A SIMPLE FORMULA AND LOWER BOUND FOR DOUBLY EVEN NORMAL MAGIC SQUARES

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Abstract. We prove a direct formula to generate a special type of $n \times n$ magic square, with the restriction that it has consecutive entries 1 to n^2 (normal) and $n = 4k$ (doubly even), having the property that its 4×4 basic squares have equal sum over rows and columns (semi-magic squares). Based on that we generate a large number of magic squares by three types of permutations giving a lower bound on the total number of magic squares $N(n)$ on the scale of the total number $N = (n^2)!$ of normal squares, i.e. $N(n) \geq O(N^c)$, $c = 1/16$. As opposed to the non-negative semi-magic squares where the precise asymptotics are known, for normal magic squares no bounds are known except an upper bound with $c = 1$ given by [12].

1. Introduction

Let $n \in \mathbb{N}$ and $\mathbf{M} = (\mathbf{M}(i,j))$, $1 \leq i,j \leq n$, be a matrix with positive integers. In the following, we shall refer to the matrix as a *square*, and its entries as *boxes.* We shall say $\mathbf{M}$ is a *semi-magic square* if the sums over rows and columns are all equal to S, the *magic sum.* If, in addition, the sums over the diagonals have the same value S, the matrix is said *magic.* If the entries are in bijection with the set $\{1, 2, \ldots, n^2\}$ the square is said *normal.* In that case, any magic square will have magic sum

$$S = \frac{n(n^2+1)}{2}\,.$$

In this paper we are interested in *normal doubly even magic squares*, when $n = 4k$, $k \in \mathbb{N}$.

Magic squares have a long history, see for example [12] for a brief and vivid account, but also more recent papers like [5], emphasizing construction algorithms. The problem of

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finding correct asymptotics for the number $M(n,t)$ of (semi-)magic squares of order $n \times n$ with *non-negative* entries and sums t on every row and column is quite different from the more restrictive normal case, whose number we denote by $N(n)$. In spite of its ancient origin and vast literature, including computational as well as recreational mathematics [9, 10, 11], no sharp bounds exist for the normal case, except the one in Ward [12] (who calls normal magic squares *classical*) showing that

$$N(n) \leq \frac{(n!)^2}{8(2n+1)!} =: N_1(n)\,. \tag{1.1}$$

Noting that the total number of normal squares is $N = (n!)^2$, this upper bound has logarithmic order equal to that of $\ln N$, appearing quite large.

For non-normal semi-magic squares, precise results exist. Early on, Read [7] proved asymptotics for a $t = 3 << n$, a setup quite different from the normal squares. An estimate for large n is established in the classic paper [2] in relation to two-way contingency tables. Stanley (see [8], also for more history of the question) proved that for a given n, the number $M(n,t)$ is a polynomial in t known as the Ehrhart polynomial of the Birkhoff polytope.

The most precise estimate of $M(n,t)$ is proven in [1]. In that formulation, the matrix is not necessarily square, instead of order $m \times n$ and has sums equal to t over rows and s over columns, with $\lambda = t/m = s/n$. Denote $M(m,s;n,t)$ their total number. The authors prove the exact correction factor to the remarkable heuristic formula obtained by Good [3, 4], who conjectured that $M(m,s;n,t)$ is approximated by $G(m,s;n,t)$ where

$$G(m,s;n,t) = \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn+\lambda mn-1}{\lambda mn}}\,.$$

Take $n = m$, $s = t = S$ and $\lambda = S/n = \frac{n^2+1}{2}$. With our notation the number $M(n,S) = M(n,S;n,S)$ is an immediate upper bound for $N(n)$. Corollary 1 to Theorem 1 in [1] can be reformulated with the asymptotic formula

$$M(n,S) = \left(1 + \frac{n^2}{2}\right)^{(n-1)^2} \frac{(n^2)!}{(n!)^{2n}} \exp\left(\frac{1}{2} + O(n^{-b})\right)\,, \quad 0 < b < \frac{1}{2}\,.$$

Unfortunately, the number $M(n,S)$ is still too large. Whereas naturally $N(n)/N < 1$, we verify that

$$M(n,S) \approx \left(\frac{en^2}{2}\right)^{n^2} = N\left(\frac{e^2}{2}\right)^{n^2} >> N > N_1(n) > N(n)$$

where the $\approx$ sign means that the ratio of the two numbers approaches one. Under these conditions, (1.1) remains the only meaningful upper bound for the normal case.

In this context, in Theorem 2 we propose a nontrivial lower bound on the scale of $N = (n^2)!$ where we show that $N(n) \geq N^c$, with $c = 1/16$.

Our proof has two steps. First, we identify a template magic square $\mathbf{A}$ defined in eq. (2.7), which can be decomposed in k^2 semi-magic squares of order 4×4 (Lemma 1). The square is mentioned in [10] for $k = 3$, as well as in [6], where the emphasis is on the generating algorithm. We provide a rigorous proof that $\mathbf{A}$ is magic (Theorem 1) and then we show there exists a very large class of transformations that produce magic squares. In this way we generate a large number of squares, which enables us to give a lower bound for the total number $N(n)$. To our knowledge, such bounds are not generally known, even in special cases, a fact mentioned in an update of [11], paragraph on *Enumeration of magic squares*.

2. Construction of the special square $\mathbf{A}$

We recall that a box (i, j) is simply the entry on row i and column j of the square. The *rank square*, denoted by $\mathbf{R}$, is the square with entries starting from 1 in the upper left corner, continuing in increasing order from left to right and top to bottom up to the last entry at the right bottom corner equal to n^2. In other words, $R(i, j)$ is the *rank of the* (i, j) *box* when counting from top to bottom, left to right, and $R'(i, j)$ the *reversed rank* (also known as complementary), when counting from bottom to top, right to left. More precisely

$$R(i,j) = n(i-1) + j\,, \qquad R'(i,j) = n^2 + 1 - R(i,j) \tag{2.1}$$

and notice that

$$R(n+1-i, n+1-j) = R'(i,j)\,. \tag{2.2}$$

For a positive integer x, we define $[x]$ as

$$[x] = r\,, \quad \text{if} \quad x = 4q + r \quad \text{and} \quad r \in \{1,2,3,4\}\,, \quad q \in \mathbb{Z}\,, \tag{2.3}$$

Remark. The bracket $[x]$ differs from the class modulo 4 in $\mathbb{Z}_4$ only for $r = 4$. The notation is justified by Proposition 2, and illustrated in Table 3.

2.1. **Basic squares.** Since $n = 4k$, $k \in \mathbb{N}$, each $1 \leq i, j \leq n$ can be written uniquely as

$$i = 4q + [i]\,, \qquad 0 \leq q \leq k-1\,, \qquad j = 4l + [j]\,, \quad 0 \leq l \leq k-1\,. \tag{2.4}$$

In a doubly even square $\mathbf{M}$, there are k^2 squares of size 4×4, denoted $B_{\mathbf{M}}(q,l)$ indexed by the pairs (q,l). They are called *basic* 4×4 *squares*. More precisely

$$B_{\mathbf{M}}(q,l) = \Big(\mathbf{M}(4q+r, 4l+s)\Big)_{1 \leq r,s \leq 4}\,, \qquad 0 \leq q, l \leq k-1\,. \tag{2.5}$$

Figure 1 depicts the case $k = 3$ for the rank square $\mathbf{R}$.

Definition 1. *In a doubly even square* $\mathbf{M}$*, we refer to the* first, respectively second diagonal modulo four *as the union of all first, respectively second diagonals belonging to all basic squares. The* diagonal modulo four *will be the union of the two diagonals of all basic squares, that is the set of entries* $\mathbf{M}(i,j)$ *corresponding to the set of indices* $(i,j) \in D$*, where*

$$D = \Big\{(i,j) \,\Big|\, [j] = [i] \quad or \quad [i] + [j] = 5\Big\}\,. \tag{2.6}$$

We note that $n + 1 = 5$ (modulo 4) and the definition corresponds to the union of the diagonals of the basic 4×4 squares.

When $[i] = [j]$, we say that (i,j) belongs to the *first diagonal* and when $[j] = [n+1-i] = 5 - [i]$ that (i,j) belong to the *second diagonal*. We see in the proposition below that they never intersect.

Proposition 1. *(i) The first and second diagonals modulo four never intersect. (ii) There are exactly* $n/2$ *diagonal modulo four terms on each row and each column, i.e. the number of diagonal and non-diagonal modulo four terms is the same. (iii) A box* $(i.j)$ *is diagonal modulo four if and only if* $(n+1-i, n+1-j)$ *is diagonal modulo four.*

Proof. In the proof, we simply refer to diagonal modulo four as diagonal.

(i) If the two diagonals intersect, $2i - 1$ must be divisible by four, a contradiction.

(ii) Each row, and by symmetry each column, can be divided in blocks of four boxes, as $n = 4k$. Each contains two diagonal boxes, and there are $k = n/4$ such blocks. Since there are $2 \times \frac{n}{4} = \frac{n}{2}$ diagonal terms and a total of n boxes on a row, we are done.

(iii) We can immediately see that $[i + j - (n+1)] = 0$ if and only if $[(n+1) - i + (n+1) - j - (n+1)] = 0$, proving the third claim. $\square$

TABLE 1. The rank square $\mathbf{R}$ for $k=3$, $n=12$. The 4×4 basic squares are delineated and the diagonal terms are marked by $*$. The gray color basic square $B_{\mathbf{R}}(1,0)$ is non-diagonal. See also Theorem 2. For $k=1,2$ all basic squares are diagonal.

*	2	3	*	*	6	7	*	*	10	11	*
13	*	*	16	17	*	*	20	21	*	*	24
25	*	*	28	29	*	*	32	33	*	*	36
*	38	39	*	*	42	43	*	*	46	47	*
*	50	51	*	*	54	55	*	*	58	59	*
61	*	*	64	65	*	*	68	69	*	*	72
73	*	*	76	77	*	*	80	81	*	*	84
*	86	87	*	*	90	91	*	*	94	95	*
*	98	99	*	*	102	103	*	*	106	107	*
109	*	*	112	113	*	*	116	117	*	*	120
121	*	*	124	125	*	*	128	129	*	*	132
*	134	135	*	*	138	139	*	*	142	143	*

Lemma 1. *Let $\mathbf{A}$ be defined based on the rank matrix $\mathbf{R}$, introduced in (2.1), as follows*

$$A(i,j)=R(i,j)\,, \quad \text{if} \quad (i,j)\notin D \qquad \text{and} \quad A(i,j)=R'(i,j) \quad \text{if} \quad (i,j)\in D\,. \tag{2.7}$$

Then, any basic square of $\mathbf{A}$ is semi-magic with sum $S_b=2(n^2+1)$ (the subscript comes from basic). The first and second diagonal sums are also equal, but depend on the index (q,l) of the basic square as seen in formula (2.8).

Remark. The basic squares of $\mathbf{A}$ are magic, but not normal since they are 4×4 and their values are not necessarily in $\{1,2,\dots,16\}$.

Proof. Let q,l,r,s are as in (2.5) and write $b=4n(q-1)+4l$, $b'=n^2+1-b$ and $c(r,s)=nr+s$. Then, the formula for $B_{\mathbf{A}}(r,s)$ shows that a basic square has the form given explicitly in Table 2, based on

$$A(i,j)=A(4q+[i],4l+[j])=\begin{cases} b'-c([i],[j])\,, & \text{if} \quad (i,j)\in D \\ b+c([i],[j])\,, & \text{if} \quad (i,j)\notin D \end{cases}.$$

It is easy to verify that such a square has the same sum over all rows and columns, equal to $2(b+b')=S_b=2(n^2+1)$. In addition, the sum over diagonals is dependent on the pair

Table 2. Generic basic square with actual values.

$b'-n-1$	$b+n+2$	$b+n+3$	$b'-n-4$
$b+2n+1$	$b'-2n-2$	$b'-2n-3$	$b+2n+4$
$b+3n+1$	$b'-3n-2$	$b'-3n-3$	$b+3n+4$
$b'-4n-1$	$b+4n+2$	$b+4n+3$	$b'-4n-4$

(q,l), because

$$4b' - (1+2+3+4)(n+1) = 4n^2 - 10n - 6 - 4b\,, \quad b = 4n(q-1) + l\,. \tag{2.8}$$

□

Theorem 1. *The matrix* **A** *defined in Lemma 1 is normal magic.*

Remarks.

1) Due to (2.2) $A(i,j) = A'(n+1-i, n+1-j)$ showing that the square $\mathbf{A}'$, with entries in reverse order from bottom to top, is also normal magic.

2) The matrix **A** is obtained by *the explicit formula (2.7)*, not by an algorithm. The only determination is whether $(i,j) \in D$ or $\notin D$, i.e. either $i \equiv j \,(mod\, 4)$ or $i + j \equiv 1 \,(mod\, 4)$, an equivalent form of (2.6).

Proof. Part 1 - bijectivity. Since $A(i,j)$ sends $\{1,2,\ldots,n\}^2$ into the set $\{1,2,\ldots n^2\}$ of equal cardinality, as well as the complementarity formula (2.2), it is sufficient to show injectivity.

The rank function $R(i,j)$ is bijective. Over the diagonal entries in D, the function is strictly decreasing, being the reversed rank, and over its complement it is strictly increasing, being the natural rank. To complete the proof, we need to show that if $(i,j) \in D$ and $(i',j') \in D^c$, then $A(i,j) \neq A(i',j')$.

Suppose they were equal. Due to (2.2), this is equivalent to

$$A(i,j) = R'(i,j) = R(n+1-i, n+1-j) = R(i',j')\,,$$

and, given the fact that the natural rank $R(\cdot,\cdot)$ is bijective,

$$n+1-i = i'\,, \qquad n+1-j = j'\,.$$

Passing to modular equalities we have

$$5 = [i] + [i']\,, \qquad 5 = [j] + [j']\,.$$

If $[i] = [j]$, then $[i'] = [j']$ and if $[i] + [j] = 5$, then $[i'] + [j'] = 5$. In both cases, we obtained that $(i',j') \in D$, a contradiction.

Part 2 - sum over rows and columns.

Lemma 1 says that the $k \times k$ basic squares are semi-magic with sum $2(n^2+1) = 2(16k^2+1)$. Then $\mathbf{A}$ is semi-magic with

$$S = \frac{n}{4} \times 2(n^2+1) = \frac{n(n^2+1)}{2} = \frac{1+2+\ldots n^2}{n}\,.$$

The sum over the first diagonal is

$$\sum_{i=1}^{n} A(i,i) = n(n^2+1) - \sum_{i=1}^{n} \Big[n(i-1)+i\Big] = n(n^2+1) + n^2 - (n+1)\left[\frac{n(n+1)}{2}\right] = S$$

and the over the second diagonal

$$\sum_{i=1}^{n} A(i,n+1-i) = n(n^2+1) - \sum_{i=1}^{n} \Big[n(i-1)+n+1-i\Big]$$

$$= n(n^2+1) - \sum_{i=1}^{n} \Big[n(i-1)+i+n+1-2i\Big] = S - \sum_{i=1}^{n} \Big[n+1-2i\Big] = S\,.$$

□

We conclude this section with Proposition 2, which is not used in the proof, but shows some interesting properties of the magic square $\mathbf{A}$. Here, modular values refer to the actual entries a being replaced by their values $[a]$.

Proposition 2. *In any basic square, the sum of all modular values is* 10. *The sum of modular values that are non-diagonal is equal to the sum of modular values belonging to the* *diagonals, thus both equal to* 5*. In other words,*

$$\forall\ 1 \le i \le n\,, \quad 0 \le l \le k-1\,, \qquad \sum_{4l<j\le 4(l+1),(i,j)\notin D} [j] = \sum_{4l<j\le 4(l+1),(i,j)\in D} [j] = 5\,.$$

Moreover, we have the formula for the sum S_i *of the entries in row* i*, not belonging to* D

$$S_i = \frac{n^2(2i-1)+n}{4}\,. \tag{2.9}$$

Remark. The sum over columns is calculated by summing up the quotients $l = 0, 1, \ldots, k-1$. Since these appear multiplied by $n = 4k$ in the row formula, the conservation of the sum is not seen modulo 4, but the analogue of (2.9) is true for columns as well.

Proof. Using Table 2, the modular values have row sum equal to 5 both over diagonal and non-diagonal terms. This can be seen noticing that $[b'-1] = [b] = 4$, and writing

TABLE 3. Basic square with modular values. Diagonal values are in bold. Sums are calculated on rows.

4	2	3	**1**	$\rightarrow$	5 and **5**
1	**3**	**2**	4	$\rightarrow$	5
1	**3**	**2**	4	$\rightarrow$	5
4	2	3	**1**	$\rightarrow$	5

The second assertion follows by calculating

$$S_i = \sum_{l=0}^{k-1} \sum_{4l<j\le 4(l+1)\,,(i,j)\notin D} \Big(4k(i-1) + 4l + [j]\Big)\,.$$

We continue with the observation that inside each basic square $\sum [j] = 5$, when $(i,j) \notin D$, obtaining

$$\begin{aligned} S_i &= 2\Big(\sum_{l=0}^{k-1} 4k(i-1) + 4l\Big) + 5k = 8k^2(i-1) + 4k(k-1) + 5k \\ &= 8k^2 i - 4k^2 + k = \frac{1}{4}\Big(2n^2 i - n^2 + n\Big)\,, \end{aligned}$$

which gives (2.9).

∎

3. The lower bound for $N(n)$

We recall that all squares discussed here are *normal*, meaning that a $m \times m$ square has entries a permutation of the first m^2 integers. A square obtained after either a row permutation or a column permutation of another square is said a *shuffle* of the original one. This is a much more restrictive type of permutation over the matrix entries as it does not allow mixing different columns or different rows.

Additionally, in a doubly even magic square $n \times n$, i.e. $n = 4k$, k integer, the basic squares (2.5) that have coordinates (q, l) with $q = l$ (main diagonal) or $q + l = k - 1$ (second diagonal) are said *diagonal basic squares.* All other basic squares are called *non-diagonal basic squares.* Figure 1 shows a highlighted non-diagonal square, when $k = 3$ and $(q, l) = (1, 0)$. For $k = 1, 2$ there are only diagonal basic squares. The number of non-diagonal basic squares is

$$k^2 - (k + k - 1) = (k - 1)^2 \,.$$

Definition 2. *A magic square of size $n = 4k$ is said* d-magic square *(from* diagonal magic square*) if its diagonal basic squares are exactly those of the square $A(i, j)$ defined in Lemma 1, eq. (2.7) with the property that its non-diagonal basic squares are simply magic (equal sum over rows and columns, not necessarily over diagonals).*

Of course, the square $A(i, j)$, proven to be magic in Theorem 1, serves as template. We notice that if we make shuffles consisting of operations like (i) an arbitrary permutation on the set of non-diagonal basic squares of a d-magic square; (ii) choosing any number among the non-diagonal basic squares and performing shuffles within each, we obtain a new d-magic square. In other words, d-magic squares are invariant to permutations and shuffles not affecting the diagonal basic squares. In addition, any d-magic square remains magic (but not necessarily d-magic) by permuting two rows or columns equidistant to the center of the square.

Proposition 3. *Let $A(i, j)$ be a doubly even d-magic square.*

1) The square obtained from $A(i, j)$ by either (i) a permutation of the non-diagonal basic squares of $A(i, j)$ among themselves, or (ii) a shuffle within any non-diagonal basic square, is again d-magic.

2) A square obtained from $A(i,j)$ by switching two rows i, i', respectively two columns j, j', that are equally distanced from the center, i.e. $i + i' == 2k+1$, respectively $j + j' = 2k+1$, is magic, but not necessarily d-magic.

Proof. 1) Since a d-magic square has the same diagonals as $A(i,j)$, we only have to prove that the sums over rows and columns are constant. This is true, since the basic squares have equal sum over rows and columns, independent on their coordinates (q,l). Successive applications of transformations of type (i) or (ii) do not change the sums over rows and columns. The bijectivity is preserved, as (i) and (ii) are bijections; a composition of such transformations is a bijection. This also proves that the new square is normal.

2) The sums over rows and columns are not changed by a switch. On the diagonals, it is sufficient to prove that

$$A(2k-i,i) + A(i,2k+1-i) = A(i,i) + A(2k+1-i,2k+1-i)\,.$$

All terms are diagonal in $\mathbf{A}$, so all have complementary values to the rank. We have to verify the same relation for $R(\cdot,\cdot)$, which means

$$R(2k-i,i) + R(i,2k+1-i) = 4k(2k-i) + i + 4k(i-1) + 2k + 1 - i$$

$$= 4k(i-1) + i + 4k(2k-i) + 2k + 1 - i = R(i,i) + R(2k+1-i,2k+1-i)\,.$$

$\square$

Theorem 2. *Let $n = 4k$, $k \geq 1$. The number of $n \times n$ magic squares, modulo rotations and symmetry, satisfies the lower bound*

$$N(n) \geq [(k-1)^2!]\,(4!)^{2(k-1)^2}\,(2^{2k}-1)^2\,. \tag{3.1}$$

Given the total number $N = (16k^2)!$ of unrestricted normal squares, the order of magnitude is

$$\lim_{k\to\infty} \frac{\ln(N(n))}{\ln N} = \frac{1}{16}\,. \tag{3.2}$$

Remark. The asymptotic value $1/16$ is not taking into account trivial transformations of the magic squares (symmetries, rotations). However, it says that the logarithmic scale is of the same order, which is generally not known in the literature.

Proof. To obtain (3.1), we multiply the following independent transformations of $\mathbf{A}$. The first two generate new, distinct d-magic squares:

• The number of permutations of $k^2 - (2k-1) = (k-1)^2$ non-diagonal basic squares;

• The number of permutations of rows within a basic square is $4! = 24$, and the same for columns. These transformations can be done independently and so repeated, again independently, for each of the $(k-1)^2$ off diagonal basic square.

The last transformation generates, from each d-magic square, a new magic square.

• We count the switches between m columns: Choose m columns from $j = 1, \dots, 2k$ and switch them pairwise with columns at equal distance from the center. This gives two distinct valid magic squares for each m, except $m = 0$, when it gives only one, as no switch occurs. We obtain a total of

$$\binom{2k}{0} + 2\sum_{m=1}^{2k} \binom{2k}{m} = 2^{2k} - 1\,.$$

This is squared since the same can be done for rows. Relation (3.2) is immediate from Stirling's formula.

□

4. A Maple program

As we pointed out in the remark following Theorem 1, the special magic square $\mathbf{A}$ is obtained in one step by using the function (2.7) and technically speaking does not require an algorithm. However, for convenience, we provide a Maple program.

Generates the square A from Lemma 1.

Given k

For j from 1 to k

do

For i from 1 to k

do

$a := 16 * (j-1)k + 4 * (i-1)\,;$

$b := a + 4 * k\,;$

$c := a + 8 * k\,;$

$d := a + 12 * k\,;$

$print(a+1, 16*k^2-a-1, 16*k^2-a-2, a+4)\,;$

$print(16*k^2-b, b+2, b+3, 16*k^2-b-3)\,;$

$print(16*k^2-c, c+2, c+3, 16*k^2-c-3)\,;$

$print(d+1, 16*k^2-d-1, 16*k^2-d-2, d+4)\,;$

$od\,;$

$od\,;$

$\square$

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