MARKOV PROCESSES WITH REDISTRIBUTION

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Abstract. We study a class of stochastic processes evolving in the interior of a set D according to an underlying Markov kernel, undergoing jumps to a random point x in D with distribution $\nu_\xi(dx)$ as soon as they reach a point ξ outside D. We give conditions on the family of measures $\nu_\xi(dx)$ preventing that infinitely many jumps occur in finite time (explosiveness), conditions for ergodicity and the existence of a the spectral gap. The setup is applied to a multitude of models considered recently, including particle systems like the Fleming-Viot branching process and a new variant of the Bak-Sneppen dynamics from evolutionary biology. The last part of the paper is expository and discusses the relation with quasi-stationary distributions.

1. Introduction

Let $S \subseteq \mathbb{R}^d$, $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$ a filtered probability space and $(x^o(t))$, $t \geq 0$ a Dynkin-Feller process on S, adapted to $(\mathcal{F}_t)$, which satisfies the usual conditions, i.e. the filtration is P - complete and right-continuous with rcll (right continuous with left side limit) paths almost surely. The process will be assumed to be *stochastic*, i.e. $P_x(x^o(t) \in S) = 1$ for any $x \in S$, $t \geq 0$; it will be referred to as the *underlying process*. In addition, $D \subseteq S$ will be an open set in the topology induced on S as a subspace of $\mathbb{R}^d$. For any $\xi \in S \setminus D$ we have a probability measure $\nu_\xi(dx)$ on D such that $\xi \to \nu_\xi(dx)$ is measurable, where $S \setminus D \ni \xi$ is endowed with its Borel σ - algebra and $M_1(D) \ni \nu_\xi(dx)$ with the Borel sets of the topology of convergence in distribution. These are the *redistribution measures*. Here $M_1(X)$ denotes

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Date: October 29, 2012.

1991 *Mathematics Subject Classification.* Primary: 60J70; Secondary: 60J75, 60J35.

Key words and phrases. Doeblin condition, jump diffusion process, Fleming-Viot, quasi-stationary distribution (qsd).

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the space of probability measures on the Polish space X. For such a process [25], the first hitting time τ^D of $S \setminus D$

$$\tau^D = \inf\{t > 0 \,|\, x^o(t) \in S \setminus D\} \tag{1.1}$$

is a stopping time (not necessarily finite) and has the property that $P_x(\tau^D > 0) = 1$ for any $x \in D$.

This setup allows the construction of a new Markov process $(x(t))$ on D. Starting from $x \in D$, $x(t)$ evolves according to $(x^o(t))$ until τ^D-, when $x^o(t)$ exits D at $\xi = x^o(\tau^D) \in S \setminus D$. After that, $x(t)$ instantaneously jumps to a random point $x_1 \in D$ with distribution $\nu_\xi(dx)$ and continues this evolution inductively, putting back to back episodes in D of total positive time length τ^*. When $\tau^* < \infty$ we say that $(x(t))$ is *explosive*.

Even though the construction is more general, the examples we discuss refer only to diffusions and to pure jump processes on the lattice $\mathbb{Z}^d$.

1.1. **Examples.** A more detailed presentation of the examples is done in the last section.

1. *Brownian motion with redistribution (rebirth).* In the simplest version, the diffusion with jumps on the connected open set D has a delta function relocation measure $\nu_\xi(dx) = \delta_{x_0}(dx)$, $x_0 \in D$ [15, 17], constant in $\xi \in \partial D$, $\nu_\xi(dx) = \nu(dx) \in M_1(D)$ [5], or with continuous dependence on the exit point $\xi \to \nu_\xi(\cdot) \in M_1(D)$ [6]. In the special case of the delta measure and bounded D, the process is generated by a Feller semigroup on a compact manifold where the boundary is glued together with the return point x_0 [15, 17] (figure eight). Other variants include the case [21] of a domain with piecewise smooth boundary and constant redistribution measure on each smooth component. The interest in the problem stems from a multitude of applications, for example in barrier options pricing theory (see [26] for the definition).

2. *An additive increase multiplicative decrease (AIMD) model for distributed resource allocation* [22]. Here a Markov process evolves in a simplex D of positive coordinates $\mathbf{x} = (x_1, \ldots, x_d)$ with $\sum x_i < C$. The particles $x_i(t)$ represent rates of flow (e.g., internet data flow) moving (possibly deterministically) on a straight line (additive increase). Upon reaching C (congestion), particles are redistributed inside the domain to a fraction of their current value (multiplicative decrease). This is reflecting the practice of *throttling* applied to outliers of data usage employed by the telecoms.

3. *Diffusive Bak-Sneppen fitness evolution model.* We consider an n particle system, a variant of the well known evolution model proposed by Bak and Sneppen in [4]. The particles follow Brownian motions $x_i(t)$ for $i \in \{1, 2, \cdots n\}$ in an interval $(0, a]$, $a > 0$ with reflection at $a \in \mathbb{R}$ and with 0 considered a boundary point, evolving independently of each other until the first one reaches 0. Each coordinate x_i represents the 'fitness state' of the i-th species and has an associated set of *neighbors* $V_i \subseteq \{1, 2, \ldots, n\}$, designated by their indices, such that $i \in V_i$. Whenever one of the fitness levels x_i reaches the boundary point zero, all x_j with $j \in V_i$ are instantaneously replaced by new i.i.d. fitness levels with distribution function $G(x)$ on the interval $(0, a)$ and the evolution continues afresh. To fix ideas, we shall assume $i \to V_i$ to be deterministic functions and $|V_i| \geq 2$, for all $1 \leq i \leq n$, where $|V|$ is the cardinality of V. It is easy to verify that $n = 1$ and $n \geq 2$ with $|V_i| = 1$ are covered in subsection 4.1 since this coincides with $V_i = \{i\}$ and particles move independently.

4. The Fleming-Viot branching particle system, studied in [8, 16, 23] for Brownian motions, and in [14, 1] in a discrete setting (where the existence of a quasi-stationary measure is obtained from the empirical measures) is perhaps the most important example. Here the redistribution probabilities $\nu_\xi(dx)$ send a particle reaching the boundary to one of the locations of the remaining particles chosen at random. This process satisfies Condition 2 but the proof is more difficult. It is rigorously presented in [19], but the main ideas are outlined in the last section of the present paper (Lemma 3). A discussion of the hydrodynamic limit (LLN) of the particle system and its connection to the Yaglom limit of the underlying process is also presented.

1.2. **General questions.** 1) If the redistribution measures are very singular, perhaps the mass migrates towards ∂D and jumps become very frequent. When is $(x(t))$ non-explosive? More precisely, under what conditions on $(\nu_\xi(dx))$ does the process not end in finite time, i.e. $P_x(\tau^* = \infty) = 1$ for all $x \in D$? This is equivalent to $(x(t))$ being *stochastic*, or equivalently having the transition probabilities satisfy $P_x(x(t) \in D) = 1$, $t > 0$. Proposition 1 gives a sufficient condition, namely that $(\nu_\xi(dx))_{\xi \in \partial D} \in M_1(D)$ be tight. However, it is easy to verify that the FV process (Example 4) for more than three particles is not tight. This difficulty is resolved with Lemma 3. Other proofs in [27], [7] are exploiting the idea that since Brownian paths meet with probability zero, the same is true for the "broken" paths of the FV process, via some path-by-path transformations. When the drift towards the boundary is very strong, explosion may occur [9].

2) A second question is, if the process is non-explosive, is it ergodic? Quite generally, the answer is positive. Due to the structural bound (3.5), it is easy to verify for many models (e.g. diffusions) that a compact set in D is a small set. If the small set is accessible, the process is Harris recurrent. Determining the speed of convergence may be harder, but we provide a criterion (2.14) for the strong Doeblin condition. How can one characterize its invariant measure? The answer is (3.13) in Theorem 3. Even though this provides a formula, it is dependent on the invariant measure of the interior chain (2.3) or, equivalently, the boundary chain (2.4).

3) The relation with particle systems. In the multi-particle systems cases, scaling limits - law of large numbers [8], hydrodynamic limits [16, 14], large deviations [18], fluctuation fields - can be considered. An important problem is to establish a lower bound for the spectral gap, uniformly over the dimension of the system (number of particles). Theorem 4 relates the spectral gap of the killed process with the spectral gap of the redistribution process. More precise results can be obtained for specific settings ([15, 17, 5, 6, 21]). Unfortunately these are one particle models. A lower bound independent of the number of particles is not yet available.

4) Relation to the existence of the quasi-stationary distribution. For particle systems, the tightness (and limit) of the empirical measure under the invariant distribution (3.13) is very important in connection to qvasi-invariant distributions. It is intuitively clear that the equilibrium measure of the n - dimensional FV system should produce an empirical measure on D converging as $n \to \infty$ to the quasi-stationary distribution of the driving process (4.11). The method is introduced in [14] and is related to an idea in [13]. This question is solved for some pure jump process in [2, 3].

2. Construction and non-explosiveness.

Since D is allowed to be unbounded, even for very well behaved underlying processes and regular domains D, we may see the possibility that $P_x(\tau^D = \infty) > 0$ for some $x \in D$. This is not a major concern, because explosiveness occurs when too many redistribution jumps occur, and not too few. However, we recall that there are no instantaneous jumps from inside D, i.e. $P_x(\tau^D > 0) = 1$. We further denote by $P^D(t, x, dy)$ the transition probabilities of the killed process $(\tilde{x}(t))_{t\geq 0}$

$$P^D(t, x, dy) = P_x(x^o(t) \in dy\,,\, \tau^D > t)\,, \qquad \forall x \in D \tag{2.1}$$

and by $(P_x^D)_{x\in D}$ its law when starting at x.

We shall construct formally a Markov process $(x(t))_{t\geq 0}$ on D based on an infinite collection $(\tilde{x}_l(t))_{l\geq 1}$ of independent processes with laws $P_\cdot^D$ for each $l \geq 1$. For each of these processes, denote τ_l^D the first hitting time of $S \setminus D$. Set $\tau_0 = 0$. Then, for any $x \in D$, define $x(t) = \tilde{x}_1(t)$ on $t \in [0, \tau_1^D)$. If $\tau_1^D = \infty$, we put $l^* = 0$, where l^* denotes the total number of jumps, and we are done. If not, we continue. When $x(t)$ reaches $S \setminus D$ at $\xi_1 = \tilde{x}(\tau_1^D -)$, it instantaneously jumps to a random point x_1, independent of the process, with probability distribution $\nu_\xi(dx_1)$ and we update $\tau_1 := \tau_1^D$. The process continues as $x(t) = \tilde{x}_2(t - \tau_1^D)$ on the time interval $\tau_1^D \leq t < \tau_1^D + \tau_2^D$, where $\tilde{x}_2(\cdot)$ has law $P_{x_1}^D$. Unless $\tau_2^D = \infty$, in which case we have $l^* = 1$, upon reaching $S \setminus D$, we set $\tau_2 := \tau_1 + \tau_2^D$ and the we continue inductively for all $l \geq 1$. If the total number of jumps l^* is finite, we set $\tau_l = \infty$ for all $l > l^*$. By construction, the sequence $(\tau_l)_{0\leq l\leq l^*}$ is strictly increasing almost surely. We denote $\tau^* = \lim_{l\to\infty} \tau_l \leq \infty$. Theorem 1 proves that under some additional conditions the process $(x(t))$ is *non-explosive*, or equivalently $P_x(\tau^* = \infty) = 1$, implying that the transition probabilities are stochastic.

Even when the underlying process is very well behaved (satisfying Conditions 1 and 3, easily verified in the diffusive case), the questions are nontrivial because of the generality of the family of redistribution measures. In the present paper and [19], we provide answers for several families $(\nu_\xi(dx))$. A direct answer is the tightness criterion from Proposition 1 (ii), applicable to the examples in subsection 4.1. The particle system subsection 4.2 is a useful example when the redistribution measures fail to be tight. In this case we provide the more general Proposition 1 (i). The geometric ergodicity results are proved with the Doeblin condition; they actually constitute good examples of a relative strength of probabilistic methods over their analytic counterparts [15, 17, 6, 23], in the sense that except for explicit calculations, Theorems 3 and 4 give the same qualitative results with a much shorter proof.

2.1. **Notations.** In the following $\lambda(B)$ will be a reference measure (Lebesgue measure when $x^o(t)$ is a diffusion) of a Borel set $B \subseteq \mathbb{R}^d$ and we shall assume that the transition probabilities of the underlying process have densities $P^D(t,x,dy) = p^D(t,x,y)dy$, where we use the notation $\lambda(dy) = dy$ for simplification. The harmonic measures on $S \setminus D$ will be denoted by

$$\lambda_0(x, d\xi) = P_x(x^o(\tau^D) \in d\xi)\,, \qquad \xi \in S \setminus D\,. \tag{2.2}$$

Naturally, when the underlying process is a diffusion and D is regular, we restrict ourselves to $\xi \in \partial D$ (the boundary of D) since $\lambda_0(x, \partial D) = 1$.

Given $\delta > 0$ we shall use the notation $D_\delta = \{x \in D \,|\, d(x, \partial D) > \delta\}$ for the open subset of D with distance at least δ from the boundary ∂D. In the lattice case state space case, $\delta = 1/2$ is a valid choice. For $\delta > 0$, we shall use the notation $D_\delta^c = D \setminus \bar{D}_\delta$ and $\tau^{D_\delta^c}$ for the first exit time from D_δ^c (note that $D_\delta^c \neq S \setminus D$).

Condition 1 (i) is immediate if we assume that the densities of (2.1) are positive. In the case of regular diffusions (3.12) on smooth domains Condition 1 is easily satisfied, as well as for pure jump processes with bounded jump rates. Nonetheless, for the sake of generality, we assume the following.

Condition 1. *For any sufficiently small $\delta > 0$ we have the properties:*
(i) For any $x \in D$, $P_x^D(\tau^D > 0) = 1$ and for any $x \in D$, $t > 0$, $P_x^D(\tau^D > t) > 0$;
(ii) for any $t > 0$, $\inf_{x \in \bar{D}_\delta} P_x^D(\tau^D > t) > 0$, and
(iii) $\lim_{t \to \infty} \sup_{x \in D_\delta^c} P_x^D(\tau^{D_\delta^c} > t) = 0$.

Two Markov chains can be constructed based on $(x(t))$. Define the *interior chain* $X_l := x(\tau_l)$, $l \geq 0$, with state space D and transition probabilities

$$S(x, dy) = \int_{S \setminus D} P_x(x(\tau^D-) \in d\xi)\nu_\xi(dy) = \int_{S \setminus D} \lambda_0(x, d\xi)\nu_\xi(dy)\,. \tag{2.3}$$

The *boundary chain* Y_l , $l \geq 0$ on $S \setminus D$ can be defined with transition probabilities

$$P(Y_1 \in d\xi' \,|\, Y_0 = \xi) = \int_D \nu_\xi(dx)\lambda_0(x, d\xi')\,. \tag{2.4}$$

We note that $S(x, D) = 1$ without any assumption on $(\nu_\xi(dx))$, thus (X_l), (Y_l) are never explosive, being well defined for any $l \geq 0$, even in case $\tau^* < \infty$.

For $f \in C_b(S)$, the semigroup $P_t f(x) = \int_D p(t, x, y) f(y) dy$ satisfies

$$P_t f(x) = P_t^D f(x) + \int_0^t \int_D P_{t-s} f(x') \int_{\partial D} P_x(\tau^D \in ds, x(\tau^D-) \in d\xi)\nu_\xi(dx')\,. \tag{2.5}$$

Due to (2.5, the densities $(p(t, x, y))$ satisfy

$$p(t, x, y) = p^D(t, x, y) + \int_0^t \int_D p(t-s, x', y) \int_{\partial D} P_x(\tau^D \in ds, x(\tau^D-) \in d\xi)\nu_\xi(dx')\,. \tag{2.6}$$

It is important to notice that, in general, the semigroup (2.5) is not Feller. When $\xi \to \nu_\xi(dx)$ is continuous with the topology of weak convergence of probability measures on ∂D, the process solves the martingale problem for the class of functions in $C_b(D)$ form the domain

$$\mathcal{D} = \{f \in \mathcal{D}^o \mid f(\xi) = \int_D f(y)\nu_\xi(dy)\,,\ \xi \in \partial D\}\,. \tag{2.7}$$

We notice that $P_x(\tau^D \in ds, x(\tau^D-) \in d\xi)$ depends exclusively of the well known process killed at the boundary (3.12). For fixed $t > 0$, this has a density $\lambda_0(x, s, \xi)$, whose marginal with respect to the exit point $\xi \in \partial D$ is the harmonic measure (2.2).

After applying the Laplace transform to (2.5), we see that the resolvent has a kernel

$$R_\beta(x, y) = \int_0^\infty e^{-\beta t} p(t, x, y)dt\,, \qquad \beta > 0$$

verifying

$$R_\beta(x, y) = R^D_\beta(x, y) + \int_D K_\beta(x, dx')R_\beta(x', y) \tag{2.8}$$

where

$$K_\beta(x, dx') = \int_0^\infty \int_{\partial D} e^{-\beta s} P_x(\tau^D \in ds, x(\tau^D-) \in d\xi)\nu_\xi(dx')\,. \tag{2.9}$$

Letting K_β denote the corresponding operator on $C_b(\bar{D})$, we have

$$(I - K_\beta)R_\beta = R^D_\beta\,, \qquad R_\beta = (I - K_\beta)^{-1}R^D_\beta\,, \tag{2.10}$$

where the first equality is rigorous, as the Laplace transform of a convolution in the time variable, while the second equality is formal. The existence conditions of the inverse are difficult to establish directly. However, one can see that the density of the invariant measure from (3.13) is obtained heuristically as

$$\lim_{\beta\to 0} \beta R_\beta(x, y) = \lim_{\beta\to 0} \int_D (\frac{I - K_\beta}{\beta})^{-1}(x, dx')R^D_\beta(x', y) = Z^{-1} \int_D G(x', y)\mu_X(dx')\,, \tag{2.11}$$

where the normalizing constant Z is the average duration of a trip to the boundary in equilibrium. We used the fact that

$$\lim_{\beta\to 0} R^D_\beta(x', y) = G(x', y)\,, \qquad \mu_X(dx') = Z \lim_{\beta\to 0}(\frac{I - K_\beta}{\beta})^{-1}(x, dx')\,. \tag{2.12}$$

Intuitively, K_β is the transition function of the interior resolvent chain (2.3).

2.2. **Non-explosion.** Given a $\delta > 0$, we denote by

$$\alpha(\delta) = \inf\{t \geq 0 \,|\, x(t) \in \bar{D}_\delta\}\,, \tag{2.13}$$

the début time of $\bar{D}_\delta$ by the process $(x(t))$, with the usual convention $\alpha(\delta) = \infty$ if the infimum is over the empty set. We define $l(\delta) = \max\{l \geq 0 \,|\, \tau_l \leq \alpha(\delta)\}$, equal to infinity if $(x(t))$ never reaches $\bar{D}_\delta$; in general, $l(\delta)$ is not a stopping time.

Condition 2. *There exists $\delta > 0$ such that $P_x(l(\delta) < \infty) = 1$.*

The condition is often uniform in $x \in D_\delta^c$ and it is easier to verify, i.e.

$$\lim_{l\to\infty} \sup_{x\in D_\delta^c} P_x(l(\delta) > l) = 0\,. \tag{2.14}$$

A useful way of looking at (i) and (ii) is to observe that $X_l := x(\tau_l)$, $0 \leq l \leq l^*$ (2.3). Even when $l^* = \infty$, it is nontrivial to show that $(x(t))$ is non-explosive since (X_l) may not be tight in D. In other words, the chain might be well defined for all $l \geq 0$, but if it migrates towards the boundary, the duration $\tau_l - \tau_{l-1}$ of an episode between boundary visits diminishes with the possibility that $\tau^* < \infty$.

The next proposition gives some concrete criteria to verify Condition 2.

Proposition 1. *(i) Condition 2 is satisfied if there exists a positive integer m and a positive real constant c_1 such that*

$$\forall x \in D_\delta^c \qquad P_x(l(\delta) \leq m) \geq c_1 > 0\,. \tag{2.15}$$

(ii) A sufficient condition for (i) with $m = 1$ is that the family of redistribution measures $(\nu_\xi(dx))_{\xi\in S\setminus D}$ be tight.

Remark. 1) If $x \in \bar{D}_\delta$, then (2.15) is trivially satisfied with $m = 0$ and any constant $c_1 \leq 1$, which implies that (2.15) can be immediately extended to all $x \in D$.
2) A sufficient condition for (2.15) is that $\nu_\xi(\bar{D}_\delta)$ is that $\xi \to \nu_\xi(dx)$ be continuous and $S \setminus D$ be bounded which works for all the applications discussed in Subsection 4.1. To summarize, Proposition 2 in subsection 4.1 $\Rightarrow$ Proposition 1 (ii) $\Rightarrow$ (2.15).
3) The particle model from Subsection 4.2 requires (2.15) with $m = n$, where $n \geq 2$ (in the nontrivial case) is the number of particles, since *the redistribution measures are not tight.*

Proof. (i) Let $x \in D_\delta^c$. From (2.15) we derive that $P_x(l(\delta) > l) \le (1-c_1)^{[\frac{l}{m}]}$ as an application of the strong Markov property to the chain $X_l := x(\tau_l)$, $l \ge 0$, which implies (2.14).

(ii) From tightness and the fact that $(\bar{D}_\delta)$, $\delta \downarrow 0$ is an exhausting sequence of compacts in D, we have that for any $\epsilon > 0$ there exists $\delta > 0$ such that $\nu_\xi(\bar{D}_\delta) > 1-\epsilon$ for all $\xi \in S \setminus D$. We may take any $0 < \epsilon < 1$. From Condition 1 (iii) we know that, starting at $x \in D_\delta^c$, the exit time τ^D, equal to τ_1 for $(x(t))$, is almost surely finite. For $x \in D_\delta^c$, we have

$$P_x(x(\tau_1) \in \bar{D}_\delta) \ge \inf_{x \in D_\delta^c} \int_{S \setminus D} P_x^D(x(\tau^D-) \in d\xi)\nu_\xi(\bar{D}_\delta) > 1-\epsilon > 0\,. \tag{2.16}$$

This proves that (2.15) is satisfied with $m = 1$ and $c_1 = 1-\epsilon$. □

By construction, the process $x(t)$ will have redistribution jumps at times $(\tau_l)_{l \ge 0}$, starting with $\tau_1 = \tau_1^D$ and continuing with $\tau_l = \sum_{1 \le j \le l} \tau_j^D$ until the last jump at τ_{l^*}. Recall that if there is a finite number of jumps, we convene that $\tau_l = \infty$ for all $l > l^*$. We have to prove that $P_x(\tau^* = \infty) = 1$ for all $x \in D$. This is the contents of Theorem 1. We start with a lemma containing the essence of the proof, and useful as an independent result. Recall $\alpha(\delta)$ is the first hitting time of $\bar{D}_\delta$ by the rebirth process $(x(t))$ and $l(\delta)$ is the number of jumps before $\alpha(\delta)$ (2.13). We note that $\alpha(\delta) \ge \tau^*$ only if $\alpha(\delta) = +\infty$.

Lemma 1. *Assume Condition 1 is satisfied and $P_x(\alpha(\delta) < \tau^*) = 1$ for all $x \in D \setminus D_\delta$. Then the process is non-explosive.*

Remark. The statement $P_x(\alpha(\delta) < \tau^*) = 1$ for all $x \in D$ is then immediate.

Proof. In view of the hypothesis, it is sufficient to prove that $P_x(\tau^* = \infty) = 1$ for any $x \in \bar{D}_\delta$. Let $S < \infty$ be a sufficiently large deterministic time; we want to show that $P_x(\tau^* \le S) = 0$. There are two possibilities: Either there are no redistribution jumps at all, in which case $\tau^* = \infty$, or there exists at least one such jump $\tau_1 < \infty$ and then we define $\alpha_1 = \inf\{t > \tau_1 \,|\, x(t) \in \bar{D}_\delta\}$. In this case we notice that since $P_x(\alpha(\delta) < \tau^*) = 1$ for any x, then $\tau^* > \alpha_1 \ge \tau_1 = \tau^D$ with probability one. Put $u(S) = \sup_{x \in \bar{D}_\delta} P_x(\tau^* \le S)$. Applying the strong Markov property to the stopping time α_1, we obtain

$$P_x(\tau^* \le S) = P_x(\tau^* \le S\,,\, \alpha_1 < \tau^*)$$

$$\le P_x(\tau^* \le S\,,\, \alpha_1 < S) = \int_0^S P_x(\tau^* \le S \,|\, \alpha_1 = s) P_x(\alpha_1 \in ds)$$

$$\leq \int_0^S E_x[P_{x(s)}(\tau^* \leq S - s \,|\, \alpha_1 = s)]\, P_x(\alpha_1 \in ds) \leq u(S) P_x(\alpha_1 \leq S)$$

after taking the supremum over $x(\alpha_1) \in \bar{D}_\delta$ in the last inequality. The supremum over $x \in \bar{D}_\delta$ on both sides of the inequality, as well as the fact that $\alpha_1 \geq \tau^D$ give

$$0 \geq u(S)(1 - \sup_{x \in \bar{D}_\delta} P_x(\alpha_1 \leq S)) \geq u(S) \inf_{x \in \bar{D}_\delta} P_x(\tau^D > S)\,.$$

Our claim is proved if we show that for sufficiently large S, $\inf_{x \in \bar{D}_\delta} P_x(\tau^D > S) > 0$, which is guaranteed by Condition 1 (ii). Then we have $u(S) = 0$ for any $S > 0$ large enough, proving the claim. ☐

Theorem 1. *Assume that Conditions 1 and 2 are satisfied for the same $\delta > 0$. Then for any $x \in D$, we have $P_x(\tau^* = \infty) = 1$.*

Proof. Condition 1 (i) implies that the sequence $(\tau_l)_{0 \leq l \leq l^*}$ is strictly increasing almost surely. Due to Lemma 1, we only have to show that $P_x(\alpha(\delta) < \tau^*) = 1$ for any $x \in D$. By construction $\alpha(\delta) = \inf\{t \geq 0 \,|\, x(t) \in \bar{D}_\delta\}$, which means that either $\alpha(\delta) < \tau^*$ (if it occurs in one of the episodes $[\tau_{l-1}, \tau_l)$, $1 \leq l \leq l^* + 1$) or $\alpha(\delta) = \infty$, or equivalently, the process never enters $\bar{D}_\delta$. If $x \in \bar{D}_\delta$, then $\alpha(\delta) = 0 < \tau^*$. Assuming $x \in D^c_\delta$, Condition 2 implies that $P_x(l(\delta) < \infty) = 1$, which means that $\alpha(\delta) \in [\tau_{l(\delta)}, \tau_{l(\delta)+1})$. We proceed to show a slightly stronger statement than needed, namely that $P_x(\alpha(\delta) < \infty) = 1$ for all $x \in D^c_\delta$.

Let $t > 0$ and $x \in D^c_\delta$. For any $l \geq 1$,

$$P_x(\alpha(\delta) > t) \leq P_x(\alpha(\delta) > t\,,\, l(\delta) \leq l) + P_x(l(\delta) > l)\,, \tag{2.17}$$

$$\leq P_x(\tau_{l(\delta)+1} > t\,,\, l(\delta) \leq l) + P_x(l(\delta) > l)\,, \tag{2.18}$$

providing the bound (we recall that $\tau^D_l = \tau_l - \tau_{l-1}$)

$$P_x(\alpha(\delta) > t) \leq (l+1) \sup_{x \in D^c_\delta} P_x(\tau^{D^c_\delta} > \frac{t}{l+1}) + P_x(l(\delta) > l)\,. \tag{2.19}$$

For any small $\epsilon > 0$, Condition 2 allows us to pick l such that $P_x(l(\delta) > l) < \epsilon$. Passing to the limit over $t \to \infty$ gives that $\limsup_{t \to \infty} P_x(\alpha(\delta) > t) < \epsilon$ (using Condition 1(iii)). Since ϵ is arbitrary, we conclude that $P_x(\alpha(\delta) < \tau^*) = 1$. From here on, the theorem is a consequence of Lemma 1. ☐

Corollary 1 states that $\bar{D}_\delta$ is an uniformly accessible set from D.

Corollary 1. *Condition 1 and (2.14) imply that*

$$\lim_{t\to\infty} \sup_{x\in D} P_x(\alpha(\delta) > t) = 0\,. \tag{2.20}$$

Proof. The proof is given by taking the supremum over $x \in D_\delta^c$ in lines (2.17)-(2.19) of the proof of Theorem 1 for D_δ^c and $\alpha(\delta) = 0$ for $x \in D_\delta$. $\square$

3. Recurrence and Ergodic properties

We shall say that a Borel set $F \in \mathcal{B}(D)$ is a *small set* (or Doeblin set) if there there exists a time $T_0 > 0$, a probability measure $\eta(dx)$ on F and a positive constant $k_0 < 1$ such that for all $x \in F$ and all $B \in \mathcal{B}(D)$, $B \subseteq F$ we have

$$P_x(Z_{T_0} \in B) \geq k_0 \eta(B)\,. \tag{3.1}$$

A sufficient condition for F to be a small set, when the transition probabilities have densities, is

$$\inf_{t\leq t'\leq 2t} \inf_{y,z\in F} p^D(t', z, y) = b_1(t,\delta) > 0\,. \tag{3.2}$$

Condition 3. *There exists $\delta > 0$ and a closed set $F \subseteq \bar{D}_\delta$ with $\lambda(F) > 0$ such that for any $t > 0$, condition (3.2) holds.*

Let $\alpha(F) = \inf\{t \geq 0 \,|\, x(t) \in F\}$. We shall say that F is *accessible* from a set $A \subseteq D$ if $P_x(\alpha(F) < +\infty) = 1$ and *uniformly accessible* from a set $A \subseteq D$ if

$$\lim_{t\to\infty} \sup_{x\in A} P_x(\alpha(F) > t) = 0\,. \tag{3.3}$$

Theorem 2. *Suppose Condition 1 and (2.14) are satisfied for the same $\delta > 0$ and F is a small set with $\lambda(F) > 0$. Then, (i) if F is accessible from $\bar{D}_\delta$, then $(x(t))$ is Harris recurrent, and (ii) if F is uniformly accessible from $\bar{D}_\delta$ and satisfies (3.2), then $(x(t))_{t\geq 0}$ satisfies the strong Doeblin condition and is uniformly exponentially ergodic.*

For the statement in discrete time, the reader is referred to [24], and Theorem 5.3 in [11] which settles the case of continuous time processes.

Remark. 1) If D is bounded, regular and the diffusion (2.1) has a uniformly elliptic infinitesimal generator with sufficiently smooth coefficients, then Conditions 1-3 are automatically satisfied with $F = \bar{D}_\delta$ since p^D are continuous in all variables and $\bar{D}_\delta$ and D_δ^c are compact.

2) In addition to the properties from 1), (3.2) is satisfied and stronger, explicit bounds [10] exist for any $t > 0$ when D is bounded connected with C^1 boundary - see also the remark following Theorem 4.

3) Conditions 1 and 3 are on the kernel of the killed process p^D only and ignore the redistribution measures $(\nu_\xi(dx))$.

If the set D is bounded, the boundary chain Y_l, $l \geq 0$ on $S \setminus D$ has at least one invariant probability measure due to the tightness of any family of measures supported on a compact set, here equal to ∂D. On the other hand, if $\mu_Y(d\xi)$ is an invariant measure for (Y_l), then $\mu_X(dx) = \int_{\partial D} \nu_\xi(dx)\mu_Y(d\xi)$ is invariant for the chain (X_l), which can be used to construct the invariant measure (3.13) under appropriate conditions.

We shall need the following lemma.

Lemma 2. *Assume Conditions 1, 3 as well as Condition 2 are satisfied for the same $\delta > 0$ and the set F in (3.2) is uniformly accessible from $\bar{D}_\delta$. Then there exists a time T_0 which may depend on δ, the redistribution measures ν_ξ and a positive constant c such that*

$$p(T_0, x, y) \geq c\,, \qquad \forall x \in D\,, \forall y \in F\,. \tag{3.4}$$

Moreover, the constant c may be chosen such that $c \geq \frac{1}{2} b_1(T_0, \delta)$.

Remark. In most applications we may choose $F = \bar{D}_\delta$ and (3.3) is no longer needed; for instance when D is bounded and the underlying process is a sufficiently regular diffusion.

Proof. Theorem 1 shows that $P_x(x(t) \in dy)$ stochastic, i.e. $P_x(x(t) \in D) = 1$ for all $x \in D$, $t \geq 0$. We recall (2.6) which implies immediately

$$p(t, x, y) \geq p^D(t, x, y)\,. \tag{3.5}$$

In view of (2.20), there exists $T_1 > 0$ with $\sup_{x \in D} P_x(\alpha(\delta) > T_1) \leq 1/4$. Also, by (3.3) there exists T_2 such that $\sup_{x \in \bar{D}_\delta} P_x(\alpha(F) > T_2) \leq 1/4$. Let $T = T_1 + T_2$, such that $\sup_{x \in D} P_x(\alpha(F) > T) \leq \frac{1}{2}$. We shall prove the lemma with $T_0 = 2T$.

Due to Condition 3 and (3.5), we see that

$$\inf_{T \leq t' \leq 2T} \inf_{y, z \in F} p(t', z, y) \geq b_1(T, \delta) > 0\,. \tag{3.6}$$

Pick $x \in D$. If $x \in F$, (3.6) implies that any $c \leq b_1(T, \delta)$ would satisfy (3.4). Suppose $x \in D \setminus F$. Let $\alpha(F)$ be as in (3.3) with $A = D_\delta$. To prove the lower bound for $p(2T, x, y)$, we start with an analogue of (2.6) with $t = 2T$. For any Borel set B,

$$P_x(x(2T) \in B) \geq P_x(x(2T) \in B, \alpha(F) \leq 2T) \tag{3.7}$$

which, after applying the Markov property to the stopping time $\alpha(F)$, implies the inequality for density functions

$$p(2T, x, y) \geq \int_0^{2T} \int_F p(2T - s, z, y) P_x(\alpha(F) \in ds, x(\alpha(F)) \in dz) \tag{3.8}$$

$$\geq \int_0^{T} \int_F p^D(2T - s, z, y) P_x(\alpha(F) \in ds, x(\alpha(F)) \in dz)\,. \tag{3.9}$$

The inequality is true for the integral on the full interval $0 \leq s \leq 2T$. In the special case when $0 \leq s \leq T$, $2T - s$ lies in the interval $[T, 2T]$, making (3.6) applicable to the integrand $p(2T - s, z, y)$, which gives

$$p(2T, x, y) \geq b_1(T, \delta) \int_0^{T} P_x(\alpha(F) \in ds, x(\alpha(F)) \in F) \geq b_1(T, \delta) P_x(\alpha(F) \leq T) \tag{3.10}$$

$$\geq b_1(T, \delta)(1 - \sup_{x \in D} P_x(\alpha(F) > T)) \geq \frac{1}{2} b_1(T, \delta)\,. \tag{3.11}$$

The last inequality is true due to the construction of T. By choosing $c = \frac{1}{2} b_1(T, \delta)$ we proved the lemma with $T_0 = 2T$. $\square$

3.1. **Proof of Theorem 2.**

Proof. Let B be a Borel subset of D and let $\lambda(\cdot|F)$ be the probability measure defined by $\lambda(B|F) = \lambda(B \cap F)/\lambda(F)$ where λ is the reference measure defined in the first paragraph of Section 2.2. Then, for any $x \in D$, according to Lemma 2

$$p(T_0, x, B) \geq p(T_0, x, B \cap F) \geq c\,\lambda(B \cap F) = c\,\lambda(F)\,\lambda(B|F)\,.$$

Setting $k_0 = \min\{c\,\lambda(F), 1\}$, we have proven the condition from Doeblin's theorem is satisfied for the Markov process $(x(t))$ with the same T_0 as in Lemma 2 and $\eta(\cdot) = \lambda(\cdot|F)$. $\square$

3.2. **The diffusive case.** The invariant measure admits a representation in terms of the Green function of the underlying process killed at the boundary $(\tilde{x}(t))$ and the invariant measure of the chain (X_l) defined in (2.3). More precise results can be given case by case. These results are not specific to diffusion processes but the presentation is more straightforward.

In the following discussion $(x^o(t))$ is a strongly elliptic diffusion on an open domain S with smooth, bounded coefficients and generator $(L, \mathcal{D}^o)$, where $\mathcal{D}^o$ is a subset of the continuous functions vanishing at infinity $f \in C_0(\bar{S})$ satisfying the appropriate boundary conditions, written as (BC). We assume $(x^o(t))$ is non-explosive, i.e. $P_x(x^o(t) \in S) = 1$, for all $x \in S$. Difusions on bounded sets, or on unbounded sets with drift towards ∂D, as well as diffusions with reflection at the boundary are natural examples.

Let $D \subset S$ be a regular domain. The process killed at ∂D with the same operator L

$$\tilde{\mathcal{D}} = \{f \in C(\bar{D}) \cap C^2(D) \,|\, f(\xi) = 0\,,\, \xi \in \partial D\,, (BC)\}\,, \tag{3.12}$$

where the closure is in the relative topology of $S \subseteq \mathbb{R}^d$. Let $G(x, x')$ be the corresponding Green function. The harmonic measures (2.2) are then concentrated on ∂D and $x^o(\tau^D) = \tilde{x}(\tau^D-)$.

Both references [17, 6] prove that the transition probabilities $P_x(x(t) \in dy)$ (2.5) have densities $p(t, x, y)$ for slightly less general redistribution measures $(\nu_\xi(dx))$, but the proofs can be extended immediately to the present case.

Theorem 3. *If the chain (2.3) has an invariant probability measure $\mu_X(dx)$, then the measure with density*

$$\frac{d\mu}{dx}(x) = Z^{-1} \int_D G(x, x')\mu_X(dx')\,, \qquad Z = \int_D E_{x'}[\tau^D]\mu_X(dx') \tag{3.13}$$

is invariant for the process $(x(t))$. When the conditions of Theorem 2 are met and the underlying process is a regular diffusion, this is the unique invariant measure of the redistribution process.

Proof. The second part of the Theorem is immediate since the invariant probability measure $\mu(dx)$ exists and is unique from the Doeblin's condition.

With (2.9) in mind, define for $g \in C_b(\partial D)$ the operator

$$\check{K}_\beta g(x) = \int_0^\infty \int_{\partial G} e^{-\beta s} P_x(\tau^D \in ds, x(\tau^D-) \in d\xi) g(\xi)\,. \tag{3.14}$$

Writing $u(x) = \check{K}_\beta g(x)$ and $h(x) = \int_{\partial G} \lambda_0(x, d\xi) g(\xi) = \check{K}_0 g(x)$, we obtained two functions in $C(\bar{D}) \cap C^2(D)$ (due to the boundary regularity), with boundary values given by g and satisfying the equations $(L - \beta)u = 0$, $Lh = 0$, respectively. The difference $u - h$ belongs to the domain of the infinitesimal generator $\tilde{\mathcal{D}}$ of the killed process $(\tilde{x}(t))$. Then $GL(u-h) = h - u$ and we have shown that

$$(\check{K}_0 - \check{K}_\beta) g = \beta G \check{K}_\beta g \,, \qquad g \in C_b(\partial D) \,. \tag{3.15}$$

All these considerations pertain to the underlying process $(x^o(t))$ killed at the boundary. The operators have no relation to the redistribution measures $\nu_\xi(dx)$. For fixed $x \in D$, their kernels can be seen as measures on ∂G satisfying inequality (3.15). We conclude that (3.15) is true for all bounded g defined on ∂D.

If $f \in C_b(D)$, then $g(\xi) = \int_D f(x') \nu_\xi(dx')$ is a bounded function on ∂D. Applying (3.15) to $g(\xi)$ and noticing that $\check{K}_0 g = Sf$ we obtain

$$Sf - K_\beta f = \beta G K_\beta f \,, \qquad f \in C_b(D) \,. \tag{3.16}$$

Since μ_X is invariant with respect to S, integrating with respect to μ_X to the left hand side we have

$$\langle \mu_X, (I - K_\beta) f \rangle = \beta \langle \mu_X, G K_\beta f \rangle \,, \qquad f \in C_b(D) \,. \tag{3.17}$$

We want to show that $\mu = \langle \mu_X, G \rangle$ is invariant. It is sufficient to show it is invariant for βR_β. Multiply by β, apply G, then μ_X to the left in (2.10),

$$\langle \mu_X, G(\beta R_\beta) f \rangle = \beta \langle \mu_X, G K_\beta R_\beta f \rangle + \beta \langle \mu_X, G R_\beta^D f \rangle \,.$$

Set $f \to R_\beta f$ and use (3.17) to replace the first term on the right hand side by

$$\langle \mu_X, (I - K_\beta) R_\beta f \rangle = \langle \mu_X, R_\beta^D f \rangle \,,$$

once again due to (2.10). Using the resolvent identity $\beta G R_\beta^D = G - R_\beta^D$ for the second term, since $G = R_0^D$, we have shown

$$\langle \mu_X, G(\beta R_\beta) f \rangle = \langle \mu_X, R_\beta^D f \rangle + \langle \mu_X, G f \rangle - \langle \mu_X, R_\beta^D f \rangle = \langle \mu_X, G f \rangle \,.$$

This proves the invariance of μ. $\square$

The next question is if we can relate the spectral gap of the killed process to the spectral gap of the jump process $(x(t))$.

Theorem 4. *Let α_D be the spectral gap of the process killed at the boundary ∂D defined by (2.1). Assume $b_1(T_0, \delta) \geq C_1 e^{-\alpha_D T_0}$, with $C_1 > 0$ depending possibly on δ but not on T_0. Under the conditions of Theorem 2, considering δ fixed and T_0 as defined in Lemma 2, there exists $c_D(T_0) > 0$ depending on the process killed at the boundary and possibly on $\nu_\xi(\cdot)$ via T_0 only, such that the convergence rate satisfies the lower bound $c_D(T_0)\alpha_D$.*

Remark. It is known [10] that assuming D is bounded connected with sufficiently smooth boundary and the diffusion (2.1) has sufficiently smooth coefficients, then the transition probabilities are bounded smooth functions in (t, x, y) for $t > 0$, $x, y \in D$ with the lower bound $b_1(T_0, \delta) \geq C_1 e^{-\alpha_D T_0}$.

Proof. We refer to Doeblin's theorem to see that the exponential rate $r \in (0, 1)$ satisfies $-\ln r \geq -\frac{1}{T_0} \ln(1 - k_0)$, with $k_0 = \min\{c\lambda(F), 1\}$ (proof of Theorem 2) and for $b_1(t, \delta)$ defined in (3.2), we have $c \geq \frac{1}{2} b_1(T_0, \delta)$ (proof of Lemma 2). Summarizing, there exists $C > 0$ such that $k_0 \geq C b_1(T_0, \delta)$. We note that C and C_1 can be chosen independently of T_0, and thus independent of $(\nu_\xi(\cdot))_{\xi \in \partial D}$. Since $b_1(T_0, \delta) \geq C_1 e^{-\alpha_D T_0}$, let $C_2 = CC_1$. Then

$$-\frac{\ln r}{\alpha_D} \geq -\frac{1}{\alpha_D T_0} \ln(1 - C_2 e^{-\alpha_D T_0}) =: c_D(T_0) > 0 \,. \tag{3.18}$$

□

4. Applications and Examples

4.1. **Brownian motion with rebirth.** In the simplest version, the diffusion with jumps on the connected open set D has a delta function relocation measure $\nu_\xi(dx) = \delta_{x_0}(dx)$, $x_0 \in D$ [15, 17], constant in $\xi \in \partial D$, $\nu_\xi(dx) = \nu(dx) \in M_1(D)$ [5, 6], or with continuous dependence on the exit point $\xi \to \nu_\xi(\cdot) \in M_1(D)$ [6]. In the special case of the delta measure and bounded D, the process is generated by a Feller semigroup on a compact manifold where the boundary is glued together with the return point x_0 [15, 17]. Other variants include the case [21] of a domain with piecewise smooth boundary and constant redistribution measure on each smooth component.

Proposition 2. *Consider a regular diffusion as in (2.7). Assuming D is bounded and the mapping $\xi \to \nu_\xi(dx)$ is piecewise continuous on ∂D, with each component having a continuous extension up to its boundary (seen as a subset of ∂D), then all Theorems 1, 2, 3 and 4 are applicable. Moreover, Condition 2 is satisfied with $m = 1$.*

Remark. 1) One such case is when the components are constant, as in [21]. 2) The components on the boundary of the domain may conflict at their intersection. However, this is a set of codimension two, and is a.s. not visited by the diffusion.

Proof. The boundary ∂D is bounded and has a finite number of compact components. On each, the local definition of $\nu_\xi(dx)$ is continuous up to the boundary of the component. A continuous function sends a compact into a compact, thus each family $(\nu_\xi(dx))$ indexed over each component of ∂D is tight, implying that the whole family is tight. From here we use Proposition 1 (ii) to ensure Condition 2. Since Condition 1 is trivially satisfied for a regular diffusion, Theorem 1 may be applied and the process is non-explosive. Condition 3 is trivially satisfied by $F = \bar{D}_\delta$, allowing us to use Theorem 2. The conditions for Theorems 3 and 4 are met for regular diffusions on a bounded domain with regular boundary, concluding the proof. □

4.2. **Diffusive Bak-Sneppen fitness evolution model.** We consider an n particle system, a variant to the well known evolution model proposed by Bak and Sneppen in [4]. The particles follow Brownian motions $x_i(t)$ for $i \in \{1, 2, \cdots n\}$ in an interval $(0, a]$, $a > 0$ with reflection at $a \in \mathbb{R}$ and with 0 considered a boundary point, evolving independently of each other until the first one reaches 0. Each coordinate x_i represents the 'fitness state' of the i-th species and has an associated set of *neighbors* $V_i \subseteq \{1, 2, \ldots, n\}$, designated by their indices, such that $i \in V_i$. Whenever one of the fitness levels x_i reaches the boundary point zero, all x_j with $j \in V_i$ are instantaneously replaced by new i.i.d. fitness levels with distribution function $G(x)$ on the interval $(0, a)$ and the evolution continues afresh. To fix ideas, we shall assume $i \to V_i$ to be deterministic functions and $|V_i| \geq 2$, for all $1 \leq i \leq n$, where $|V|$ is the cardinality of V. It is easy to verify that $n = 1$ and $n \geq 2$ with $|V_i| = 1$ are covered in subsection 4.1 since this coincides with $V_i = \{i\}$ and particles move independently.

In the framework laid out in the Introduction, the dimension d is equal to the number of particles n, the vector-valued process $x^o(t) = \{x_i(t)\}_{1 \leq i \leq n}$ is a diffusion on $S = (-\infty, a]^n$ with reflecting boundary conditions on each component at a, $D = (0, a]^n$ and $\partial D = \partial(-\infty, a]^n$ in $\mathbb{R}^n$. Then $\tilde{x}(t)$ is the process killed at the part of the boundary of the hypercube D containing at least one zero component.

To be specific, the reflection takes place on the upper hyper-surface $U = \cup_{i=1}^n U_i$ where $U_i = \{x \in D \,|\, x_i = a\,,\, x_j \neq a \text{ for all } j \neq i\}$. On the other hand, redistribution is triggered on the lower hyper-surface $L = \cup_{i=1}^n L_i$ where $L_i = \{x \in D \,|\, x_i = 0\,,\, x_j \neq 0 \text{ for all } j \neq i\}$. We note that more than one boundary hits at the same time occur with probability zero in this setup. When one of the particles hits $\partial D = \{0\}$, the redistribution is carried out through a measure $\nu_{x(\tau^D-)}(dz)$ where $x(\tau^D-) = \xi = (\xi_1, \ldots, \xi_n) \in L$

$$\nu_\xi(dz) = \sum_{i=1}^n \mathbf{1}_{L_i}(\xi) \cdot \Big(\otimes_{j \notin V_i} \delta_{\xi_j}(dz_j) \Big) \otimes \Big(\otimes_{j' \in V_i} dG(z_i) \Big)\,, \tag{4.1}$$

with i in the periodic lattice $\mathbb{Z}_n$ and $\delta_x(\cdot)$ the delta measure at x.

For any sufficiently small $\delta > 0$ we let

$$F_i = \{\, x \in \bar{D} \,|\, \sum_{j=1}^n \mathbf{1}_{[0,\delta)}(x_j) = i \,\}\,, \tag{4.2}$$

the set on which there are exactly i coordinates less than δ. We notice that $F_0 = \bar{D}_\delta$ and $\cup_{i=0}^n F_i = \bar{D}$. We shall prove the bound (2.15) with $m := n$ inductively; the set F_n the worst case scenario and $F_0 = \bar{D}_\delta$, the set we want to enter almost surely.

Proposition 3. *Assume that G is concentrated on $(0, a)$, i.e. $G(0^+) = 0$ and $G(a-) = 1$. For all $1 \leq k \leq n$, there exists a positive real w_k such that*

$$\inf_{x \in F_k} P_x(x(\tau^D) \in A_{k-1}) \geq w_k\,, \qquad A_l = \cup_{0 \leq j \leq l} F_j\,, \quad 0 \leq l \leq n\,. \tag{4.3}$$

Remark. Since G is concentrated in $(0, a)$, there exists $\delta \in (0, a)$ such that G charges $[\delta, a]$, which is equivalent to $G(\delta) < 1$. The key observation is that since the particle hitting the boundary is for sure in $[0, \delta)$ and G charges $[\delta, a]$, then with a positive probability independent of the current configuration, right after the jump there will be at least one more particle in $[\delta, a]$.

Proof. First we note that there exist a positive real $v_k = v_k(\delta, a)$, $0 \leq k \leq n$, satisfying

$$\inf_{x \in F_k} P_x(x(\tau^D-) \in F_k \cap \partial D) \geq v_k > 0\,. \tag{4.4}$$

To see that, the harmonic functions $u(x) = P_x(x(\tau^D-) \in F_k \cap \partial D)$ have limit one at interior points of $F_k \cap \partial D$ and zero at exterior points. Since the boundary function is piecewise continuous (indicator function), the solution $u(x)$ is equal to its Fourier series and has limit

$1/2$ at points of discontinuity of the boundary. Thus over A_k, a compact set, the minimum is strictly positive.

Following the distribution after the redistribution jump, when $x \in F_k$

$$(4.5)\quad P_x(x(\tau^D) \in A_{k-1}) \geq \inf_{x \in F_k} \int_{\partial D} P_x^D(x(\tau^D-) \in d\xi)\nu_\xi(A_{k-1}) \geq v_k \inf_{\xi \in F_k \cap \partial D} \nu_\xi(A_{k-1})\,.$$

According to (4.1), $\xi \in F_k$ will imply that $\nu_\xi(A_{k-1}) \geq (1 - G(\delta))^{|V_i|}$ for all $1 \leq k \leq n$. This is based on the fact that $x_i = 0 \in [0, \delta)$ to begin with, and then the event that all neighbors V_i go to $[\delta, a]$ implies that the number of particles in $[0, \delta)$ has diminished by at least one. Using the fact that $G(0^+) = 0$, $G(\delta) < 1$ and $l(V) = \max_{1 \leq i \leq n} |V_i| \leq n$ we proved the lower bound $\nu_\xi(A_k) \geq (1 - G(\delta))^{l(V)} > 0$, uniformly in k. This proves the proposition with $w_k = v_k(1 - G(\delta))^{l(V)}$. □

Proposition 4. *Under the same assumptions as in Proposition 3, condition (2.15) is satisfied with $m = n$ and $c_1 = (\min\{w_k\})^n$ and Theorems 1, 2, 3 and 4 are in force.*

Proof. Put $w = \min\{w_k\}$. Since $F_0 = \bar{D}_\delta$, it is clear in view of Proposition 3 that if $x \in F_k$, then $P_x(x(\tau_k) \in \bar{D}_\delta) \geq w^k$, meaning that $P_x(l(\delta) \leq n) \geq w^k$. We only have to choose $m := n$ and $c_1 = w^n$. Once Condition (2.15) is satisfied, Condition 2 is satisfied. The Brownian motions satisfy Condition 1, 3 so all theorems in Section 2.2 are in force. □

4.3. **The FV branching model.** In order to concentrate on the idea of the proof, we adopt the simpler setup of a family of $n \geq 2$ independent d - dimensional diffusions $x^o(t) = (x_1(t), \ldots, x_n(t))$. Each is generated by a second order strongly elliptic operator A with smooth, bounded coefficients and evolves in a bounded C^2 domain $G \subset \mathbb{R}^q$. We then set $D = G^n$ and $d = nq$ and $L = \sum_{i=1}^n A_{x_i}$, where A_{x_i} acts on the variable i, meaning that particles are killed at the boundary and redistributed inside G.

The boundary ∂D is the set of vectors $\xi \in \bar{D}$ with exactly one component in ∂G, designating the particle "killed" at the boundary ∂G. Instantaneously, one of the $n-1$ remaining particles, chosen with a probability proportional to a function of the configuration, branches into two independent offsprings. The branching mechanism is meaningful even when more than one offspring is produced. The standard model is critical with n fixed. It is then more convenient to think of a pure jump of particle i to the location of particle j, which is a special case of the type of process studied in this paper.

Note that the "edges" $\mathcal{K}$, where at least two components are on the boundary at the same time, are hit with probability zero. When ξ is an element on ∂D with $x_i \in \partial G \setminus \mathcal{K}$, let $\xi^{ij} \in G^n$ be the vector with the same components as ξ with the exception of ξ_i which is replaced by ξ_j. We shall assume there exist Borel measurable functions $\partial D \ni \xi \to p_{ij}(\xi) \in [0,1]$, indexed by $1 \leq i, j \leq n$, $i \neq j$, such that $\sum_j p_{ij}(\xi) = 1$ and satisfying

$$\forall \xi \in \partial D \setminus \mathcal{K}\,, x_i \in \partial G \qquad \nu(\xi, d\mathbf{x}) = \sum_{j \neq i} p_{ij}(\xi) \delta_{\xi^{ij}}(d\mathbf{x})\,. \tag{4.6}$$

In the uniform distribution case $p_{ij} = 1/(n-1)$.

Non-explosiveness. The only condition for non-explosion on the redistribution measures $(\nu(\cdot, dx))$ will be that the probability to choose the most distant particle from the boundary be uniformly bounded away from zero. To make this precise, let $r > 0$ chosen sufficiently small such that $G_r \neq \emptyset$ and its boundary is smooth.

We assume there exists $p_0 > 0$, independent of $\xi \in \partial G^n$, such that when $x_i \in \partial G$ and j is a maximizer of the distance to the boundary, we have the lower bound

$$p_{ij}(\xi) \geq p_0\,, \qquad \xi_j = \max_{k \neq i} d(\xi_k, \partial G) \wedge r\,. \tag{4.7}$$

Since $\delta \leq r$ implies $(G_r)^n \subseteq D_\delta$, we may simply pick $\delta = r$.

A multi-dimensional analogue of Proposition 3 in the Bak-Sneppen fitness model can be formulated by replacing the interval $[0, \delta)$ by $\bar{G} \setminus \bar{G}_\delta$ in (4.2).

The system can move form a configuration with a number k (i.e. in F_k) to one with at most $k-1$ particles within distance δ from the boundary (in A_{k-1}) with a probability bounded away from zero, uniformly on configuration. The proof has two steps: 1) shows that starting from a configuration in the interior of F_k, the probability to exit through $F_k \cap \partial D$ has a uniform lower bound v_k, and 2) once on the boundary of $F_k \cap \partial D$, the probability to be redistributed to a configuration in A_{k-1} has a uniform lower bound w_k.

Part 1) does not depend on the redistribution mechanism. It can be simplified substantially as we did above in the Bak-Sneppen case, or opt for a slight modification of the claim for more general domains. The idea is that $F_k = F_k(\delta)$ depends on δ. We may prove that under very general conditions for G and the diffusion coefficients, the passage from $F_k(\delta)$ to $A_{k-1}(\frac{\delta}{2})$ has a uniform lower bound. Since there are a finite number of particles (steps) the loss in distance to the boundary is not changing the conclusion.

In the FV case, the step $k = n$ is non-trivial, due to the singularity of the redistribution measures, i.e. F_n leads to A_n, and not necessarily to A_{n-1}. Particles may stay trapped in A_n (all within δ distance form the boundary). More and more degenerate configurations could form, with particles congregating closer and closer to ∂G. In a cascading effect, it appears conceivable that infinitely many jumps occur in finite time.

Suppose for a moment that the system has entered A_{n-1}. In that case Proposition 3 can be applied for all $1 \le k \le n-1$. Part 1) was not a problem in any case, and 2) can be carried out due to (4.7). We need *at least one particle* in G_δ to make sure that the particle killed on ∂G will be redistributed at that location, which increases by one the number of particles in G_δ and thus moves the configuration to A_{k-1}.

This discussion shows that once in A_{n-1}, the system will enter $A_0 = G_\delta^n$ in a finite number of steps, with probability one, by using a geometric distribution argument. Let $\alpha'(\delta)$, $l'(\delta)$ be the time, respectively the number of jumps until the system enters A_{n-1}. In order to prove Condition 2, which is sufficient for Theorem 1, we have to show that

$$P_x(l'(\delta) < \infty) = 1\,. \tag{4.8}$$

The greatest effort is to prove the following lemma. Let $J(t)$ be the number of jumps up to time t.

Lemma 3. *For any $\phi \in C(\bar{G})$, ϕ positive on G, let $\Phi(x) = \sum_{i=1}^n \phi^2(x_i)$ and*

$$U = \inf_{\xi \partial G^n \setminus \mathcal{K}} \left\{ \int_D \ln \Phi(x)\nu(\xi, dx) - \ln \Phi(\xi) \right\}. \tag{4.9}$$

There exists ϕ such that $U > 0$ and, if we denote $M(t) = \ln \Phi(t) - UJ(t)$, then for sufficiently large n, $M(t \wedge \alpha'(\delta))$ is a $\mathcal{F}_t$ sub-martingale. Moreover, $E_x[J(t \wedge \alpha'(\delta))] < \infty$ implying (4.8).

Remark. The dependence on n can be removed but the proof involves another estimate and another setting up the proof in two stages. The first shows that $E_x[\alpha'(\delta) \wedge \tau^*] < \infty$, $x \in F_n$, and the second uses a version of Lemma (3) based on a test function ϕ satisfying weaker conditions. The construction can be achieved for Lipschitz domains G with integrable Martin kernel.

We sketch the proof of the lemma. Let $G' = G_\delta^c = G \setminus \bar{G}_\delta$ called a vicinity of the boundary. Let $\psi \in C(\overline{G'}) \cap C^2(G')$ be the solution to $L\psi = 0$, $\psi(x) = 1$ on ∂G_δ and $\psi(x) = 0$ on ∂G. We then check that $\phi(x) = \psi^2(x)$ satisfies the following conditions.

(i) $L\phi(x) \geq 0$, (ii) $\phi(x) > 0$ on G', (iii) $\phi(x) = 0$ on ∂G and (iv) there exists a constant $c_2 > 0$ such that $||\nabla\phi(x)||^2 \leq c_2\phi(x)$ for any $x \in G'$.

The smooth boundary of G ensures that ψ can be chosen continuous up to the boundary which ensures (iv). For sets with the uniform exterior cone condition, a construction based on the modulus of continuity up to the boundary can produce the necessary test function.

The logarithm of the radial process $r(t) = \Phi(x(t))$, with $\Phi(\mathbf{x}) = \sum_{i=1}^{n} \phi^2(x_i)$ will verify Lemma 3.

Calculating the Ito formula for the Bessel-type process $r(t)$ and then $\ln r(t)$, we obtain that all coefficients are bounded and, for sufficiently large n, the sub-martingale condition $L \ln \Phi(t) \geq 0$ is satisfied. We recall that L is the n-dimensional generator of the n particle process. The size of n depends on the *eccentricity* of the operator. As mentioned in the remark, this limitation can be removed, but we chose to focus on the idea.

Due to the smoothness ∂G and conditions (i)-(iv), between jumps the terms in the Ito formula are bounded. Again, under less regularity, the construction can be localized and produce a local semi-martingale. In our setting, it remains to verify that $U > 0$ in (4.9).

Due to (4.7), when particle i hits the boundary and $j \neq i$ is the maximizer of the distance to ∂G, we have the non-random lower bound

$$\int_{G^N} \ln \Phi(\mathbf{x})\nu_\xi(d\mathbf{x}) - \ln \Phi(\xi) = p_0 \ln\Big(1 + \frac{\phi^2(x_j)}{\sum_{k\neq i}\phi^2(x_k)}\Big) \geq p_0 \ln(\frac{n}{n-1}) > 0\,.$$

It follows that

$$E_x[J(t\wedge\alpha'(\delta))] \leq U^{-1}E_x\Big[\ln\frac{\Phi(t\wedge\alpha'(\delta))}{\Phi(0)}\Big] < +\infty\,. \tag{4.10}$$

This formula proves that the number of jumps until hitting $\partial G'$ (exiting the vicinity of the boundary) is finite with probability one.

Since the boundary hits can be totally ordered in a sequence almost surely, the system must hit $F_{n-1} = G^n \setminus (G')^n$ and thus D_δ.

Uniqueness of the invariant measure. A straightforward coupling argument proves that the process will access with probability one the set $\bar{D}_\delta$. Denoting $y(t) = \phi(x_i(t))$ for one generic index i, we see that between jumps the process $(y(t))$ is a continuous semi-martingale with bounded coefficients in its differential formula. By construction, $0 \leq y(t) \leq 1$. We know that $0 < y(t)$ except at jump times (technically it is at $y(t-)$) and from the maximum principle for ϕ, $y(t) < 1$ as long as $x_i(t)$ does not enter $\bar{G}_\delta$. Jumps occur only in the positive direction. This implies that a diffusion with the same coefficients (these are continuous and adapted to the original driving process, and the diffusion matrix remains strongly elliptic) having all jumps suppressed will be pathwise dominated by $y(t)$. The probability that a diffusion on $(-\infty, 1)$ hits the right side endpoint is positive, as long as the drift is bounded. This proves that $x_i(t)$ enters $\bar{G}_\delta$, and as described in the proof of non-explosiveness, then all particles will eventually enter $\bar{G}_\delta$, and thus the n-dimensional process will enter $\bar{D}_\delta$. Any compact in D is a small set due to (3.5). The estimate on the hitting time of the ineterior set can be made uniform as long as the initial point belongs to a compact set. Exponential ergodicity follows from the local Doeblin theorem.

Relation to quasi-invariant measures. Let $p^G(t, x', x)dx$ be (2.1) for the one particle process with generator L on G (the driving process), L^* be its adjoint operator.

Between jumps, the corresponding FV n-particle system is non-interacting. The only times particles "see" each other are when one hits ∂G, via redistribution. Suppose $\nu_\xi(dx)$ are given by the uniform distribution among the $n-1$ particles which are not at the boundary. It is intuitively clear that the correlation introduced this way are of the same scale (with respect to n) as to allow a weak law of large numbers. Let $J_n(t)$ be the number of jumps up to time $t > 0$, where we introduced the subscript n to emphasize the scaling variable.

Let $\mu^n(t, dx) = \frac{1}{n}\sum_{i=1}^n \delta_{x_i(t)}(dx)$ be the empirical measure associated with the system. Evidently $\mu^n(dx) \in M_1(G)$ is a probability measure on G. In [16] we prove that if $\lim_{n\to\infty} \mu^n(0, dx)$ converges weakly to $\mu_0(dx) = v_0(x)dx$, then as $n \to \infty$,

1) $t \longrightarrow \exp\{-J_n(t)\}\mu^n(t, dx)$, $t \geq 0$ is a tight measure - valued process converging in probability (in the weak topology of probability measures, i.e. as integrals against test functions) to the deterministic, unique smooth solution of the forward heat equation $v(t, x)dx$ with zero boundary conditions, $\partial_t v = L^* v$, $v_{|\partial G} = 0$, $v(0, x) = v_0(x)$, i.e.

$$v(t, x) = \int_G p^G(t, x', x)\mu_0(dx') = p^G(t, \mu_0, x)$$

and

2) $\exp\{-J_n(t)\}$ converges to the normalizing factor

$$\int_G v(t, x)dx = p^G(t, \mu_0, G) = P_{\mu_0}(\tau^G > t)\,.$$

As a consequence we have the *hydrodynamic limit*

$$\lim_{n\to\infty} \mu^n(t,dx) = u(t,x)dx\,, \qquad u(t,x) = \frac{p^G(t,\mu_0,x)}{p^G(t,\mu_0,G)}\,. \tag{4.11}$$

On one hand, as $t \to \infty$, $\mu^n(t,dx)$ converges to the empirical measure under the equilibrium (stationary) measure of the n-particle FV system from (3.13). On the other hand, the limit in t for the right hand side of (4.11) is the *Yaglom limit* of the diffusion killed on the boundary. This limit, when it exists, is a *quasi-stationary distribution.*

For Brownian motions $L = L^* = \frac{1}{2}\Delta$ and the Yaglom limit is the normalized first eigenfunction of L with zero boundary conditions. It is apparent that by following the same route

FV Particle system $\to$ hydrodynamic limit $\to$ limit in t $\to$ Yaglom limit

we should be able to obtain the quasi-stationary measure (left-eigenfunction) in cases when it is not known apriori. The following conjecture is natural.

Under its equilibrium distribution, the empirical measure of the n particle FV system is tight and, as $n \to \infty$, converges to the Yaglom limit of the kernel p^G.

Since the equilibrium measure is not a product measure, and (3.13) is is not a readily computable formula, the question remains open in general.

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