the region on only one side of the half-parallelepiped. The boundary $\partial\mathcal{R}$ contains points x such that either a small neighborhood F_x of x intersected with $\partial\mathcal{R}$ is nonplanar or is planar but has a different normal direction. The second fact is a consequence of the finiteness of $\mathcal{R}$, which cannot have only one direction for the boundary. The first fact means that the intersection of F_x with any hyperplane has surface area zero. We shall take $\mathbf{n} = x/\|x\|$ and r sufficiently small such that the intersection of $S_0(\mathbf{n})$ with $\partial\mathcal{R}$ be a subset of F_x, setting $F = F_x$. □

If T_x and T_y are the first hitting times of the boundary for the Brownian motion starting at $x, y \in \mathcal{R}$, we define

$$
\begin{aligned}
K_1 &= \Big\{ k \in \mathbb{R}^d \,:\, \exists\, x, y \in \mathcal{R} \text{ with } k = y - x \text{ and } P(T_x = T_y) > 0 \Big\} \\
K_n &= \Big\{ k \in \mathbb{R}^d \,:\, P(Y_n = 0 \,|\, Y_0 = k) > 0 \Big\}\,.
\end{aligned}
\tag{3.2}
$$

Proposition 4. *If the region $\mathcal{R}$ is convex, $d > 1$ and $k \in K_1$ then k is parallel to one of the planar subsets of $\partial\mathcal{R}$.*

Proof. We want to show that two paths $z_x(t, \omega)$ and $z_y(t, \omega)$, starting at $x, y \in \mathcal{R}$, respectively, collapse *in one step* or at the first boundary hit with positive probability only if $y - x = k$ is parallel to one of the *flat* subsets of the boundary. This is a necessary but not sufficient condition, as Example 1 shows.

If $P(T_x = T_y) > 0$ then $\lambda\Big((\partial\mathcal{R} \pm k) \cap \partial\mathcal{R}\Big) > 0$ (from Proposition 6) which implies that there exists a point $x_0 \in \partial\mathcal{R}$ and a ball $B(x_0, \delta)$ with $\delta > 0$ such that $F_0 = B(x_0, \delta) \cap \partial\mathcal{R}$ has the property $\lambda\Big(F_0\Big) > 0$ and $F_0 \pm k \subseteq \partial\mathcal{R}$. To fix ideas let's assume $F_0 + k \subseteq \partial\mathcal{R}$. We can take x_0 in the interior of a smooth component of the boundary and δ sufficiently small such that F_0 be smooth. By convexity we have $\mathcal{R}_0 = \{x + \alpha\, k \,:\, \alpha \in (0,1)\,,\ x \in F_0\} \subseteq \mathcal{R}$. We want to show that F_0 is a subset of a hyperplane. Suppose this is not true. We pick a point x_0 in the interior of F_0 and consider the tangent hyperplane to $\mathcal{R}$ at both x_0 and $x_0 + k$. The two hyperplanes must coincide, otherwise convexity would be violated. We can take another point x_1 arbitrarily close to x_0 and repeat the argument. If the neighborhood of x_0 contained in F_0 is not contained in a hyperplane, one of the pairs of tangent planes must be distinct, bringing a contradiction.

There are two possibilities left: either F_0 and $F_0 + k$ are included in the same hyperplane, or in distinct parallel hyperplanes. We show that the second case is impossible.

Assume F_0 is included in H_1 and $F_0 + k$ is included in H_2 and $H_2 = H_1 + k$. Since $\lambda\Big(H_i \cap \partial\mathcal{R}\Big) > 0$ for both $i = 1, 2$, again by convexity, the region $\mathcal{R}$ is on *only one* side of H_1 and on *only one* side of H_2. To collapse when the path reaches F_0 the starting points x, y with $y - x = k$ must be located in the interior of the region. It follows that the segment $[x, y]$ is contained in $int(\mathcal{R})$ and so there exist x', y' in $int(\mathcal{R})$ with $y' - x'$ parallel to k such that $\|y' - x'\| > \|k\|$. This is impossible for the set situated between the two hyperplanes. We obtain a contradiction. The only possibility remains that F_0 and $F_0 + k$ be subsets of the same hyperplane. This concludes our proof. $\square$

Proposition 5. *If the region $\mathcal{R}$ is convex or a polyhedron, then $K_n \subseteq G(\mathcal{R})$.*

Proof. Proposition 4 proves the case $n = 1$ because in the case of a polyhedron the statement is trivial. We shall proceed inductively over $n \in \mathbb{Z}_+$ to prove the statement. We denote $P(k \to dk') = P(Y_{m+1} \in dk' \,|\, Y_m = k)$, the transition probability of the chain $Y_n(\omega)$. Assume Proposition 5 is true for $n = 1, 2, \ldots m$ and we want to show it for $n = m + 1$. Then

$$\Big\{k \,:\, P(Y_{m+1} = 0 \,|\, Y_0 = k) > 0\Big\} = \Big\{k \,:\, \int_{\mathcal{R}} P(k \to dk') P(Y_{m+1} = 0 \,|\, Y_1 = k') > 0\Big\}$$

$$= \Big\{k \,:\, \int_{\mathcal{R}} P(k \to dk') P(Y_m = 0 \,|\, Y_0 = k') > 0\Big\}$$

by the Markov property. The integral can be positive only if the integrand $P(Y_m = 0 \,|\, Y_0 = k')$ is positive, which implies that $k' \in K_m$. A necessary condition that an integral of a nonnegative function f with respect to a measure μ be positive is that $\mu(supp(f)) > 0$. In this case, K_m is a closed set and $P(k \to K_m) > 0$. We have shown that

$$K_{m+1} \subseteq \Big\{k \,:\, P(k \to K_m) > 0\Big\} \subseteq \Big\{k \,:\, \lambda\Big((\partial\mathcal{R} \pm k) \cap K_m\Big) > 0\Big\}.$$

We shall conclude the proof if we notice that in order to have $\lambda\Big((\partial\mathcal{R} \pm k) \cap K_m\Big) > 0$, given that $K_m \subseteq G(\mathcal{R})$, the vector k will be the difference between points situated in two parallel hyperplanes with direction $\mathbf{n}_j$ for a given $j \le d(\mathcal{R})$ from Definition 1. Recall that $\mathbf{n}_1, \ldots, \mathbf{n}_{d(\mathcal{R})}$ are all the possible normals to the planar subsets of the boundary $\partial\mathcal{R}$ defined previously. In other words, there exists a direction $\mathbf{n}_j$ such that $\mathbf{n}_j \cdot k \in Z^j$. Therefore $k \in G(\mathcal{R})$. Without loss of generality, we can choose only one such direction $\mathbf{n}_j$ due to the fact that the surface area of sets corresponding to more than one direction is zero (codimension greater or equal to 2 in $\mathbb{R}^d$). $\square$

Proposition 6. *Let $\mathcal{R}$ be a bounded open region in $\mathbb{R}^d$ with piecewise smooth boundary. Let x, y two points in $\mathcal{R}$ and $k = y - x$. Then, the probability that the two paths $z_x(\cdot,\omega)$ and $z_y(\cdot,\omega)$ of the coupled process of Brownian motion with rebirth collapse at the first hitting time of the boundary $T = \min(T_x, T_y)$ is*

$$P\Big(T_x = T_y\Big) = u(x, (\mathcal{R} - k) \cap \mathcal{R}, (\partial\mathcal{R} - k) \cap \partial\mathcal{R}) \tag{3.3}$$

where u is given by (2.3) with $\mathcal{A} = (\mathcal{R} - k) \cap \mathcal{R}$ and $U = (\partial\mathcal{R} - k) \cap \partial\mathcal{R}$.

Remark. The probability given in (3.3) can be expressed in a symmetric fashion by considering the shift by $k' = x - y$. Note that $U = (\partial\mathcal{R} - k) \cap \partial\mathcal{R} \subset \partial((\mathcal{R} - k) \cap \mathcal{R}) = \partial A$.

Proof. There are two ways to collapse in one step (at the first boundary hit). The first is to hit the boundary simultaneously with the two paths and the second is to hit the boundary with one path while the other is at zero. The second case implies that $z_x(t,\omega)$ reaches $x - y$ or $y - x$ and at least one of them belongs to the boundary. We have to calculate (2.3) for a set $U = \{\pm k\}$ over the region $\mathcal{R}$, which is zero in $d > 1$. The $d = 1$ case is treated in [2], [3]. This reduces the problem to the first case, when the probability of collapse in one step is given by (3.3). $\square$

Proof of Theorem 1. We have to prove three facts: 1) that if two paths collapse with positive probability, then their relative distance must enter the set of *grid* points $G(\mathcal{R})$ after a finite number of steps, 2) that the probability to collapse is zero if the initial points were such that $x-y \notin G(\mathcal{R})$ and 3) that the probability that starting at points x, y in $\mathcal{R}$ with $x-y=k \in G(\mathcal{R})$, the chain $\{Y_n\}_{n\geq 0}$ from (2.5) with Y_0 defined as the position of the particle which did not hit the boundary in one step, will exit the set $G(\mathcal{R})$ in finite time with positive probability.

1) This is a consequence of Proposition 4 and Proposition 5.

2) In view of 1) it is clear that it would be enough to show that the probability that starting from $k \in \mathcal{R} \setminus G(\mathcal{R})$ we enter $G(\mathcal{R})$ *in one step* is zero. Assume the contrary. From the definition (2.5) it follows that

$$
\begin{aligned}
P\Big(Y_{n+1} \in G(\mathcal{R}) \,|\, Y_n = k\Big) &= u(0, (\mathcal{R}-k)\cap\mathcal{R}, (G(\mathcal{R})-k)\cap(\mathcal{R}-k)\cap\partial\mathcal{R}) \\
&+ u(0, (\mathcal{R}-k)\cap\mathcal{R}, (\partial\mathcal{R}-k)\cap\mathcal{R}\cap G(\mathcal{R})) \\
&+ u(0, (\mathcal{R}-k)\cap\mathcal{R}, (\partial\mathcal{R}-k)\cap\partial\mathcal{R}) \cdot \mathbf{1}_{\{0\}}(G(\mathcal{R})) > 0\,.
\end{aligned}
$$

Since $k \in \mathcal{R}\setminus G(\mathcal{R})$, $\lambda((\partial\mathcal{R}-k)\cap\partial\mathcal{R}) = 0$. Hence $u(0,(\mathcal{R}-k)\cap\mathcal{R},(\partial\mathcal{R}-k)\cap\partial\mathcal{R}) = 0$. Also we notice that both sets $(G(\mathcal{R})-k)\cap\partial\mathcal{R}$ and $(\partial\mathcal{R}-k)\cap G(\mathcal{R})$ are the intersection of two surfaces, hence of measure zero unless at least one subset of positive area coincides. According to Propositions 4, 5 and 6, this can only happen if the subset is included in a hyperplane parallel to one component of $G(\mathcal{R})$. That makes the vector k to be the difference of elements of two parallel planes contained in the *grid* $G(\mathcal{R})$ which implies that its projection on the common normal, say $\mathbf{n}_j$ to the planes is in Z^j. This is the definition of $k \in G(\mathcal{R})$.

3) We notice that $k \in G(\mathcal{R})$ is equivalent to $k \cdot \mathbf{n}_j \in Z^j$ for a subset of normal directions $\mathbf{n}_j$. First we notice that if we start with a *node* we shall enter $G(\mathcal{R}) \setminus N(\mathcal{R})$ in one step with positive probability, due to the codimension two of the set allowing it to remain a node. Assume that $k \cdot \mathbf{n}_i \notin Z^i$ for a given normal direction $\mathbf{n}_i$, or that there exist nonplanar subsets on the boundary with positive area (in the convex but non polyhedral region case). We want to show that, with positive probability, we can determine an evolution of the chain $\{Y_n\}$ which exits the grid $G(\mathcal{R})$. Naturally, if the two paths exit the grid in one step (at the first ever boundary hit) with positive probability, we are done. Assume that this does not happen. Without loss of generality, we shall denote by k the initial position of the Markov chain $\{Y_n\}$.

There are two possible outcomes for k while the successive boundary hits occur:

(i) Y_n enters the set $\mathcal{R} \setminus G(\mathcal{R})$ with positive probability, in which case we are done,

(ii) Y_n stays in $G(\mathcal{R})$ with positive probability.

In case (ii) we shall hit the boundary over a *flat* portion parallel to one of the facets of the region at each time. The reason for this is that we have to ensure that $(\partial\mathcal{R} \pm k) \cap G(\mathcal{R})$ has positive surface area. The grid has codimension one in the case of both general polyhedral regions and convex regions $\mathcal{R}$. The reasoning is the same as in Proposition 5.

This implies that no matter how many planar directions from the grid k is parallel to, with positive probability, after one step, we can transfer $k \to k'$ with k' parallel to only one planar direction of the grid. Let the normal to that planar direction be denoted by $\mathbf{n}_i$. Proposition 3 shows that there will be a solid radial domain $S_0(\mathbf{n})$ such that the intersection with $\mathcal{R}$ is not perpendicular to $\mathbf{n}_i$. The algorithm from Proposition 2 enables us to bring k' inside $S_0(\mathbf{n})$ with positive probability. Then, with positive probability, the first boundary hit of the Brownian paths will be through the unique part of the boundary of $S_0(\mathbf{n})$ not perpendicular on $\mathbf{n}_i$. This is equivalent to having Y_n exit the grid, and concludes the proof of the theorem. $\square$

References

[1] Duffie, D. (1996) *Dynamic Asset Pricing Theory.* Princeton University Press, Princeton, N.J.

[2] Grigorescu, I., Kang, M. (2002) *Brownian Motion on the Figure Eight.* J. Theoret. Probab. 15, no. 3, 817–844.

[3] Grigorescu, I., Kang, M. (2003) *Path Collapse for an Inhomogeneous Random Walk.* J. Theoret. Probab. 16, no. 1, 147–159.

[4] Grigorescu, I., Kang, M. (2001) *Ergodic Properties of Multidimensional Brownian Motion with Rebirth.* Preprint.

[5] Jerison, D., Kenig, C. (1981) *The Dirichlet problem in Nonsmooth Domains.* Ann. of Math.(2) 113, no. 2, 367–382.