PATH COLLAPSE FOR AN INHOMOGENEOUS RANDOM WALK†

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Abstract. On an open interval we follow the paths of a Brownian motion which returns to a fixed point as soon as it reaches the boundary and restarts afresh indefinitely. We determine that two paths starting at different points either cannot collapse or they do so almost surely. The problem can be modelled as a spatially inhomogeneous random walk on a group and contrasts sharply with the higher dimensional case in that if two paths may collapse they do so almost surely.

Date: February 12, 2002.

1991 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 60J50, 35K15 .

Key words and phrases. Absorbing Brownian motion, inhomogeneous random walks, recurrence.

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† The first author was supported in part by the University of Utah and the University of Miami. The second author was supported in part by Northwestern University.

1. Introduction.

In [2] we defined a diffusion process obtained by piecing together a countable sequence of coupled Brownian motions killed at the boundary of an interval (a, b). Starting at a point $x \in (a, b)$ we follow a Brownian motion path until the boundary is hit. Then, as soon as either a or b have been reached, we *instantaneously* go to a fixed point of return in the interval, which we shall assume to be the origin, for convenience, and restart the motion until the next boundary hit. The indefinite continuation of the evolution after killing suggests the name of *Brownian motion with rebirth.* In this perspective, a particle is created at the origin each time its predecessor was absorbed at the boundary.

Paths starting at distinct points in the interval, driven by the same Brownian motion, stay parallel to each other between the boundary hits. A consequence is that they either never intersect, or they *coincide* after a finite time T_c, the so called *time of collapse.* Two paths are naturally coupled, but due to the rebirth dynamics, they are not simple translates one of the other, like in the case of an unbounded Brownian flow. It is easy to see that two paths may only meet, if they do at all, at the point of return. This paper proves that only paths starting at points x, y from (a, b) with $x - y \in Z_{a,b}$, the additive subgroup of R generated by the two endpoints, have a chance to join each other and in fact they *will* meet with probability one. In dimension $d > 1$ the situation is similar in that there exists a *grid set* $\mathcal{G}$ analogous to $Z_{a,b}$ but the positive probability of collapse is never equal to one (see [3]).

The case when b/a is rational is proven in [2]. It is much more difficult to see why the same conclusion is true if b/a is irrational. The state space of all possible differences between the paths becomes infinite. The problem can be converted into establishing recurrence properties for a spatially inhomogeneous random walk on the subgroup $Z_{a,b}$ of $(R, +)$ generated by the two endpoints.

We can look at the evolution of the paths indexed by the starting points $x \in (a, b)$ as a stochastic flow. Suppose we start with a smooth mass profile. It would be interesting to check if at some finite time the support of the random profile becomes countable, which is equivalent to studying whether the sequence of collapsing times for the different classes has finite limit. The answer should be negative. On the average over all paths, we know from [2] that the process is exponentially ergodic, henceforth the initial empirical distribution will converge in law to a smooth profile, more exactly the Green function for the Dirichlet Laplacian on the interval.

There is another view over the eventual collapse of two paths. We have seen (again in [2]) that the *rebirth* process evolves on a pair of circles with one common point (the origin), which is the figure eight (see [5]). Two points x, y in (a, b) are *commensurate* if their difference $x - y$ belongs to $Z_{a,b}$. Reaching zero from $x - y$ is equivalent to winding around the circle of perimeters $|a|$ and b, respectively, the exact number of times needed to "untangle" the distance between x and y. It is remarkable that Brownian motion will perform any combination of integer *winding numbers* on each circle with positive probability and will find itself at zero with probability one.

2. The results.

Let $\{W\}$ be a family of Brownian motions indexed by points $x \in R$, more precisely the collection of $W_x = (w_x(t,\omega), \{\mathcal{F}_t\}_{t\geq 0})$ such that $P(w_x(0,\omega) = x) = 1$ and let (a, b) be an open interval with $0 \in (a, b)$. We shall define inductively the increasing sequence of stopping times $\{\tau_n\}_{n\geq 0}$, together with the pair of adapted nondecreasing processes $\{N_x^a(t,\omega)\}_{t\geq 0}$ and $\{N_x^b(t,\omega)\}_{t\geq 0}$ and the process $\{z_x(t,\omega)\}_{t\geq 0}$, starting at $x \in (a, b)$. Let $\tau_0 = T_x = \inf\{t \,:\, w_x(t,\omega) \notin (a,b)\}$, while for $t \leq \tau_0$ we set $N_x^a(t,\omega) = 1_{\{a\}}(w_x(t,\omega))$, $N_x^b(t,\omega) = 1_{\{b\}}(w_x(t,\omega))$ and $z_x(t,\omega) = w_x(t,\omega) - aN_x^a(t,\omega) - bN_x^b(t,\omega)$. By induction on $n \in Z_+$

$$\tau_{n+1} = \inf\{t > \tau_n \,:\, w_x(t,\omega) - aN_x^a(\tau_n,\omega) - bN_x^b(\tau_n,\omega) \notin (a,b)\} \tag{2.1}$$

which enables us to define

$$\begin{aligned} N_x^a(t,\omega) &= N_x^a(\tau_n,\omega) + 1_{\{a\}}(z_x(t,\omega))\,, \\ N_x^b(t,\omega) &= N_x^b(\tau_n,\omega) + 1_{\{b\}}(z_x(t,\omega))\,, \end{aligned} \tag{2.2}$$

as well as

$$z_x(t,\omega) = w_x(t,\omega) - aN_x^a(t,\omega) - bN_x^b(t,\omega) \tag{2.3}$$

for $\tau_n < t \leq \tau_{n+1}$. We notice that $z_x(t,\omega) = 0$ for all $t = \tau_n$. The construction is well defined due to the following result from [2].

Proposition 1. *The sequence of stopping times $\tau_0 < \tau_1 < \ldots < \tau_n < \ldots$ are finite for all n and $\lim_{n\to\infty} \tau_n = \infty$, both almost surely. Also, the processes $N_x^a(t,\omega)$ and $N_x^b(t,\omega)$ defined for $t \geq 0$ have the properties*

(i) they are nondecreasing, piecewise constant, predictable and right-continuous

(ii) $P(N_x^a(t,\omega) < \infty) = P(N_x^b(t,\omega) < \infty) = 1$.

The law of the process $\{z_x(t,\omega)\}_{t\geq 0}$, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ will be denoted by Q_x and the family of processes $\{Q_x\}_{x\in(a,b)}$ will be denoted simply by $\{Q\}$.

We shall recall the notation $Z_{a,b} = \{k = mb + na \,:\, m,n \in Z\}$ the subgroup of $(R,+)$ generated by the numbers a and b. The case b/a rational is solved in [2]. If b/a is irrational the correspondence between $(m,n) \in Z \times Z$ and $k = mb + na$ is one-to-one and onto. Let $Z_{a,b}^* = Z_{a,b} \cap (a,b)$. For two points x and y in (a,b) we shall consider the coupled paths $\{z_x(t,\omega)\}_{t\geq 0}$ and $\{z_y(t,\omega)\}_{t\geq 0}$ defined by construction in (2.2) and (2.3).

Theorem 1. *Let $\{z_x(t,\omega)\}_{t\geq 0}$ and $\{z_y(t,\omega)\}_{t\geq 0}$ be two elements of the family of processes defined by $\{Q\}$ starting from x and y, two points in (a,b).*

(i) In case $x - y \notin Z_{a,b}^*$ the paths will never collapse.

(ii) In case $x - y \in Z_{a,b}^*$ the paths will collapse almost surely.

Corollary 1. *A finite family of paths starting at points included in $Z_{a,b}^*$ will collapse almost surely.*

3. Proof of Theorem 1

We shall assume that b/a is irrational, $b > 0 > a$ and without loss of generality that $b > |a|$. We recall that $Z_{a,b} = \{k = mb + na \,:\, m,n \in Z\}$ the subgroup of $(R,+)$ generated by the numbers a and b, $Z_{a,b}^* = Z_{a,b} \cap (a,b)$.

The idea of the proof can be outlined in four steps. First we note that the collapse of the original processes, starting from x and y respectively, can be considered as the recurrence of the Markov chain $\{Y_n\}_{n\geq 0}$ defined as the non-zero location of the original processes in $Z_{a,b}^*$ at each hitting times and described in (3.5). In the second step we reduce the Markov chain $\{Y_n\}_{n\geq 0}$ on $Z_{a,b}^*$ to the equivalent random walk $\{L_n\}_{n\geq 0}$ on $Z \setminus \{+1\}$ with transition probabilities (3.8)-(3.10). The third step establishes the general recurrence criterion of Proposition 3. Finally, the fourth step calculates explicitly the general term of (3.12) using the notion of descending cycle (given Definition 1) and shows that the series has the same nature as a series with terms equal to $|k|$, where $\{k\}$ is a certain enumeration of $Z^*_{a,b}$.

We start with the following lemma, essentially an exercise in [4], proved in [2].

Lemma 1. *Let* $T(a) = \inf\{t > 0 \,:\, w_x(t,\omega) \le a\}$*,* $T(b) = \inf\{t > 0 \,:\, w_x(t,\omega) \ge b\}$ *and* $T_x = \min\{T(a)\,,\, T(b)\}$*. Then,*

(i)

$$E_x[e^{-\alpha T(a)} 1_{T(a)<T(b)}] = \frac{\sinh \sqrt{2\alpha}(b-x)}{\sinh \sqrt{2\alpha}(b-a)}\,, \tag{3.1}$$

$$E_x[e^{-\alpha T(b)} 1_{T(b)<T(a)}] = \frac{\sinh \sqrt{2\alpha}(x-a)}{\sinh \sqrt{2\alpha}(b-a)} \tag{3.2}$$

and

(ii)

$$P_x(T(a) < T(b)) = \frac{b-x}{b-a} \quad , \quad P_x(T(b) < T(a)) = \frac{x-a}{b-a}\,. \tag{3.3}$$

(i) The difference between the two paths $z_x(t,\omega)$ and $z_y(t,\omega)$ will stay piecewise constant between successive hits to the boundary by either of them. If the two were to collapse, this could only happen at zero. From (2.3) we can see that

$$\begin{aligned} z_x(t,\omega) - z_y(t,\omega) &= x - y - bN^b_x(t) - aN^a_x(t) + bN^b_y(t) + aN^a_y(t) \\ &= x - y + a(N^a_y(t) - N^a_x(t)) + b(N^b_y(t) - N^b_x(t)) \end{aligned}$$

which proves (i).

(ii) We denote by T_c the time of collapse of two paths $z_x(t,\omega)$ and $z_y(t,\omega)$

$$T_c = \inf\{t \,:\, z_x(t,\omega) = 0 \,\text{and}\, z_y(t,\omega) = 0\} \tag{3.4}$$

with the convention that $T_c = \infty$ if the paths never collapse.

Since the initial distance between the piecewise parallel paths is $x - y < b - a$ we notice that after each time the boundary is hit the distance will change into a new value from $Z^*_{a,b}$. The boundary is hit by one of the paths at a time, otherwise their distance would have already been $b - a$ which is impossible. This implies that the union of increasing sequences of a.s. finite times of hitting the boundary $\{\tau^x_n\}$ and $\{\tau^y_n\}$, corresponding to x and y from the interval (a,b) can be rearranged in increasing order. The new increasing sequence of stopping times will be simply denoted by $\{\tau_n\}$.

At each such hitting time, one of the paths will go back to zero, while the other one will be in the set $Z^*_{a,b}$. It is important to recall that the two paths can only collapse at 0, since they evolve in parallel fashion between the times τ_n. In other words, $T_c \in \{\tau_n\}_{n\geq 0}$.

These considerations allow us to define a Markov chain $Y_n(\omega) = z_r(\tau_n,\omega)$, where r is either x or y in such a way that $z_r(\tau_n,\omega)$ is the point which is *not* situated at zero at time τ_n for $\tau_n < T_c$ and $Y_n(\omega)=0$ for $\tau_n \geq T_c$.

The chain has transition probability

$$P_{k,j} = \begin{cases} \frac{|a|-|k|}{b+|a|-|k|} & \text{if} \quad j = b-|k| \;\text{ and }\; k<0 \\ \frac{b}{b+|a|-|k|} & \text{if} \quad j = a+|k| \;\text{ and }\; k<0 \\ \frac{|a|}{b+|a|-|k|} & \text{if} \quad j = b-|k| \;\text{ and }\; k>0 \\ \frac{b-|k|}{b+|a|-|k|} & \text{if} \quad j = a+|k| \;\text{ and }\; k>0 \\ 1 & \text{if} \quad j=k=0 \\ 0 & \quad \text{otherwise} \end{cases} . \tag{3.5}$$

The probabilities are derived from the formulas (3.3) applied to the intervals associated to the strip determined by the two paths. We only have to prove that $P_{k_0}(Y_n = 0 \,:\, n<\infty) = 1$ for any initial state $k_0 \in Z^*_{a,b}$ of the Markov chain, since we know that the hitting times $\{\tau_n\}$ are finite almost surely.

Proposition 2. *If two paths of the process $\{Q\}$ starting at $x,y \in (a,b)$ never collapse then the chain $Y_n(\omega)$ generated by the pair x,y will never reach zero. More precisely,*

$$\Big\{\omega \,:\, T_c(\omega) = \infty\Big\} \subseteq \Big\{\omega \,:\, \min\{n>0 \,:\, Y_n(\omega)=0\} = \infty\Big\}.$$

Proof. Since the two paths never collapse, they will not collapse at the time of the first boundary hit. From the construction of the chain $Y_n(\omega)$ it follows that $Y_n \neq 0$ for any $n>0$. $\square$

Due to the uniqueness of the representation $k = mb+na$ with $m,n \in Z$ for any $k \in Z^*_{a,b}$ we can denote the two integer coefficients as functions of k: $m=m(k)$ and $n=n(k)$. Let $l = l(k) = m(k)+n(k)$ be the sum of the two integer coefficients.

The following lemma establishes the one-to-one correspondence between $Z^*_{a,b}$ and $Z\setminus\{+1\}$ in a constructive fashion. The proof is elementary and will be omitted.

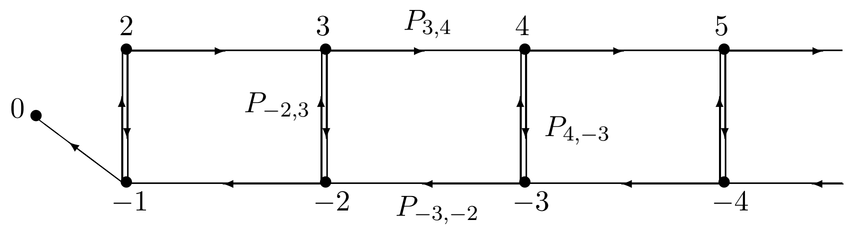

FIGURE 1. Ladder structure of $\{L_n\}_{n\geq 0}$

Lemma 2. *The function $l = l(k) = m(k) + n(k)$, defined for all $k \in Z^*_{a,b}$ is one-to-one and takes all the values in $Z \setminus \{+1\}$. Its inverse $k = k(l)$ is defined as follows. We construct the increasing sequence of positive integers $1 = q_0 < q_1 < q_2 < \dots q_n < q_{n+1} \dots$ by*

$$q_n = \max_{q \in Z_+} \{nb + qa > a\} \qquad n \geq 1\,. \tag{3.6}$$

For a given $l \in Z_+ \setminus \{+1\}$ we identify n such that

$$(q_n + n) + 1 \leq l < (q_{n+1} + n + 1) + 1 \qquad \text{then} \qquad k(l) = (n+1)b + (l - n - 1)a\,, \tag{3.7}$$

while for $l < 0$ we generate $l' = -l+1 \geq 2$ and $k(l) = b - k(l')$ if $k(l') > 0$ and $k(l) = a - k(l')$ if $k(l') < 0$. Finally the case for $l = 0$ is obvious, with $m = n = 0$.

Definition 1. *According to the construction of the function $k(l)$ described in (3.6) and (3.7), after every addition of the term $b > 0$ the sequence of actual values of k, corresponding to the numbers l which follow until we cross over into the negative side again, is decreasing by $|a|$ at each step. A sequence of such l for which k sees a decreasing row of $k(l)$ will be called a descending cycle.*

We define the derived Markov chain $\{L_n(\omega)\}_{n\geq 0}$ with state space $Z \setminus \{+1\}$ as $L_n(\omega) = l(Y_n(\omega))$. The chain is a random walk with transition probabilities

$$P(L_{n+1} = l' \,|\, L_n = l) = \begin{cases} \frac{|a| - |k(l)|}{b + |a| - |k(l)|} & \text{if} \quad l' = l + 1 \\ \frac{b}{b + |a| - |k(l)|} & \text{if} \quad l' = 1 - l \end{cases} \tag{3.8}$$

when $k(l) < 0$,

$$P(L_{n+1} = l' \,|\, L_n = l) = \begin{cases} \frac{b - |k(l)|}{b + |a| - |k(l)|} & \text{if} \quad l' = l + 1 \\ \frac{|a|}{b + |a| - |k(l)|} & \text{if} \quad l' = 1 - l \end{cases} \tag{3.9}$$

when $k(l) > 0$ and

$$(3.10) \qquad P(L_{n+1} = l' \,|\, L_n = l) = \begin{cases} 1 & \text{in case} \quad l' = l = 0 \\ 0 & \text{in all cases other than (3.9) and (3.10)} \end{cases} .$$

To help understanding the special structure of the random walk $\{L_n\}_{n\geq 0}$ we provide the Figure 1 representing $\{L_n\}_{n\geq 0}$ as a Markov chain on a ladder. The arrows connect all the possible positions which a given site (initial point of the arrow) can access in one step.

For any $l, l' \in Z \setminus \{+1\}$ we shall write briefly $p_{l,l'} = P(Y_{n+1} = l' \,|\, Y_n = l)$ and notice that for $l \neq 0$, $p_{l,l'} > 0$ if and only if $l' = l+1$ or $l' = -(l-1)$. The passage from l to $l+1$ shall be called the *short* jump, while the passage from l to $1-l$ shall be called the *long* jump.

Let $T_l = \inf\{n > 0 \,:\, Y_n = l\}$ the first hitting time of the state $l \in Z \setminus \{+1\}$. We denote by $P_u(d\omega)$ the conditional probability $P(\cdot \,|\, Y_0 = u)$, for $u \in Z \setminus \{+1\}$. Also, denote

$$(3.11) \qquad c_l = \frac{p_{l,l+1}}{p_{-l,-l+1}} \qquad d_l = \frac{p_{l,-l+1}}{p_{-l,-l+1}}$$

for $l \geq 2$ and set $c_1 = 1$ and $d_1 = (1 - \frac{|a|}{b})^{-1}$.

Proposition 3. *The probability that two paths of the process $\{Q\}$ never collapse is*

$$(3.12) \qquad P(T_c = \infty) = \lim_{l\to\infty} P_{-1}(T_{l+1} < T_0) = \Big(\frac{|a|}{b-|a|}\Big)\Big(\sum_{l=1}^{\infty} \frac{d_l}{\Pi_{j=1}^{l} c_j}\Big)^{-1} .$$

Remark: Note that the formula (3.12) is actually a general recurrence criterion for Markov chains on a ladder. The transition probabilities of our chain $\{L_n\}_{n\geq 0}$ are not homogeneous, but it is interesting to compare this case with the homogeneous Markov chains on a ladder, which reduce to *correlated random walks*, where the probability to move up or down depends on whether the last step was up or down. For a recent reference in this direction, see Böhm [1].

Proof. From (3.8), (3.9) and the construction of $k(l)$ given in Lemma 2, equation (3.7), we see that we can reach $l = -1$ and $l = 0$ with positive probability from any initial state of the chain $\{L_n\}$. Denote $P_u(T_l < T_0)$ the probability that starting from u we get to l before getting to zero. For $l \geq 2$,

$$P_{-1}(T_{l+1} < T_0) = P_{-1}(T_l < T_0) P_{back} \,,$$

where P_{back} is the probability of making a finite number of excursions in the interval $[-l+1, l]$ from l and coming back. It is to prove that once at $-l+1$, we cannot reach $\tilde{l} > l$ for the first time without *passing through* all l' such that $l \leq l' \leq \tilde{l}$.

Let $M \in Z_+$ and $-l+1 = l_0, l_1, \dots, l_M, l_{M+1} = \tilde{l}$ a realization of the process $\{Y_n\}$ with positive probability going from $-l+1$ to $\tilde{l}$, $\tilde{l} > l$, such that $l_j \neq \tilde{l}$ for any $j = 0, 1, \dots, M$. This implies that the transitions between successive states have positive probability (according to (3.8) and (3.9)). We want to show that $|l_j| < \tilde{l}$ for any $j = 0, 1, \dots, M$. If $j = 1$ we see that $l_1 = -l+2$ or $l_1 = l$, henceforth the statement is true. Let j' with $1 < j' \leq M$ the first rank in the sequence which would not satisfy the property.

1) If $l_{j'} > 0$ then $l_{j'} \geq \tilde{l}$ and then $l_{j'-1} = l_{j'} - 1$ or $l_{j'-1} = -(l_{j'} - 1)$. In both cases we shall have $|l_{j'} - 1| < \tilde{l}$ or simply $l_{j'} - 1 \leq \tilde{l} - 1$, hence $l_{j'} = \tilde{l}$. This is impossible.

2) If $l_{j'} < 0$ then $-l_{j'} \geq \tilde{l}$ and then $l_{j'-1} = l_{j'} - 1$ or $l_{j'-1} = -(l_{j'} - 1)$. In both cases we shall have $|l_{j'} - 1| = -l_{j'} + 1 \leq \tilde{l} - 1$ implying that $\tilde{l} + 1 \leq \tilde{l} - 1$, again impossible.

Now we want to show that if $\tilde{l} = l+1$ then the state l is reached by the sequence $-l+1 = l_0, l_1, \dots, l_M$. Since $l_{M+1} = l+1$ we have the possibility that $l_M = l$, in which case we are done, or that $l_M = -l$. The case $l_{M-1} = -l-1$ cannot happen since $|-l-1| = \tilde{l} = l+1$. Hence $l_{M-1} = l+1$, again impossible.

We have shown that if $l \geq 2$, the only possible ways to reach $l+1$ for the first time if we start from $-l+1$ are the union over $k = 0, 1, 2, \dots$ of realizations of the chain which reach l and then return to $-l+1$, reach l again from $-l+1$, repeat the excursion k times in total and then go directly to $l+1$ from l. By Markov property, this implies that

$$
\begin{aligned}
P_{back} &= p_{l,l+1}\Big(P(\text{no excursion}) + P(\text{one excursion})\dots\Big) \\
&= p_{l,l+1}\Big(1 + (p_{l,-l+1}P(l))^1 + (p_{l,-l+1}P(l))^2 \dots\Big) = \frac{p_{l,l+1}}{1 - p_{l,-l+1}P(l)}
\end{aligned}
$$

where $P(l) = P_{-l+1}(T_l < T_0)$. We also have

$$
P_{-1}(T_{l+1} < T_0) = P_{-1}(T_l < T_0)\frac{1 - p_{l,-l+1}}{1 - p_{l,-l+1}P(l)}\,. \tag{3.13}
$$

We want to calculate

$$
\begin{aligned}
P(l+1) &= p_{-l,-l+1}\Big(P(l)p_{l,l+1} + P(l)^2 p_{l,-l+1}p_{l,l+1} + \dots P(l)^j p_{l,-l+1}^{j-1}p_{l,l+1} + \dots\Big) + p_{-l,l+1} \\
&= p_{-l,-l+1}\frac{P(l)p_{l,l+1}}{1 - p_{l,-l+1}P(l)} + p_{-l,l+1}\,.
\end{aligned}
$$

Notice that $p_{-l,l+1} = 1 - p_{-l,-l+1}$ hence

$$1 - P(l+1) = p_{-l,-l+1} \frac{1 - P(l)}{1 - p_{l,-l+1} P(l)} \,, \tag{3.14}$$

that is

$$1 - P(l+1) = p_{-l,-l+1} \Big(\frac{1}{1 - P(l)} - p_{l,-l+1} \frac{P(l)}{1 - P(l)} \Big)^{-1}$$

or

$$(1 - P(l+1))^{-1} = (1 - P(l))^{-1} \frac{p_{l,l+1}}{p_{-l,-l+1}} + \frac{p_{l,-l+1}}{p_{-l,-l+1}} \,.$$

This gives $y_{l+1} = c_l y_l + d_l$ for the sequence $y_l = (1 - P(l))^{-1}$ with $c_l = p_{l,l+1}/p_{-l,-l+1}$ and $d_l = p_{l,-l+1}/p_{-l,-l+1}$. From (3.13) we know that

$$P_{-1}(T_{l+1} < T_0) = P_{-1}(T_2 < T_0) \Pi_{k=2}^{l} \frac{1 - p_{k,-k+1}}{1 - p_{k,-k+1} P(k)} \tag{3.15}$$

and $P_{-1}(T_2 < T_0) = P(2) = |a|/b$. We see that $\Pi_{k=2}^{l} (1 - p_{k,-k+1} P(k))^{-1}$ is

$$\Big(\Pi_{k=2}^{l} \frac{1 - P(k+1)}{1 - P(k)} \Big) \Pi_{k=2}^{l} p_{-k,-k+1}^{-1} = \frac{1 - P(l+1)}{1 - P(2)} \Pi_{k=2}^{l} p_{-k,-k+1}^{-1}$$

from equation (3.14). Combining the two we have

$$P_{-1}(T_{l+1} < T_0) = (\frac{|a|}{b})(1 - P(2))^{-1} \frac{1}{y_{l+1}} \Pi_{k=2}^{l} \frac{1 - p_{k,-k+1}}{p_{-k,-k+1}} \tag{3.16}$$

$$= \Big(\frac{|a|}{b - |a|} \Big) \frac{1}{y_{l+1}} \Pi_{k=2}^{l} \frac{p_{k,k+1}}{p_{-k,-k+1}} \,.$$

We recall that $\frac{p_{k,k+1}}{p_{-k,-k+1}} = c_k$ from (3.11) which permits to calculate y_l by recursion

$$y_{l+1} = d_l + d_{l-1} c_l + d_{l-2} c_{l-1} c_l + \dots d_{l-j} \Pi_{j'=l-j+1}^{l} c_{j'} + \dots + (\Pi_{j'=2}^{l} c_j) y_2 \tag{3.17}$$

while $y_2 = (1 - |a|/b)^{-1}$. It follows that

$$P_{-1}(T_{l+1} < T_0) = \Big(\frac{|a|}{b - |a|} \Big) \Big((1 - |a|/b)^{-1} + d_2 c_2^{-1} + \dots d_{l-1} (\Pi_{j=2}^{l-1} c_j)^{-1} \dots d_l (\Pi_{j=2}^{l} c_j)^{-1} \Big)^{-1}$$

which implies (3.12). $\square$

Proposition 4. *The series from equation (3.12) has the same nature as* $\sum_{l=2}^{\infty} |k(l)|$. *As a consequence, the paths will collapse with probability one if and only if the series* $\sum_{l=2}^{\infty} |k(l)|$ *diverges, which is equivalent to the recurrence of the inhomogeneous random walk* $\{L_n(\omega)\}$.

Proof. We want to write explicitly the general term of the series (3.12), namely

$$\frac{p_{l,-l+1}}{p_{-l,-l+1}}\Big(\frac{p_{-2,-1}}{p_{3,4}}\frac{p_{-3,-2}}{p_{4,5}}\cdots\frac{p_{-l+1,-l+2}}{p_{l,l+1}}\Big)\frac{p_{-l,-l+1}}{p_{2,3}}\,.$$

It can be simplified to

$$p_{2,3}^{-1}(1-p_{l,l+1})\Pi_{j=3}^{l}r_j \qquad \text{with} \qquad r_j=\frac{p_{-j+1,-j+2}}{p_{j,j+1}}=\frac{1-p_{-j+1,j}}{1-p_{j,-j+1}}\,.$$

The product $p_{2,3}^{-1}(1-p_{l,l+1})$ is bounded above and below by positive constants independent of $l \in Z_+$. The nature of the series is the same as for the product $\Pi_{j=3}^{l}r_j$. Lemma 3 and Proposition 5 complete the calculation of the general term. From (3.21) and (3.22) we can see that $\Pi_{j=3}^{l}r_j$ is equal to a constant times the factor $|k(l)|(b+|a|-|k(l)|)$, and for all l, $|a| \le b+|a|-|k(l)| \le b+|a|$. ☐

Lemma 3. *If* $k=k(l)<0$,

$$r_l=\frac{|k|(b+|a|-|k|)}{(b+|k|)(|a|-|k|)} \tag{3.18}$$

and if $k=k(l)>0$

$$r_l=\frac{|k|(b+|a|-|k|)}{(b-|k|)(|a|+|k|)}\,. \tag{3.19}$$

Proof. If $k=k(l)<0$, then

$$1-p_{l,-l+1}=\frac{|a|-|k|}{(b+|a|-|k|)}$$

since the chain must do the *long* jump to reach $-l+1$ which is equivalent to hitting a. It implies that $k'=k(-l+1)=a+|k|=a-k<0$. To reach l from $-l+1$ we need to perform once again a *long* jump, hence

$$1-p_{-l+1,l}=\frac{|a|-|k'|}{(b+|a|-|k'|)}=\frac{|k|}{b+|k|}$$

which implies (3.18). We derive (3.19) analogously. ☐

We recall the Definition 1 of a *descending cycle.*

For a given $n \in Z_+$, let's denote by

$$k=nb-q_n|a| \qquad k'=(n+1)b-q_n|a| \qquad k''=(n+1)b-(q-1)|a| \quad q\le q_{n+1} \tag{3.20}$$

the first negative element of the n^{th} descending cycle, the first positive element of the $(n+1)^{th}$ descending cycle and a given positive element of the $(n+1)^{th}$ descending cycle, respectively.

Proposition 5. *The factors r_k and r_j, for $k' \leq j \leq k''$ are*

$$r_k = \frac{|k|(b+|a|-|k|)}{(b+|k|)(|a|-|k|)} \quad \textit{and} \quad \Pi_{j=k'}^{k''} r_j = \frac{k''(b+|a|-k'')}{(k'+|a|)(b-k')}$$

while

$$\Pi_{j=k}^{k''} r_j = \Big(\frac{(n+1)b-(q-1)|a|}{nb-(q_n-1)|a|}\Big)\Big(\frac{-nb+q|a|}{-(n-1)b+q_n|a|}\Big)\,. \tag{3.21}$$

If $q = q_{n+1}$, $k'' = k''_{n+1}$ denotes the last positive element from the descending cycle $n+1$ and, with this notation,

$$\Pi_{j=k}^{k''} r_j = \frac{k''_{n+1}(b+|a|-k''_{n+1})}{k''_n(b+|a|-k''_n)}\,. \tag{3.22}$$

Lemma 4. *If a and b are two real numbers such that b/a is negative and irrational, then the set $Z^+_{a,b} = \{mb+na \,:\, m,n \in Z_+\}$ is dense on the real line.*

Proof. It is well known that the additive subgroup of R generated by a rational and an irrational, with no restriction on the coefficients, is dense in R. We shall make sure that we can take positive coefficients and prove the result on the way. Without loss of generality we assume $b > |a|$. Let $r_0 = b$ and $r_1 = |a|$. Set $m_1 = 1$ and n_1 will be the largest positive integer such that $r_0 - n_1 r_1 \in (0, |a|)$. Let $r_2 = r_0 - n_1 r_1$ and we proceed by defining r_{j+1} as the remainder of the integer division of r_{i-1} by r_i, for any $i \geq 2$. The coefficients m_j and n_j of b and a, respectively, are derived from the recursion by noticing that all $\{r_j\}_{j\geq 1} \subseteq Z_{a,b}$. More explicitly, if $r_j = m_j b + n_j a$, $r_{j-1} = m_{j-1}b + n_{j-1}a$ then $r_{j+1} = (m_{j-1}b + n_{j-1}a) - q_j(m_j b + n_j a)$ implying $m_{j+1} = m_{j-1} - q_j m_j$ and $n_{j+1} = n_{j-1} - q_j n_j$, where $q_j = [r_{j+1}/r_j]$. The sequence $\{r_j\}$ is positive and decreasing and is bounded by $|a|$. This implies that the pair (m_j, n_j) can only be in the first quadrant or in the third quadrant. The first pair is in the first quadrant. It is easy to check that the second pair will be in the third quadrant. This implies, according to the recursion, that all the odd terms of the sequence of pairs are in the first quadrant and, moreover, the positive sequences $\{m_{2j+1}\}$ and $\{n_{2j+1}\}$ are increasing. A similar argument makes the negative even sequences decreasing. We want to show that we can find a sequence of positive numbers in $Z^+_{a.b}$ which converges to zero. If the decreasing sequence $\{r_j\}$ converges to zero we are done. If it converges to a positive constant, we use the monotonicity of the sequences of coefficients and take differences of terms approaching that positive limit. The rest of the argument is standard. □

We are ready to conclude the proof of Theorem 1.

Proof. The problem is reduced by Proposition 2 to showing that $Y_n(\omega)$ reaches zero in finite time almost surely. Lemma 2 converts the problem into the question of recurrence for the random walk $\{L_n\}$. Proposition 3 gives the recurrence condition (3.12) and Proposition 4 simplifies the problem to the study of the nature of the series $\sum_{l=2}^{\infty} |k(l)|$. Lemma 4 says that $k = mb + na$ is dense everywhere, in particular in (a, b). But any point in $Z_{a,b} \cap (a, b)$ is a valid $k = k(l)$ for our summation. The terms of the series cannot converge to zero. □

Acknowledgements. The authors would like to thank the anonymous referee for his careful reading and useful suggestions as well as the recent reference [1].

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