THE DOEBLIN CONDITION FOR A CLASS OF DIFFUSIONS WITH JUMPS†

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Abstract. We prove non-explosiveness and a lower bound of the spectral gap via the strong Doeblin condition for a large class of stochastic processes evolving in the interior of a region $D \subseteq \mathbb{R}^d$ with boundary ∂D according to an underlying Markov process with transition probabilities $p(t, x, dy)$, undergoing jumps to a random point x in D with distribution $\nu_\xi(dx)$ as soon as they reach a boundary point ξ. Besides usual regularity conditions on $p(t, x, dy)$, we require a tightness condition on the family of measures ν_ξ, preventing mass from escaping to the boundary. The setup can be applied to a multitude of models considered recently, including a particle system with the Bak-Sneppen dynamics from evolutionary biology.

1. Introduction

Let $S \subseteq \mathbb{R}^d$ be a closed set, $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$ a filtered probability space and $(x^o(t))$, $t \geq 0$ a Markov process on S, adapted to $(\mathcal{F}_t)_{t\geq 0}$. The usual conditions will be assumed without loss of generality, i.e. the filtration is P - complete and right-continuous. For example, S is the closure of an open set and $(x^o(t))$ is a diffusion with the appropriate boundary conditions. Alternatively, S may be a countable set (e.g. a subset of $\mathbb{Z}^d$). In addition, D will be an open connected set in the topology induced on S and ∂D its boundary. Besides $(x^o(t))$, two more processes will be considered: the process killed at the boundary $(\tilde{x}(t))$ and the derived jump process $(x(t))$.

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In the present construction, the underlying process $(x^o(t))$ does not need be a diffusion. However, it is assumed *stochastic* (honest), i.e. $P_x(x^o(t) \in S) = 1$ for any $x \in S$, $t \geq 0$. Some results, as Theorem 1 and 2 are general, while the other two, Theorems 3 and 4, are formulated for diffusions. For the pure jump process on a discrete set D we adopt the discrete topology.

Let $\tau^D = \inf\{t > 0 \,|\, x^o(t) \in \partial D\}$ be the first hitting time of the boundary ∂D by the underlying process. Since D is allowed to be infinite, even a very regular D may see the possibility that $P_x(\tau^D = \infty) > 0$ for some $x \in D$. However, we assume that no instantaneous jumps are allowed from inside D, i.e. $P_x(\tau^D > 0) = 1$. We further denote by $P^D(t, x, dy)$ the transition probabilities of $(\tilde{x}(t))_{t \geq 0}$, the process killed at the boundary ∂D

$$P^D(t, x, dy) = P_x(x^o(t) \in dy \,,\, \tau^D > t)\,, \qquad \forall x \in D \tag{1.1}$$

and by $(P^D_x)_{x \in D}$ its law when starting at x. In the following $\lambda(B)$ will be a reference measure (Lebesgue measure when $x^o(t)$ is a diffusion) of a Borel set $B \subseteq \mathbb{R}^d$ and we shall assume they have densities $P^D(t, x, dy) = p^D(t, x, y)dy$, where we use the notation $\lambda(dy) = dy$ for simplification. Additionally, the harmonic measures will be denoted by $\lambda_0(x, d\xi) = P_x(\tilde{x}(\tau^D -) \in d\xi)$, $\xi \in \partial D$.

For any $\xi \in \partial D$ we have a probability measure $\nu_\xi(dx)$ on D such that $\xi \to \nu_\xi(dx)$ is measurable, where $\partial D \ni \xi$ is endowed with the Borel σ - algebra induced as a subspace in $\mathbb{R}^d$ and $M_F(D) \ni \nu_\xi(dx)$ with the Borel sets of the topology of convergence in distribution. Here $M_F(X)$, $M_1(X)$ denote the space of finite, respectively probability measures on the Polish space X.

We shall construct a Markov process $(x(t))_{t \geq 0}$ on D with jumps at the boundary, based on an infinite collection $(\tilde{x}_l(t))_{l \geq 1}$ of independent processes with laws $P^D_\cdot$ for each $l \geq 1$. For each of these processes, denote τ^D_l the first hitting time of the boundary. Set $\tau_0 = 0$. Then, for any $x \in D$, define $x(t) = \tilde{x}_1(t)$ on $t \in [0, \tau^D_1)$. If $\tau^D_1 = \infty$, we put $l^* = 0$, where l^* denotes the total number of jumps, and we are done. If not, we continue. When $x(t)$ reaches the boundary at $\xi_1 = \tilde{x}(\tau^D_1 -)$, it instantaneously jumps to a random point x_1, independent of the process, with probability distribution $\nu_\xi(dx_1)$ and we update $\tau_1 := \tau^D_1$. The process continues as $x(t) = \tilde{x}_2(t - \tau^D_1)$ on the time interval $\tau^D_1 \leq t < \tau^D_1 + \tau^D_2$, where $\tilde{x}_2(\cdot)$ has law $P^D_{x_1}$. Unless $\tau^D_2 = \infty$, in which case we have $l^* = 1$, upon reaching the boundary, we set $\tau_2 := \tau_1 + \tau^D_2$ and the we continue inductively for all $l \geq 1$. If the total number of jumps l^* is finite, we set $\tau_l = \infty$ for all $l > l^*$. By construction, the sequence $(\tau_l)_{0\le l\le l^*}$ is strictly increasing almost surely (Condition 1 (i)). We denote $\tau^* = \lim_{l\to\infty} \tau_l \le \infty$. Theorem 1 proves that under some additional conditions the process $(x(t))$ is *non-explosive*, or equivalently $P_x(\tau^* = \infty) = 1$, implying that the transition probabilities are stochastic.

We give a brief outline of the paper. The results are described in the next section, where we give general conditions for both non-explosiveness (Theorem 1, with proof in Section 3) and geometric ergodicity (Theorem 2, with proof in Section 4) for $(x(t))$. The two theorems are our main results in general setting. Theorem 3 introduces an explicit formula for the invariant measure and Theorem 4 gives a lower bound of the spectral gap in terms of the spectral gap of the process $(\tilde{x}(t))$ killed at the boundary. Their proofs are left to Section 4.

Subsection 5.1 generalizes a class of models studied recently [2, 3, 9, 11, 13, 14, 15] by identifying a unifying condition (2.3) which guarantees existence (non-explosiveness) and convergence to equilibrium, providing the motivation for this paper. These results are based on the Doeblin condition; they actually constitute good examples of a relative advantage of probabilistic methods over their analytic counterparts [9, 11, 3, 15].

The interest in the problem stems from a multitude of applications, for example in barrier options pricing theory [9, 17], but also in particle systems. A new such example is described in Subsection 5.2 where we prove that a Bak-Sneppen type of interaction fits in this framework. Most interesting features of this model from evolutionary biology are observed in equilibrium (self-organizing criticality) but the relevant literature mostly concentrates on Markov chain simulation, which assumes a fast rate of convergence to equilibrium. The present result appears to be the only formal proof of geometric ergodicity in any variant of the model except the finite state discrete version, where it is trivial. Here the particles (fitness levels) are diffusing while confined to a finite interval $(0, a)$, with reflection at the upper end and jumps at the lower end. With minor changes, one can consider diffusions with negative drift on the positive half-line.

Finally, the construction (but not the proof) presented here includes a particle system with Fleming-Viot branching mechanism studied in [4, 10, 15] for Brownian motions, and in [8] in a discrete setting (where the existence of a quasi-stationary measure is obtained from the empirical measures). In this model the redistribution probability $\nu_\xi(dx)$ sends a particle reaching the boundary to one of the locations of the remaining particles chosen at random. This process satisfies Condition 2 (i) but the proof is more elaborate and is left to an upcoming paper [12].

2. Results

In the following $\delta > 0$ and $D_\delta = \{x \in D \,|\, d(x, \partial D) > \delta\}$ is an open subset of D with distance at least δ from the boundary ∂D. In the countable state space case, $D_\delta = D$ is a valid choice. For $\delta > 0$, we shall use the notation $D_\delta^c = D \setminus D_\delta$ and $\tau^{D_\delta^c}$ for the first hitting time of the boundary of D_δ^c.

Condition 1. *For any sufficiently small $\delta > 0$ we have the properties:*

(i) For any $x \in D$, $t > 0$, $P_x^D(\tau^D > 0) = 1$ and $P_x^D(\tau^D > t) > 0$;

(ii) for any $t > 0$, $\inf_{x \in \bar{D}_\delta} P_x^D(\tau^D > t) > 0$, and

(iii) $\lim_{t\to\infty} \sup_{x \in D_\delta^c} P_x^D(\tau^{D_\delta^c} > t) = 0$.

Remark. Condition (i) is immediate if we assume that the densities of (1.1) are positive. In the case of regular diffusions on smooth domains Condition 1 is easily satisfied.

Given a $\delta > 0$, we denote by $\alpha(\delta) = \inf\{t \ge 0 \,|\, x(t) \in \bar{D}_\delta\}$, the first hitting time of $\bar{D}_\delta$ by the process $(x(t))$, and $\alpha(\delta) = \infty$ if the infimum is over the empty set. We define $l(\delta) = \max\{l \ge 0 \,|\, \tau_l \le \alpha(\delta)\}$, equal to infinity if $(x(t))$ never reaches $\bar{D}_\delta$; in general, $l(\delta)$ is not a stopping time.

Condition 2. *(i) There exists $\delta > 0$ such that $P_x(l(\delta) < \infty) = 1$.*

(ii) The condition (i) is uniform in $x \in D_\delta^c$, i.e.

$$\lim_{l\to\infty} \sup_{x \in D_\delta^c} P_x(l(\delta) > l) = 0\,. \tag{2.1}$$

A useful way of looking at (2.1) is to observe that $X_l := x(\tau_l)$, $0 \le l \le l^*$ is a Markov chain with state space D and transition probabilities

$$S(x, dy) = \int_D P_x(x(\tau^D-) \in d\xi)\nu_\xi(dy) = \int_D \lambda_0(x, d\xi)\nu_\xi(dy)\,. \tag{2.2}$$

We note that $S(x, D) = 1$ without any assumption on ν_ξ, thus (X_l) is well defined for any $l \le l^*$. Even when $l^* = \infty$, it is nontrivial to show that $(x(t))$ is non-explosive since (X_l) may not be tight in D. In other words, if the chain migrates towards the boundary, the duration $\tau_l - \tau_{l-1}$ of an episode between boundary visits diminishes with the possibility that $\tau^* < \infty$.

Proposition 1. *Condition 2 is satisfied if there exists a positive integer m and a positive real constant c_1 such that*

$$\inf_{x \in D_\delta^c} P_x(l(\delta) \le m) \ge c_1 > 0\,. \tag{2.3}$$

Remark. 1) If $x \in \bar{D}_\delta$, then (2.3) is trivially satisfied with $m = 0$ and any constant $c_1 \le 1$, which implies that (2.3) can be immediately extended to all $x \in D$.
2) A sufficient condition for (2.3) is that $\nu_\xi(\bar{D}_\delta)$ be bounded away from zero, uniformly in $\xi \in \partial D$, as in (5.1). This corresponds to $m = 1$ in (2.3) and works for all the applications discussed in Subsection 5.1. The particle model from Subsection 5.2 requires (2.3) with $m = n$, where $n \ge 2$ (in the nontrivial case) is the number of particles. To summarize, (5.1) $\Rightarrow$ (2.3) $\Rightarrow$ (2.1).

Proof. Let $x \in D_\delta^c$. From (2.3) we derive that $P_x(l(\delta) > l) \le (1 - c_1)^{[\frac{l}{m}]}$ as an application of the strong Markov property to the chain $X_l := x(\tau_l)$, $l \ge 0$, which implies (2.1). $\square$

We may state our first result, with proof given in Section 3.

Theorem 1. *Assume that Conditions 1 and 2 (i) are satisfied for the same $\delta > 0$. Then for any $x \in D$, we have $P_x(\tau^* = \infty) = 1$.*

We state the strong Doeblin condition, that ensures uniform exponential ergodicity. For the statement in discrete time, the reader is referred to [16], and Theorem 5.3 in [6] which settles the case of continuous time processes.

Doeblin's Theorem. *Let X be a locally compact metric space and Z_t, $t \ge 0$ a continuous time Markov process with state space X. Assume there exists a time $T_0 > 0$, a probability measure $\eta(dx)$ on X and a positive constant $k_0 < 1$ such that for all $x \in X$ and all $B \in \mathcal{B}(X)$ we have $P_x(Z_{T_0} \in B) \ge k_0 \eta(B)$. Then, (i) the process has an invariant probability measure $\mu(dx)$ and there exist positive constants C and $r < 1$, independent of x, such that*

$$||P_x(Z_t \in \cdot) - \mu(\cdot)|| \le C r^t\,, \tag{2.4}$$

where $||\cdot||$ denotes the total variation norm on the space of finite measures and (ii) the convergence rate satisfies $r \le (1 - k_0)^{\frac{1}{T_0}}$.

An interesting observation is that a second Markov chain Y_l, $l \ge 0$ on ∂D can be defined with transition probabilities $P(Y_1 \in d\xi' \,|\, Y_0 = \xi) = \int_D \nu_\xi(dx) \lambda_0(x, d\xi')$. If the set D is bounded, the chain has at least one invariant probability measure due to the tightness of any family of measures supported on a compact set. On the other hand, if $\mu_Y(d\xi)$ is an invariant measure for (Y_l), then $\mu_X(dx) = \int_{\partial D} \nu_\xi(dx)\mu_Y(d\xi)$ is invariant for the chain (X_l), which can be used to construct the invariant measure (2.9) under appropriate conditions.

To establish exponential ergodicity of the process $(x(t))$ we shall make an additional assumption, natural when D is bounded (which we don't require). Let $\alpha(F) = \inf\{t \geq 0 \,|\, x(t) \in F\}$. We shall say that F is *uniformly accessible* from a set $A \subseteq D$ if

$$\lim_{t\to\infty} \sup_{x\in A} P_x(\alpha(F) > t) = 0\,. \tag{2.5}$$

Condition 3. *There exists $\delta > 0$ and a closed set $F \subseteq \bar{D}_\delta$ with $\lambda(F) > 0$ such that for any $t > 0$*

$$\inf_{t\leq t'\leq 2t} \inf_{y,z\in F} p^D(t', z, y) = b_1(t,\delta) > 0\,. \tag{2.6}$$

Remark. 1) If D is bounded, regular and the diffusion (1.1) has a uniformly elliptic infinitesimal generator with sufficiently smooth coefficients, then Conditions 1-3 are automatically satisfied with $F = \bar{D}_\delta$ since p^D are continuous in all variables and $\bar{D}_\delta$ and D^c_δ are compact.

2) In addition to the properties from 1), (2.6) is satisfied and stronger, explicit bounds [5] exist for any $t > 0$ when D is bounded connected with C^1 boundary - see also the remark following Theorem 4.

3) Conditions 1 and 3 are on the kernel of the killed process p^D only and ignore the jumps at the boundary. Condition 1 is required for non-explosiveness while Condition 3 is required for the Doeblin condition, that is, for ergodicity.

The proof of the following theorem is given in Section 4.

Theorem 2. *Suppose Conditions 1, 2 and 3 satisfied for the same $\delta > 0$ and the set F in Condition 3 is uniformly accessible from $\bar{D}_\delta$ in the sense of (2.5). Then the process $(x(t))_{t\geq 0}$ is uniformly exponentially ergodic.*

In many cases, the invariant measure admits a representation in terms of the Green function of the underlying process killed at the boundary $(\tilde{x}(t))$ and the invariant measure of the chain (X_l) defined in (2.2). More precise results can be given case by case. We make a few supplementary assumptions in order to formulate this rigorously.

In the following discussion, preparing for Theorems 3 and 4, D will be a regular bounded domain. We shall denote by $(L, \mathcal{D}^o)$ the infinitesimal generator of the diffusion $(x^o(t))$, where L is a second-order uniformly elliptic differential operator with uniformly bounded smooth coefficients and the domain $\mathcal{D}^o \subseteq C_b(S)$ is the set of bounded, twice differentiable functions with continuous derivatives up to ∂S with some set of boundary conditions compatible with Conditions 1 and 3. In that case the underlying process $(x^o(t))$ is said *regular*. Then, the process $(\tilde{x}(t))$ has an infinitesimal generator with the same operator L and domain

$$\tilde{\mathcal{D}} = \{ f \in \mathcal{D}^o \,|\, f(\xi) = 0\,,\ \xi \in \partial D \}\,. \tag{2.7}$$

Let $G(x, x')$ be the corresponding Green function. By construction, the process $(x(t))$ has infinitesimal generator with operator L acting on the domain

$$\mathcal{D} = \{ f \in \mathcal{D}^o \,|\, f(\xi) = \int_D f(y)\nu_\xi(dy)\,,\ \xi \in \partial D \}\,. \tag{2.8}$$

Theorem 3. *If, in addition to the conditions of Theorem 2 the underlying process is a regular diffusion and the chain (2.2) has an invariant probability measure $\mu_X(dx)$, then the invariant probability measure $\mu(dx)$ of the process $(x(t))$ is absolutely continuous with respect to the Lebesgue measure $\lambda(dx) = dx$ with density*

$$\frac{d\mu}{dx}(x) = Z^{-1} \int_D G(x, x')\mu_X(dx')\,, \qquad Z = \int_D E_{x'}[\tau^D]\mu_X(dx')\,, \tag{2.9}$$

where Z is the normalizing constant.

To better understand (2.9), it is useful to calculate the resolvent

$$R_\beta(x, y) = \int_0^\infty e^{-\beta t} p(t, x, y)dt\,, \qquad \beta > 0$$

of the transition semigroup $p(t, x, dy)$ of the process $(x(t))$. One can establish [11, 3] that $P_x(x(t) \in dy)$ have densities $p(t, x, y)$. They satisfy

$$p(t, x, y) = p^D(t, x, y) + \int_0^t \int_D p(t-s, x', y) \int_{\partial D} P_x(\tau^D \in ds, x(\tau^D -) \in d\xi)\nu_\xi(dx')\,. \tag{2.10}$$

Applying the Laplace transform we see that the kernel of the resolvent verifies

$$R_\beta(x, y) = R^D_\beta(x, y) + \int_D K_\beta(x, dx')R_\beta(x', y) \tag{2.11}$$

where

r-02 (2.12)
$$K_\beta(x,dx') = \int_0^\infty e^{-\beta s} P_x(\tau^D \in ds, x(\tau^D -) \in d\xi)\nu_\xi(dx')\,. \tag{2.12}$$

With a minor abuse of notation between the operator K_β on $C_b(\bar{D})$ and its kernel we have

r-03 (2.13)
$$(I - K_\beta)R_\beta = R_\beta^D\,, \qquad R_\beta = (I - K_\beta)^{-1} R_\beta^D\,, \tag{2.13}$$

where the second equality is formal. The existence conditions of the inverse are difficult to establish directly. However, one can see that the density of the invariant measure from (2.9) is (again formally) obtained as

r-04 (2.14)
$$\lim_{\beta\to 0} \beta R_\beta(x,y) = \lim_{\beta\to 0} \int_D (\frac{I - K_\beta}{\beta})^{-1}(x,dx') R_\beta^D(x',y) = Z^{-1}\int_D G(x',y)\mu_X(dx') \tag{2.14}$$

since

r-05 (2.15)
$$\lim_{\beta\to 0} R_\beta^D(x',y) = G(x',y)\,, \qquad \mu_X(dx') = Z \lim_{\beta\to 0} (\frac{I - K_\beta}{\beta})^{-1}(x,dx')\,. \tag{2.15}$$

Intuitively, K_β is the transition function of the interior resolvent chain (2.2) and Z accounts for the average duration of a trip to the boundary in equilibrium. Under stronger assumptions, the spectral gap is calculated in [9, 11] when the inverse Laplace transform is obtainable. We shall not pursue this venue in this paper.

The next question is if we can relate the spectral gap of the killed process to the spectral gap of the jump process $(x(t))$. The following result is proven in Section 4.

cor-1 **Theorem 4.** *Let α_D be the spectral gap of the process killed at the boundary ∂D defined by (1.1). Assume $b_1(T_0,\delta) \geq C_1 e^{-\alpha_D T_0}$, with $C_1 > 0$ depending possibly on δ but not on T_0. Under the conditions of Theorem 2, considering δ fixed and T_0 as defined in Lemma 2, there exists $c_D(T_0) > 0$ depending on the process killed at the boundary and possibly on $\nu_\xi(\cdot)$ via T_0 only, such that the convergence rate from (2.4) satisfies $-\ln r \geq c_D(T_0)\alpha_D$.*

Remark. It is known [5] that assuming D is bounded connected with sufficiently smooth boundary and the diffusion (1.1) has sufficiently smooth coefficients, then the transition probabilities are bounded smooth functions in (t,x,y) for $t > 0$, $x,y \in D$ with the lower bound $b_1(T_0,\delta) \geq C_1 e^{-\alpha_D T_0}$.

3. Proof of non-explosiveness

By construction, the process $x(t)$ will have a boundary jump at times $(\tau_l)_{l\geq 0}$, starting with $\tau_1 = \tau_1^D$ and continuing with $\tau_l = \sum_{1\leq j\leq l} \tau_j^D$ until the last jump at τ_{l^*}. Recall that if there is a finite number of jumps, we convene that $\tau_l = \infty$ for all $l > l^*$. We have to prove that $P_x(\tau^* = \infty) = 1$ for all $x \in D$, or equivalently, that $x(t)$ is *non-explosive.* This is the contents of Theorem 1. We start with a lemma containing the essence of the proof, and useful as an independent result. Recall $\alpha(\delta)$ is the first hitting time of $\bar{D}_\delta$ by the jump-process $(x(t))$ and $l(\delta)$ is the number of jumps before $\alpha(\delta)$ (defined right before Condition 2). We note that $\alpha(\delta) \geq \tau^*$ only if $\alpha(\delta) = +\infty$.

Lemma 1. *Assume Condition 1 is satisfied and $P_x(\alpha(\delta) < \tau^*) = 1$ for all $x \in D \setminus D_\delta$. Then the process is non-explosive.*

Remark. The statement $P_x(\alpha(\delta) < \tau^*) = 1$ for all $x \in D$ is then immediate.

Proof. In view of the hypothesis, it is sufficient to prove that $P_x(\tau^* = \infty) = 1$ for any $x \in \bar{D}_\delta$. Let $S < \infty$ be a sufficiently large deterministic time; we want to show that $P_x(\tau^* \leq S) = 0$. There are two possibilities: Either there are no jumps at all, in which case $\tau^* = \infty$, or there exists at least one jump $\tau_1 < \infty$ and then we define $\alpha_1 = \inf\{t > \tau_1 \,|\, x(t) \in \bar{D}_\delta\}$. In this case we notice that since $P_x(\alpha(\delta) < \tau^*) = 1$ for any x, then $\tau^* > \alpha_1 \geq \tau_1 = \tau^D$ with probability one. Put $u(S) = \sup_{x\in\bar{D}_\delta} P_x(\tau^* \leq S)$. Applying the strong Markov property to the stopping time α_1, we obtain

$$P_x(\tau^* \leq S) = P_x(\tau^* \leq S\,,\, \alpha_1 < \tau^*)$$
$$\leq P_x(\tau^* \leq S\,,\, \alpha_1 < S) = \int_0^S P_x(\tau^* \leq S \,|\, \alpha_1 = s) P_x(\alpha_1 \in ds)$$
$$\leq \int_0^S E_x[P_{x(s)}(\tau^* \leq S - s \,|\, \alpha_1 = s)]\, P_x(\alpha_1 \in ds) \leq u(S) P_x(\alpha_1 \leq S)$$

after taking the supremum over $x(\alpha_1) \in \bar{D}_\delta$ in the last inequality. The supremum over $x \in \bar{D}_\delta$ on both sides of the inequality, as well as the fact that $\alpha_1 \geq \tau^D$ give

$$0 \geq u(S)(1 - \sup_{x\in\bar{D}_\delta} P_x(\alpha_1 \leq S)) \geq u(S) \inf_{x\in\bar{D}_\delta} P_x(\tau^D > S)\,.$$

Our claim is proved if we show that for sufficiently large S, $\inf_{x\in\bar{D}_\delta} P_x(\tau^D > S) > 0$, which is guaranteed by Condition 1 (ii). Then we have $u(S) = 0$ for any $S > 0$ large enough, proving the claim. $\square$

3.1. **Proof of Theorem 1.**

Proof. Condition 1 (i) implies that the sequence $(\tau_l)_{0\le l\le l^*}$ is strictly increasing almost surely. Due to Lemma 1, we only have to show that $P_x(\alpha(\delta)<\tau^*)=1$ for any $x\in D$. By construction $\alpha(\delta)=\inf\{t\ge 0\,|\,x(t)\in\bar D_\delta\}$, which means that either $\alpha(\delta)<\tau^*$ (if it occurs in one of the episodes $[\tau_{l-1},\tau_l)$, $1\le l\le l^*+1$) or $\alpha(\delta)=\infty$, or equivalently, the process never enters $\bar D_\delta$. If $x\in\bar D_\delta$, then $\alpha(\delta)=0<\tau^*$. Assuming $x\in D_\delta^c$, (2.1) implies that $P_x(l(\delta)<\infty)=1$, which means that $\alpha(\delta)\in[\tau_{l(\delta)},\tau_{l(\delta)+1})$. We proceed to show a slightly stronger statement than needed, namely that $P_x(\alpha(\delta)<\infty)=1$ for all $x\in D_\delta^c$.

Let $t>0$ and $x\in D_\delta^c$. For any $l\ge 1$,

03 (3.1)
$$P_x(\alpha(\delta)>t)\le P_x(\alpha(\delta)>t\,,\,l(\delta)\le l)+P_x(l(\delta)>l)\,,$$

030 (3.2)
$$\le P_x(\tau_{l(\delta)+1}>t\,,\,l(\delta)\le l)+P_x(l(\delta)>l)\,,$$

providing the bound (we recall that $\tau_l^D=\tau_l-\tau_{l-1}$)

03-2 (3.3)
$$P_x(\alpha(\delta)>t)\le(l+1)\sup_{x\in D_\delta^c}P_x(\tau^{D_\delta^c}>\frac{t}{l+1})+P_x(l(\delta)>l)\,.$$

For any small $\epsilon>0$, Condition 2 allows us to pick l such that $P_x(l(\delta)>l)<\epsilon$. Passing to the limit over $t\to\infty$ gives that $\limsup_{t\to\infty}P_x(\alpha(\delta)>t)<\epsilon$ (using Condition 1(iii)). Since ϵ is arbitrary, we conclude that $P_x(\alpha(\delta)<\tau^*)=1$. From here on, the theorem is a consequence of Lemma 1. □

Corollary 1 states that $\bar D_\delta$ is an uniformly accessible set from D.

cnon **Corollary 1.** *Conditions 1 and 2 imply that*

ec100 (3.4)
$$\lim_{t\to\infty}\sup_{x\in D}P_x(\alpha(\delta)>t)=0\,.$$

Proof. The proof is given by taking the supremum over $x\in D_\delta^c$ in lines (3.1)-(3.3) of the proof of Theorem 1 for D_δ^c and $\alpha(\delta)=0$ for $x\in D_\delta$. □

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4. Proof of exponential ergodicity.

We shall need the following lemma.

Lemma 2. *Assume Conditions 1, 3 as well as Condition 2 are satisfied for the same $\delta > 0$ and the set F in (2.6) is uniformly accessible from $\bar{D}_\delta$. Then there exists a time T_0 which may depend on δ, the redistribution measures ν_ξ and a positive constant c such that*

$$p(T_0, x, y) \geq c\,, \qquad \forall x \in D\,, \forall y \in F\,. \tag{4.1}$$

Moreover, the constant c may be chosen such that $c \geq \frac{1}{2} b_1(T_0, \delta)$.

Remark. In most applications we may choose $F = \bar{D}_\delta$ and (2.5) is no longer needed; for instance when D is bounded and the underlying process is a sufficiently regular diffusion.

Proof. Theorem 1 shows that $P_x(x(t) \in dy)$ stochastic, i.e. $P_x(x(t) \in D) = 1$ for all $x \in D$, $t \geq 0$. We recall (2.10) which implies immediately

$$p(t, x, y) \geq p^D(t, x, y)\,. \tag{4.2}$$

In view of (3.4), there exists $T_1 > 0$ with $\sup_{x \in D} P_x(\alpha(\delta) > T_1) \leq 1/4$. Also, by (2.5) there exists T_2 such that $\sup_{x \in \bar{D}_\delta} P_x(\alpha(F) > T_2) \leq 1/4$. Let $T = T_1 + T_2$, such that $\sup_{x \in D} P_x(\alpha(F) > T) \leq \frac{1}{2}$. We shall prove the lemma with $T_0 = 2T$.

Due to Condition 3 and (4.2), we see that

$$\inf_{T \leq t' \leq 2T} \inf_{y, z \in F} p(t', z, y) \geq b_1(T, \delta) > 0\,. \tag{4.3}$$

Pick $x \in D$. If $x \in F$, (4.3) implies that any $c \leq b_1(T, \delta)$ would satisfy (4.1). Suppose $x \in D \setminus F$. Let $\alpha(F)$ be as in (2.5) with $A = D_\delta$. To prove the lower bound for $p(2T, x, y)$, we start with an analogue of (2.10) with $t = 2T$. For any Borel set B,

$$P_x(x(2T) \in B) \geq P_x(x(2T) \in B, \alpha(F) \leq 2T) \tag{4.4}$$

which, after applying the Markov property to the stopping time $\alpha(F)$, implies the inequality for density functions

$$p(2T, x, y) \geq \int_0^{2T} \int_F p(2T - s, z, y) P_x(\alpha(F) \in ds, x(\alpha(F)) \in dz) \tag{4.5}$$

$$\geq \int_0^{T} \int_F p^D(2T - s, z, y) P_x(\alpha(F) \in ds, x(\alpha(F)) \in dz)\,. \tag{4.6}$$

The inequality is true for the integral on the full interval $0 \le s \le 2T$. In the special case when $0 \le s \le T$, $2T - s$ lies in the interval $[T, 2T]$, making (4.3) applicable to the integrand $p(2T - s, z, y)$, which gives

$$p(2T, x, y) \ge b_1(T, \delta) \int_0^T P_x(\alpha(F) \in ds, x(\alpha(F)) \in F) \ge b_1(T, \delta) P_x(\alpha(F) \le T) \tag{4.7}$$

$$\ge b_1(T, \delta)(1 - \sup_{x \in D} P_x(\alpha(F) > T)) \ge \frac{1}{2} b_1(T, \delta)\,. \tag{4.8}$$

The last inequality is true due to the construction of T. By choosing $c = \frac{1}{2} b_1(T, \delta)$ we proved the lemma with $T_0 = 2T$. □

Lemma 2 leads to our second result.

4.1. **Proof of Theorem 2.**

Proof. Let B be a Borel subset of D and let $\lambda(\cdot|F)$ be the probability measure defined by $\lambda(B|F) = \lambda(B \cap F)/\lambda(F)$ where λ is the reference measure defined right after (1.1). Then, for any $x \in D$, according to Lemma 2

$$p(T_0, x, B) \ge p(T_0, x, B \cap F) \ge c\, \lambda(B \cap F) = c\, \lambda(F)\, \lambda(B|F)\,.$$

Setting $k_0 = \min\{c\, \lambda(F), 1\}$, we have proven the condition from Doeblin's theorem is satisfied for the Markov process $(x(t))$ with the same T_0 as in Lemma 2 and $\eta(\cdot) = \lambda(\cdot|F)$. □

4.2. **Proof of Theorem 3.**

Proof. The invariant probability measure $\mu(dx)$ exists and is unique from the Doeblin's condition. We want to show that if $f \in \mathcal{D}$

$$\int_D \Big(\int_D G(x, x')\mu_X(dx') \Big) Lf(x) dx = 0\,. \tag{4.9}$$

The operator G commutes with L on the domain $\tilde{\mathcal{D}}$. Let $g(x) = \int_{\partial D} f(\xi)\lambda_0(x, d\xi)$ be the solution to $Lg = 0$, $g(\xi) = f(\xi)$ for all $\xi \in \partial G$. Then $f - g \in \tilde{\mathcal{D}}$ and we have

$$\int_D \Big(\int_D G(x, x')\mu_X(dx') \Big) Lf(x) dx = \int_D \Big(\int_D G(x, x')\mu_X(dx') \Big) L(f(x) - g(x)) dx \tag{4.10}$$

$$= \int_D \mu_X(dx') L \Big(\int_D G(x, x')(f(x) - g(x)) \Big) dx = \int_D \mu_X(dx') \Big(g(x') - f(x') \Big) \tag{4.11}$$

$$= \int_D \mu_X(dx') \int_{\partial D} \lambda_0(x', d\xi) \int_D \nu_\xi(dy) f(y) - \int_D \mu_X(dx') f(x') \tag{4.12}$$

$$= \int_D \mu_X(dx') \int_D S(x', dy) f(y) - \int_D \mu_X(dx') f(x') = 0\,, \tag{4.13}$$

where the equality between (4.11) and (4.12) is due to $f(\xi) = \int_D f(x') \nu_\xi(dx')$. $\square$

4.3. **Proof of Theorem 4.**

Proof. We refer to the last part of Doeblin's theorem to see that $-\ln r \geq -\frac{1}{T_0} \ln(1 - k_0)$, with $k_0 = \min\{c\lambda(F), 1\}$ (proof of Theorem 2) and for $b_1(t, \delta)$ defined in (2.6), we have $c \geq \frac{1}{2} b_1(T_0, \delta)$ (proof of Lemma 2). Summarizing, there exists $C > 0$ such that $k_0 \geq C b_1(T_0, \delta)$. We note that C and C_1 can be chosen independently of T_0, and thus independent of $(\nu_\xi(\cdot))_{\xi \in \partial D}$. Since $b_1(T_0, \delta) \geq C_1 e^{-\alpha_D T_0}$, let $C_2 = CC_1$. Then

$$-\frac{\ln r}{\alpha_D} \geq -\frac{1}{\alpha_D T_0} \ln(1 - C_2 e^{-\alpha_D T_0}) =: c_D(T_0) > 0\,. \tag{4.14}$$

$\square$

5. Examples

5.1. **Brownian motion with rebirth.** In the simplest version, the diffusion with jumps on the connected open set D has a delta function relocation measure $\nu_\xi(dx) = \delta_{x_0}(dx)$, $x_0 \in D$ [9, 11, 13], constant in $\xi \in \partial D$, $\nu_\xi(dx) = \nu(dx) \in M_1(D)$ [2, 3], or with continuous dependence on the exit point $\xi \to \nu_\xi(\cdot) \in M_1(D)$ [3]. In the special case of the delta measure and bounded D, the process is generated by a Feller semigroup on a compact manifold where the boundary is glued together with the return point x_0 [9, 11]. Other variants include the case [14] of a domain with piecewise smooth boundary and constant redistribution measure on each smooth component.

Proposition 2. *Consider a diffusion as in (2.8). Assuming D is bounded and the redistribution measure has the property*

$$\liminf_{\delta \to 0} b_2(\delta) > 0\,, \qquad \textit{where} \quad b_2(\delta) = \inf_{\xi \in \partial D} \nu_\xi(D_\delta)\,. \tag{5.1}$$

Then (a) condition 2 is satisfied with $m = 1$; (b) a sufficient condition for (5.1) is the piecewise continuity of the function $\xi \to \nu_\xi(dx)$, where the space $M_1(D)$ of probability measures on D is endowed with the topology of convergence in distribution.

Remark. An immediate nontrivial example of (5.1) is when $\nu_\xi(dx)$ are continuous in ξ over each smooth component of ∂D, even though not continuous over ∂D. One such case is when the components are constant, as in [14]. But simply put, it is sufficient that $\inf_{\xi\in\partial D}\nu_\xi(K)>0$ for some compact $K\subset\subset D$.

Proof. *(a)* For $x\in D_\delta^c$, since $\tau_1=\tau^D$ we have

ec10
$$P_x(x(\tau_1)\in\bar D_\delta)\geq\inf_{x\in D_\delta^c}\int_{\partial D}P_x^D(x(\tau^D-)\in d\xi)\nu_\xi(\bar D_\delta)\geq b_2(\delta)\,. \tag{5.2}$$

This proves that (2.3) is satisfied with $m=1$ and $c_1=b_2(\delta)$, which proves part *(a)*.

(b) We shall assume that $\xi\to\nu_\xi(dx)$ is continuous; it is easy to generalize to the case of piecewise continuity by taking the minima of $b_2(\delta)$ over all the finite number of continuous portions of $\partial D\ni\xi$. Let $\psi_\delta=\mathbf{1}_{\bar D_{2\delta}}*\rho_\delta$ where ρ_δ is a smooth approximation to the delta function at the origin. We choose ρ_δ in such a way that $0\leq\psi_\delta\leq 1$ is smooth, $\psi_\delta(x)=1$ on $\bar D_{3\delta}$ and $\psi_\delta(x)=0$ on $D\setminus D_\delta$ which ensures $\mathbf{1}_{\bar D_\delta}(z)\geq\psi_\delta(z)$. Let $\{\delta_k\}_{k\geq 1}$ be a sequence $\delta_k\downarrow 0$ and $u_k(\xi)=\int_D\psi_{\delta_k}(z)\nu_\xi(dz)$. Since $u_k(\xi)$ are 1) continuous in ξ on the compact ∂D, 2) nondecreasing in k, 3) $\lim_{k\to\infty}u_k(\xi)=1$ for all $\xi\in\partial D$, then we have the limit $\lim_{k\to\infty}\inf_{\xi\in\partial D}u_k(\xi)=1$. □

s:bksn

5.2. **Diffusive Bak-Sneppen fitness evolution model.** We consider an n particle system, a variant to the well known evolution model proposed by Bak and Sneppen in [1]. The particles follow Brownian motions $x_i(t)$ for $i\in\{1,2,\cdots n\}$ in an interval $(0,a]$, $a>0$ with reflection at $a\in\mathbb{R}$ and with 0 considered a boundary point, evolving independently of each other until the first one reaches 0. Each coordinate x_i represents the 'fitness state' of the i-th species and has an associated set of *neighbors* $V_i\subseteq\{1,2,\ldots,n\}$, designated by their indices, such that $i\in V_i$. Whenever one of the fitness levels x_i reaches the boundary point zero, all x_j with $j\in V_i$ are instantaneously replaced by new i.i.d. fitness levels with distribution function $G(x)$ on the interval $(0,a)$ and the evolution continues afresh. To fix ideas, we shall assume $i\to V_i$ to be deterministic functions and $|V_i|\geq 2$, for all $1\leq i\leq n$, where $|V|$ is the cardinality of V. It is easy to verify that $n=1$ and $n\geq 2$ with $|V_i|=1$ are covered in subsection 5.1 since this coincides with $V_i=\{i\}$ and particles move independently.

In the framework laid out in the Introduction, the dimension d is equal to the number of particles n, the vector-valued process $x^o(t)=\{x_i(t)\}_{1\leq i\leq n}$ is a diffusion on $S=(-\infty,a]^n$ with reflecting boundary conditions on each component at a, $D=(0,a]^n$ and

$\partial D = \partial(-\infty, a]^n$ in $\mathbb{R}^n$. Then $\tilde{x}(t)$ is the process killed at the part of the boundary of the hypercube D containing at least one zero component.

To be specific, the reflection takes place on the upper hyper-surface $U = \cup_{i=1}^n U_i$ where $U_i = \{x \in D \,|\, x_i = a\,,\, x_j \neq a \;\text{ for all } j \neq i\}$. On the other hand, redistribution is triggered on the lower hyper-surface $L = \cup_{i=1}^n L_i$ where $L_i = \{x \in D \,|\, x_i = 0\,,\, x_j \neq 0 \;\text{ for all } j \neq i\}$. We note that more than one boundary hits at the same time occur with probability zero in this setup. When one of the particles hits $\partial D = \{0\}$, the redistribution is carried out through a measure $\nu_{x(\tau^D-)}(dz)$ where $x(\tau^D-) = \xi = (\xi_1, \ldots, \xi_n) \in L$

nxbk
$$\nu_\xi(dz) = \sum_{i=1}^n \mathbf{1}_{L_i}(\xi) \cdot \Big(\otimes_{j \notin V_i} \delta_{\xi_j}(dz_j) \Big) \otimes \Big(\otimes_{j' \in V_i} dG(z_i) \Big)\,, \tag{5.3}$$

with i in the periodic lattice $\mathbb{Z}_n$.

For any sufficiently small $\delta > 0$ we let $F_i = \{\, x \in \bar{D} \,|\, \sum_{j=1}^n \mathbf{1}_{[0,\delta)}(x_j) = i \,\}$, the set on which there are exactly i coordinates less than δ. We notice that $F_0 = \bar{D}_\delta$ and $\cup_{i=0}^n F_i = \bar{D}$. We shall prove the bound (ec1 2.3) with $m := n$ inductively; the set F_n the worst case scenario and $F_0 = \bar{D}_\delta$, the set we want to enter almost surely.

p-i **Proposition 3.** *Assume that G is concentrated on $(0, a)$, i.e. $G(0^+) = 0$ and $G(a-) = 1$. For all $1 \leq k \leq n$, there exists a positive real w_k independent of x, such that, if $x \in F_k$, then*

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$$P_x(x(\tau^D) \in A_{k-1}) \geq w_k\,, \qquad A_l = \cup_{0 \leq j \leq l} F_j\,, \quad 0 \leq l \leq n\,. \tag{5.4}$$

Remark. Since G is concentrated in $(0, a)$, there exists $\delta \in (0, a)$ such that G charges $[\delta, a]$, which is equivalent to $G(\delta) < 1$. The key observation is that since the particle hitting the boundary is for sure in $[0, \delta)$ and G charges $[\delta, a]$, then with a positive probability independent of the current configuration, right after the jump there will be at least one more particle in $[\delta, a]$.

Proof. First we note that there exist a positive real $v_k = v_k(\delta, a)$, $0 \leq k \leq n$, satisfying

n11
$$\inf_{x \in F_k} P_x(x(\tau^D-) \in F_k \cap \partial D) \geq v_k > 0\,. \tag{5.5}$$

To see that, the harmonic functions $u(x) = P_x(x(\tau^D-) \in F_k \cap \partial D)$ have limit one at interior points of $F_k \cap \partial D$ and zero at exterior points. Since the boundary function is piecewise continuous (indicator function), the solution $u(x)$ is equal to its Fourier series and has limit 1/2 at boundary points. Thus over A_k, a compact set, the minimum is strictly positive.

Following the distribution after jump as in (5.2), when $x \in F_k$

n10
$$P_x(x(\tau^D) \in A_{k-1}) \geq \inf_{x\in F_k} \int_{\partial D} P_x^D(x(\tau^D-) \in d\xi)\nu_\xi(A_{k-1}) \geq v_k \inf_{\xi \in F_k \cap \partial D} \nu_\xi(A_{k-1})\,. \tag{5.6}$$

According to (5.3), $\xi \in F_k$ will imply that $\nu_\xi(A_{k-1}) \geq (1-G(\delta))^{|V_i|}$ for all $1 \leq k \leq n$. This is based on the fact that $x_i = 0 \in [0,\delta)$ to begin with, and then the event that all neighbors V_i go to $[\delta, a]$ implies that the number of particles in $[0,\delta)$ has diminished by at least one. Using the fact that $G(0^+) = 0$, $G(\delta) < 1$ and $l(V) = \max_{1\leq i\leq n} |V_i| \leq n$ we proved the lower bound $\nu_\xi(A_k) \geq (1-G(\delta))^{l(V)} > 0$, uniformly in k. This proves the proposition with $w_k = v_k(1-G(\delta))^{l(V)}$. □

p-ii **Proposition 4.** *Under the same assumptions as in Proposition 3, condition (2.3) is satisfied with $m = n$ and $c_1 = (\min\{w_k\})^n$.*

Proof. Put $w = \min\{w_k\}$. Since $F_0 = \bar{D}_\delta$, it is clear in view of Proposition 3 that if $x \in F_k$, then $P_x(x(\tau_k) \in \bar{D}_\delta) \geq w^k$, meaning that $P_x(l(\delta) \leq n) \geq w^k$. We only have to choose $m := n$ and $c_1 = w^n$. □

References

BS1 [1] Bak, P., Sneppen K. (1993) *Punctuated equilibrium and criticality in a simple model of evolution*, Phys. Rev. Let. Vol. 74

BAP1 [2] Ben-Ari, Iddo; Pinsky, Ross G. *Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure.* J. Funct. Anal. 251 (2007), no. 1, 122–140.

BAP2 [3] Ben-Ari, Iddo; Pinsky, Ross G. *Ergodic behavior of diffusions with random jumps from the boundary.* `http://www.math.uconn.edu/~benari/pdf/Iddo-Ross1_1006.pdf`

BHM [4] Burdzy, K., Hołyst, R., March, P. (2000) *A Fleming-Viot particle representation of the Dirichlet Laplacian.* Comm. Math. Phys. 214, no. 3.

Da [5] Davies, E. B. *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, UK, 1989.

D-M-T [6] Down, D., Meyn, P., Tweedie, R. L. (1995) *Exponential and uniform ergodicity of Markov processes.* Ann. Probab., Vol. 23, No. 4, 1671–1691.

Duflo [7] Duflo, M. *Random iterative models.* Applications of Mathematics (New York), 34. Springer-Verlag, Berlin, 1997.

FerrariMaric [8] Ferrari, P. A.; Maric, N. *Quasi stationary distributions and Fleming-Viot processes in countable spaces.* Electron. J. Probab. 12 (2007), no. 24, 684–702

GK1 [9] Grigorescu, I., Kang, M. *Brownian motion on the figure eight*, J. Theoret. Probab. 15 (2002) 817–844.

GK2 [10] Grigorescu, I., Kang, M. (2004) *Hydrodynamic Limit for a Fleming-Viot Type System.* Stochastic Process. Appl. 110, no. 1, 111-143.

GK3 [11] Grigorescu, I.; Kang, M. *Ergodic properties of multidimensional Brownian motion with rebirth.* Electron. J. Probab. 12 (2007), no. 48, 1299–1322

GKimm [12] Grigorescu, I.; Kang, M. *Immortal particle for a catalytic branching process.* (2009) preprint.

E [13] Kosygina, E. *Brownian flow on a finite interval with jump boundary conditions.* Discrete Contin. Dyn. Syst. Ser. B 6 (2006), no. 4, 867–880

WLi1 [14] Li, W.; Leung, Y.; Rakesh. *Spectral analysis of Brownian motion with jump boundary*, Proceedings of the American Mathematical Society. Vol 136, 4427-4436, (2008)

Uwe [15] Lobus, J-U. *A Stationary Fleming-Viot Type Brownian Particle System*, Mathematische Zeitschrift, (2008) 1432-1823.

M-T [16] Meyn, P., Tweedie, R. L. (1993) *Markov chains and stochastic stability.* Springer-Verlag, London.

Shreve [17] Shreve, Steven E. *Stochastic calculus for finance. II. Continuous-time models.* Springer Finance. Springer-Verlag, New York, 2004.