DIFFUSIVE SCALING LIMITS OF MUTUALLY INTERACTING PARTICLE SYSTEMS*

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Abstract. We prove the diffusive scaling limits of some interacting particle systems in random dynamical environments. The limits are identified as nonlinear parabolic systems, with coefficients given by equilibrium variational problems. Three related models are studied that correspond to different environments. All the models are of nongradient type, and one is nonreversible. The proofs involve techniques of entropy production estimates, the nongradient method and asymmetric tools, in particular a proof of the strong sector condition.

Key words. hydrodynamic limit, nongradient system

AMS subject classifications. Primary 60F10, 60J75, 60K35; secondary 82C26

1. Introduction. In this article we study the diffusive scaling limits of three models of random walks with simple exclusion on a multidimensional lattice subject to rapidly fluctuating jump rates determined by another system of similar walks. In each of the three models there are two types of particles which we denote by η and ξ. Each particle, independently of the others, waits a random, exponentially distributed length of time and then attempts to jump to a neighbouring site. The interaction enters in two ways. If a particle attempts to jump to a site already occupied by a particle of the same type, the jump is suppressed. This hard core interaction between particles of the same type is called simple exclusion. The second interaction is through a speed change. The expected waiting time for a given particle depends on the local configuration of particles of the *other* type. The three models differ in the exact form of this *speed change*. It is more convenient to think of this in terms of the inverse of the expected waiting time, or the rate of jumping. In *Model 1*, the rate for an η particle to jump from a site x to a neighbouring site y is $\gamma_1 + \xi_x + \xi_y$ where ξ_x and ξ_y are the numbers of ξ particles at x and y, while the ξ particles all jump at rate γ_2. Here γ_1 and γ_2 are two positive numbers. In other words, the ξ particles perform the symmetric simple exclusion process and the η particles perform a "simple exclusion in a symmetric simple exclusion environment". In *Model 2*, the rate for an η particle to jump from x to the nearest neighbour y is $\gamma_1 + \frac{1}{2}(\xi_x + \xi_y)$, and the rate for a ξ particle to jump is $\gamma_2 + (1 - \frac{1}{2}(\eta_x + \eta_y))$. Hence the two processes dynamically drive each other through the interdependence of the jump rates. *Model 1* and *2* are in some sense warm-ups for *Model 3* in which an η particle jumps from x to nearest neighbour y at rate $\gamma_1 + \xi_x$ and a ξ particle does the same at rate $\gamma_2 + 1 - \eta_x$.

Such models can be thought of as microscopic pursuit and evasion predator-prey models. In *Models 2* and *3*, for example, the η particles represent prey and the ξ particles represent predators. The predators jump fast until they find a prey and then slow down, while the prey jump slowly until they see a predator, at which time they speed up to run away. Very little work has been done on pursuit and evasion

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systems, as opposed to birth and death predator prey systems, where birth and death rates are functions of the other species. Both types of predator prey systems are usually modelled by continuum equations – systems of partial differential equations – though particle systems are more realistic. There is quite a bit of work in the probability literature on systems of birth and death processes with random walks [1] and on speed change models with a scalar conservation law, but as far as we know, no previous results on speed change (i.e. pursuit and evasion) systems. Parabolic systems have much more interesting behaviour than scalar parabolic equations [9], and are more relevant to biology.

With respect to modelling in biology using parabolic systems a question arises as to whether one should use divergence or non-divergence form. Our models shed some light on this issue: Microscopically they are in (discrete) non-divergence form, but macroscopically the bulk equations take divergence form.

Another motivation for these models comes from the theory of homogenisation. Let us recall two well known examples.

1. To each bond $x, x+e$ of the multidimensional integer lattice $\mathbf{Z}^d$ is associated an independent random variable $a(x, x+e) \geq \delta > 0$. We let $x(t)$ be a continuous time random walk on $\mathbf{Z}^d$ with generator $Lf(x) = \sum_{|e|=1} a(x, x+e)(f(x+e) - f(x))$ and we ask for the asymptotic behaviour of $x_\epsilon(t) = \epsilon x(\epsilon^{-2}t)$. This is the reversible case, in which the rate of jumping from x to nearest neighbour $x+e$ is the same as the rate of jumping back, that is, $a(x, x+e) = a(x+e, x)$. The uniform measure is an invariant and reversible (unnormalized) measure and by standard methods of homogenisation one finds that the limiting process is a Brownian motion with covariance $\ell \cdot \bar{a}\ell = \inf_f \sum_e E[a(0,e)(e \cdot \ell + \tau_e f - f)^2]$, the infimum ranging over stationary processes.

2. To each site x of $\mathbf{Z}^d$ is associated an independent random variable $a(x) \geq \delta > 0$ and $y(t)$ is a continuous time random walk with generator $Lf(x) = \sum_{|e|=1} a(x)(f(x+e) - f(x))$. This is the non-reversible case, in which the rate of jumping from x, $a(x)$ depends on x alone. Here the (unnormalized) invariant measure gives mass $a^{-1}(x)/E[a^{-1}]$ to site x. The rescaled process $y_\epsilon(t) = \epsilon y(\epsilon^{-2}t)$ again converges to Brownian motion. The variance of $y_\epsilon(t)$ can be computed explicitly and an application of the ergodic theorem tells us that the asymptotic variance in this case is $E[a^{-1}(0)]^{-1}$. Now in each of the two models, suppose that we replaced the static random field by one varying in time. Similar questions can be answered in the reversible case of example 1, but in the non-reversible case of example 2 little is known. The problem is that in the second case there is no invariant measure.

Interacting systems with two types of particles provide examples of dynamic random environments which can be analyzed. In particular, the speed change in our *Model 3* is of the type of the second example.

The main results of the article are scaling limits for the diffusively rescaled density fields in the three models. The limits are coupled parabolic systems, with diffusion matrices which can be obtained from certain variational problems. We use the non-gradient method ([10], [13]), and its adaptation to the mean zero non-reversible setting [15]. The main idea is to consider the models as bounded perturbations of symmetric simple exclusion and for this we have to assume $\gamma_1, \gamma_2 > 0$. The work [15] is unpublished. The only other case we know treating the non-reversible, non-gradient case using the strong-sector estimate to bound the asymmetry in terms of the symmetry is in [6] where a very interesting model related to vortex flow ([7]) is studied. The method used is the relative entropy method, which requires certain a priori regularity for solutions of the hydrodynamic equation. However, to prove this one needs first some regularity of the diffusion coefficient as a function of the density, and this has not been obtained for the model in [6] at the present time. Hence the proof is not complete. For parabolic systems as considered in the present article, even some regularity of the coefficients would not help, as the needed regularity results for solutions are not available. Hence one is forced to use the method of [10],[13],[15]. Because of the lack of references we have provided a sketch of the argument, refering to the existing literature whenever possible.

2. The models. In each of the three models there are two types of particles which we call η particles and ξ particles. The particles perform symmetric nearest neighbour random walks on the multidimensional integer lattice $\mathbf{Z}^d$, with exclusion within their type. In other words, each particle waits an exponential amount of time, then attempts a jump to a neighbouring site chosen with equal probabilities. The jump is only executed if the target site is free of a particle of the same type. If we start with at most one particle of each type at each site, it will stay so forever, so the state space of all three models is $X = (\{0,1\} \times \{0,1\})^{\mathbf{Z}^d}$. Configurations will be denoted (η, ξ), and for each $x \in \mathbf{Z}^d$, $\eta_x \in \{0,1\}$ and $\xi_x \in \{0,1\}$ denote the presence or absence of a particle at that site.

The interaction is through the expected length of the holding time, which will depend on the local environment. Let us introduce some notation. The operations $\eta \mapsto \eta^{x,y}$ and $\xi \mapsto \xi^{x,y}$ exchange the occupation numbers at the two sites x and y. More precisely, they are defined as $\eta^{x,y}_x = \eta_y$, $\eta^{x,y}_y = \eta_x$ and $\eta^{x,y}_z = \eta_z$ otherwise, and analogously for ξ. It is convenient to use the η and ξ lattice gradients acting on functions on X, which are given by

$$\nabla^\eta_{x,y} f(\eta,\xi) = f(\eta^{x,y},\xi) - f(\eta,\xi), \qquad \nabla^\xi_{x,y} f(\eta,\xi) = f(\eta,\xi^{x,y}) - f(\eta,\xi). \tag{2.1}$$

We can now describe the three models (from easiest to hardest).
Model 1. The ξ particles attempt jumps to each neighbour at rate γ_2. An η particle at x attempts to jump to nearest neighbour y at rate $\gamma_1 + \frac{\xi_x+\xi_y}{2}$. The infinitesimal generator is

$$L^{(1)} f = \sum_{x\sim y} (\gamma_1 + \frac{\xi_x+\xi_y}{2}) \nabla^\eta_{x,y} f + \gamma_2 \nabla^\xi_{x,y} f. \tag{2.2}$$

The sum is over ordered nearest neighbour pairs $x \sim y$.
Model 2. A ξ particle at x attempts to jump to nearest neighbour y at rate $\gamma_2 + 1 - \frac{\eta_x+\eta_y}{2}$. An η particle at x attempts to jump to nearest neighbour y at rate $\gamma_1 + \frac{\xi_x+\xi_y}{2}$. The infinitesimal generator is

$$L^{(2)} f = \sum_{x\sim y} (\gamma_1 + \frac{\xi_x+\xi_y}{2}) \nabla^\eta_{x,y} f + (\gamma_2 + 1 - \frac{\eta_x+\eta_y}{2}) \nabla^\xi_{x,y} f. \tag{2.3}$$

Model 3. A ξ particle at x attempts to jump to each nearest neighbour site at rate $\gamma_2 + 1 - \eta_x$. An η particle at x attempts to jump to each nearest neighbour site at rate $\gamma_1 + \xi_x$. The infinitesimal generator is

$$L^{(3)} f = \sum_{x\sim y} (\gamma_1 + \xi_x \eta_x (1-\eta_y)) \nabla^\eta_{x,y} f + (\gamma_2 + (1-\eta_x)\xi_x(1-\xi_y)) \nabla^\xi_{x,y} f. \tag{2.4}$$

We shall use the generic notation L for the infinitesimal generator of the three models, unless we need to differentiate between them (especially in Section 5.)

Models 1 and *2* are reversible with respect to the family of product [Bernoulli] measures $\pi_{u,v} = (m_u \times m_v)^{\otimes \mathbf{Z}_N^d}$, $u, v \in [0,1]$ where $m_u(1) = u$ and $m_u(0) = 1-u$. The corresponding Dirichlet forms

$$\mathcal{D}_{u,v}(f) = -E^{\pi_{u,v}}[fLf]$$

are given by

$$\frac{1}{2}\sum_{x\sim y} E^{\pi_{u,v}}[(\gamma_1 + \frac{\xi_x+\xi_y}{2})(\nabla^{\eta}_{x,y}f)^2 + \gamma_2(\nabla^{\xi}_{x,y}f)^2] \qquad (Model\ 1)$$

and

$$\frac{1}{2}\sum_{x\sim y} E^{\pi_{u,v}}[(\gamma_1 + \frac{\xi_x+\xi_y}{2})(\nabla^{\eta}_{x,y}f)^2 + (\gamma_2 + 1 - \frac{\eta_x+\eta_y}{2})(\nabla^{\xi}_{x,y}f)^2] \ \ (Models\ 2\,,3).$$

We learned about *Model 3* from Donatis Surgailis who also indicated the following key fact, which is easy to check.

Proposition 2.1. (**Surgailis**) *The product measures $\pi_{u,v}$, $u, v \in [0,1]$ are invariant for $L^{(3)}$ in Model 3.*

However $L^{(3)}$ is *not* reversible with respect to the $\pi_{u,v}$. The generator of *Model 2* is nothing but the symmetric part of the generator in *Model 3*.

One could of course consider much more general speed change models, where the holding time of a particle is a general function of the local configuration. The basic problem then becomes one of finding the set of invariant measures, which is extremely hard in general.

On the other hand one can start with a family of invariant measures, and construct appropriate Dirichlet forms. This produces dynamics for which the measures are guaranteed to be reversible and invariant. However, dynamics for which we can determine a nice family of measures which are invariant but *not* reversible are rare, a fact underlying the importance of *Model 3*.

For each of the three models one can check that for $\gamma_1, \gamma_2 > 0$ the two particle densities are the only conserved quantities. A consequence is that, on a box of side length ϵ^{-1} with periodic or reflecting boundary conditions, once we fix the number of η and the number of ξ particles, then the continuous time Markov chain $(\eta(\cdot), \xi(\cdot))$ is ergodic and the distribution converges to the uniform distribution on configurations with those numbers of particles.

We also have the obvious lower bound

$$\mathcal{D}(f) \geq \gamma \mathcal{D}^{(0)}(f)\,, \quad \gamma \leq \gamma_1 \wedge \gamma_2$$

for each of the three Dirichlet forms in terms of the Dirichlet form $\mathcal{D}^{(0)}$ of two independent copies of the symmetric simple exclusion process,

$$\mathcal{D}^{(0)}_{u,v}(f) = \frac{1}{2}\sum_{x\sim y} E^{\pi_{u,v}}[(\nabla^{\eta}_{x,y}f)^2 + (\nabla^{\xi}_{x,y}f)^2].$$

We can also rewrite the Dirichlet form as $\mathcal{D}^{(2)}(f) = \sum_{x\sim y}\mathcal{D}_{x,y}(f)$ where

$$D_{x,y}(f) = \frac{1}{2}E[(\gamma_1 + \frac{\xi_x+\xi_y}{2})(\nabla^{\eta}_{x,y}f)^2 + (\gamma_2 + 1 - \frac{\eta_x+\eta_y}{2})(\nabla^{\xi}_{x,y}f)^2]\,. \tag{2.5}$$

Since it is well known that on a box of side length ϵ^{-1}, the spectral gap of symmetric simple exclusions is bounded below by some constant multiple of ϵ^2, we immediately obtain for our three Dirichlet forms the Poincaré inequalities $Var(f) \leq C\epsilon^{-2}\mathcal{D}(f)$ on boxes of side length ϵ^{-1}, uniformly in the density. For fixed $\gamma_1, \gamma_2 > 0$ we will prove diffusive scaling limits for the joint empirical densities of particles. The limits are coupled systems of parabolic partial differential equations. The diffusion matrices for limits of such systems cannot in general be expected to be elementary functions of the densities. However we can obtain variational formulae for the diffusion matrices and these can be used to show some structure of the equations.

This is made precise by the *hydrodynamic scaling limit.* To avoid technicalities we work on the torus $\mathbf{T}^d$ instead of $\mathbf{R}^d$, though it is known how to deal with infinite systems [3]. We are given functions $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$ of $\mathbf{x} \in \mathbf{T}^d$ taking values in $[0,1]$. The small scaling parameter $\epsilon > 0$ represents the separation between macroscopic and microscopic pictures. To keep ourselves on the torus we assume that ϵ^{-1} is an integer. Macroscopic space and time variables $\mathbf{x} \in \mathbf{T}^d$ and $\mathbf{t} \geq 0$ are related to microscopic variables $x \in \mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d$ and $t \geq 0$ by

$$x = \lfloor \epsilon^{-1}\mathbf{x} \rfloor, \qquad t = \epsilon^{-2}\mathbf{t}.$$

We assume that the initial distribution μ_0^ϵ of the process running on $\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d$ is such that the following law of large numbers holds: *As* $\epsilon \to 0$*, in* μ_0^ϵ*-probability, the empirical density fields* $(\eta_{\lfloor \epsilon^{-1}\mathbf{x} \rfloor}, \xi_{\lfloor \epsilon^{-1}\mathbf{x} \rfloor})$ *converge weakly to* $(u_0(\mathbf{x}), v_0(\mathbf{x}))$ where $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$ are some nice functions on the torus. Consider $\hat{P}_\epsilon$, the distributions of

$$\mathbf{t} \longrightarrow (\eta_{\lfloor \epsilon^{-1}\mathbf{x} \rfloor}(\epsilon^{-2}\mathbf{t})d\mathbf{x}, \xi_{\lfloor \epsilon^{-1}\mathbf{x}(\epsilon^{-2}\mathbf{t}) \rfloor})d\mathbf{x}) \,, \tag{2.6}$$

seen as measures on $D([0,\infty); M(\mathbf{T}^d) \times M(\mathbf{T}^d))$, the Skorohod space of left-limit and right-continuous maps from $[0,\infty)$ into $M(\mathbf{T}^d) \times M(\mathbf{T}^d))$, the space of pairs of probability measures with the topology of weak convergence, indexed by the scaling parameter $\epsilon > 0$.

We shall denote by $L_l^{(i)}$, for $i = 0,1,2,3$, the restrictions of the infinitesimal generators of the processes confined to a box Λ_l of size $l \in Z_+$ centered at the origin. For fixed numbers of particles m and n, we denote by $P^{n,m,l}$ the product Bernoulli measure π_ϱ conditional on the hyperplane $\sum_{x\in\Lambda_l} \zeta_x = (m,n) = \lfloor (2l+1)^d \varrho \rfloor$, where $\zeta_x = (\xi_x, \eta_x)$, $\varrho = (u,v) \in [0,1] \times [0,1]$.

Let $\mathcal{F}$ be the class of local functions f on the state space $\{0,1\}^{Z^d} \times \{0,1\}^{Z^d}$ satisfying the bound

$$E^{n,m,l}[fh] \leq C \sum_{|x-y|=1\,,\, |x|,|y| \leq l'} \mathcal{D}_{x,y}^{(0)}(h) \,, \tag{2.7}$$

with a constant $C > 0$, uniformly over boxes of size $l \in Z_+$ for functions with finite support h (*local functions*). The integer $l' \leq l$ stands for the largest integer such that the box $\Lambda_{l'} + supp(f)$ be included in Λ_l. In particular, mean-zero local functions like the gradients $\nabla\zeta$, the currents W_{0,e_i} and the fluctuations Lg for g local satisfy the property.

We shall see in equation (3.25), Section 3 that, for any $\varrho = (u,v) \in [0,1] \times [0,1]$ and for $i = 0,1,2,3$ we can define the equivalent semi-norms

$$\langle f\,,\, f \rangle_{-1,\varrho}^{(i)} = \lim_{(n(2l+1)^{-d}\,,\, m(2l+1)^{-d}) \to \varrho} (2l)^{-d} E^{n,m,l}[\sum_{x \leq l'} \tau_x f\,,\, (-L_l^{(i)})^{-1}(\tau_x f)] \,. \tag{2.8}$$

If $\mathcal{N}$ is the null space corresponding to $i=0$, we denote the completion of the quotient space $\mathcal{F}/\mathcal{N}$ by $\mathcal{H}^{(i)}_{-1,\varrho}$, a Hilbert space for the symmetric cases $i=0$ and $i=2$. The null space is the same for all i due to the fact that $L^{(3)}_{sym} = L^{(2)}$ and equivalence of the norms warranted by the *strong sector condition* described in Lemma 2.5.

We need the compressibility matrix

$$\chi(\varrho) = \chi(u,v) = \begin{pmatrix} u(1-u)I_d & 0 \\ 0 & v(1-v)I_d \end{pmatrix}, \tag{2.9}$$

where I_d is the d-dimensional identity matrix.

THEOREM 2.2. *(Model 1) Assume $\gamma_1, \gamma_2 > 0$. Then $\hat{P}_\epsilon \Rightarrow \delta_{u,v}$, the Dirac mass on the trajectory $(u(\mathbf{t},\mathbf{x}), v(\mathbf{t},\mathbf{x}))d\mathbf{x}$ where (u,v) is the unique weak solution of*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \nabla \begin{pmatrix} e(u,v) & 0 \\ 0 & \gamma_2 I_d \end{pmatrix} \nabla \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{x} \in \mathbf{T}^d,\ t \geq 0 \tag{2.10}$$

with $(u(0,\mathbf{x}), v(0,\mathbf{x})) = (u_0(\mathbf{x}), v_0(\mathbf{x}))$, satisfying $\int_0^T \int_{\mathbf{T}^d} [|\nabla u|^2 + |\nabla v|^2] d\mathbf{x} d\mathbf{t} < \infty$. The matrix $e(u,v)$ is continuous in u and v and is given by the variational formula, for any $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbf{R}^d$:

$$\mathbf{r} e(u,v) \mathbf{r}' =$$

$$\frac{1}{2u(1-u)} \inf_{g \in \mathcal{F}} E^{\pi_{u,v}} [\sum_{i=1}^d (\gamma_1 + \frac{\xi_0 + \xi_{e_i}}{2})(r_i(\eta_{e_i} - \eta_0) - \nabla^\eta_{0,e_i}\Omega_g)^2 + \gamma_2 (\nabla^\xi_{0,e_i}\Omega_g)^2].$$

Here $\Omega_g = \sum_{x \in \mathbf{Z}^d} \tau_x g$ with τ_x the shift operator.

THEOREM 2.3. *(Model 2) Assume $\gamma_1, \gamma_2 > 0$. Then $\hat{P}_\epsilon$ are tight and any limit point is supported on the set of weak solutions of*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \nabla D^{(2)}(u,v) \nabla \begin{pmatrix} u \\ v \end{pmatrix},$$

where $\mathbf{x} \in \mathbf{T}^d,\ t \geq 0$ with $(u(0,\mathbf{x}), v(0,\mathbf{x})) = (u_0(\mathbf{x}), v_0(\mathbf{x}))$ satisfying $\int_0^T \int_{\mathbf{T}^d} [|\nabla u|^2 + |\nabla v|^2] d\mathbf{x} d\mathbf{t} < \infty$. The diffusion matrix $D^{(2)}(u,v)$ is continuous in u and v, and is given by

$$\begin{aligned} D^{(2)}(u,v) = &\begin{pmatrix} (\gamma_1 + v) I_d & 0 \\ 0 & (\gamma_2 + 1 - u) I_d \end{pmatrix} \\ &+ \frac{1}{4}\left[B - (u(1-u)(\gamma_1+v) + v(1-v)(\gamma_2+1-u)) I_d\right] \chi^{-1}(u,v) \begin{pmatrix} I_d & I_d \\ I_d & I_d \end{pmatrix}, \end{aligned} \tag{2.11}$$

where for any $\mathbf{r} = (r_1, \cdots, r_d)$,

$$\begin{aligned} \mathbf{r} B \mathbf{r}' = \frac{1}{2} \inf_{g \in \mathcal{F}} E^{\pi_{u,v}} [&\sum_{i=1}^d (\gamma_1 + \frac{\xi_0 + \xi_{e_i}}{2})(r_i(\eta_{e_i} - \eta_0) - \nabla^\eta_{0,e_i}\Omega_g)^2 \\ &+ (\gamma_2 + 1 - \frac{\eta_0 + \eta_{e_i}}{2})(r_i(\xi_{e_i} - \xi_0) - \nabla^\xi_{0,e_i}\Omega_g)^2]. \end{aligned}$$

Before stating the third hydrodynamic limit, we need to recall that on hyperplanes $\sum\zeta = (n,m)$ for fixed nonnegative integers m and n, the generators $L_l^{(i)}$ are invertible.

THEOREM 2.4. *(Model 3) Assume $\gamma_1, \gamma_2 > 0$. Then $\hat{P}_\epsilon$ are tight and any limit point is supported on the set of weak solutions of*

$$\frac{\partial}{\partial t}\left(\begin{array}{c} u \\ v \end{array}\right) = \nabla D^{(3)}(u,v)\nabla\left(\begin{array}{c} u \\ v \end{array}\right), \qquad \mathbf{x}\in\mathbf{T}^d,\ t\ge 0 \tag{2.12}$$

with $(u(0,\mathbf{x}), v(0,\mathbf{x})) = (u_0(\mathbf{x}), v_0(\mathbf{x}))$ satisfying $\int_0^T\int_{\mathbf{T}^d}[|\nabla u|^2 + |\nabla v|^2]d\mathbf{x}d\mathbf{t} < \infty$ where $D^{(3)}(u,v)$ is a $2d\times 2d$ matrix valued function continuous in u and v given by

$$\Big(D^{(3)}(u,v)\Big)^{-1}\chi(\varrho) = \lim_{(n(2l+1)^{-d},\, m(2l+1)^{-d})\to(u,v)} (2l)^{-d}E^{n,m,l}[\sum_{x\le l'}\tau_x\nabla\zeta\ (-L_l^{(3)})^{-1}(\tau_x\nabla\zeta)]\,. \tag{2.13}$$

Furthermore, there exist $2d\times 2d$ matrices Q and V, with V symmetric, such that

$$D^{(3)}(u,v)Q = D^{(2)}(u,v)V \tag{2.14}$$

and $Q_{sym} < V$ in the sense of quadratic forms.

A comment related to the asymmetric diffusion coefficient $D^{(3)}(u,v)$ is included at the end of Section 5.

Remark 1. (on uniqueness). Uniqueness of the hydrodynamic equations for Models 2 and 3 is a hard problem and we have not pursued it here.

Remark 2. (on the degenerate case). If $\gamma_1 = \gamma_2 = 0$, then *Model 2* and *3* are no longer ergodic. For example, any configuration in which every site where there is a ξ particle is also occupied by an η particle and there are η but no ξ particles in all nearest neighbour[ing] sites is an absorbing state for *Model 3.* We can construct such configurations which have macroscopic profiles, and since every state in our systems has bounded specific entropy, it follows that the diffusion coefficients simply vanish. It is an interesting question whether the scaling limit could hold after removing some bad configurations from the space, but we do not know how to answer this. On the other hand if only one of γ_1 and γ_2 vanish the situation is not so bad. One can check, for example in *Model 2*, that the spectral gap on a box of side length ϵ^{-1} is correct, say if $\gamma_1 = 0$ but $\gamma_2 > 0$, but with a factor $Cv\epsilon^2$ where v is the density of ξ particles and with a factor $C(1-u)\epsilon^2$ if $\gamma_1 > 0$ but $\gamma_2 = 0$. Analogous results hold for *Model 1.* In a similar way, one can check that the diffusion matrices of *Model 2* and *3* dominate

$$\left(\begin{array}{cc} C(\gamma_2)vI_d & 0 \\ 0 & \gamma_2 I_d \end{array}\right) \tag{2.15}$$

if $\gamma_1 = 0$ and

$$\left(\begin{array}{cc} \gamma_1 I_d & 0 \\ 0 & C(\gamma_1)(1-u)I_d \end{array}\right) \tag{2.16}$$

if $\gamma_2 = 0$ for some $C(d) > 0$ for $d > 0$. For the rest of the article we concentrate exclusively on the case

$$\gamma_1, \gamma_2 \ge \gamma > 0.$$

Remark 3. (on the birth-death model). Let $a(\eta,\xi), b(\eta,\xi)$ be positive local functions, and

$$L_{reaction}f(\eta,\xi) = \sum_x a(\tau_x\eta,\tau_x\xi)(f(\eta^x,\xi) - b(\eta,\xi)) + d(\tau_x\eta,\tau_x\xi)(f(\eta,\xi^x) - f(\eta,\xi))$$

with $\eta^x_x = 1 - \eta_x$ and $\eta^x_y = \eta_y$ otherwise, and analogously for ξ. Let $L^{(i)}_\epsilon = \epsilon^{-2}L^{(i)} + L_{reaction}$, $i = 1,2,3$. The hydrodynamic limit is a nonlinear reaction-diffusion equation of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \nabla D^{(i)}(u,v)\nabla \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F(u,v) \\ G(u,v) \end{pmatrix},$$

where $F(u,v) = E^{\pi_{u,v}}[a(\eta,\xi)(1-2\eta_0)]$, $G(u,v) = E^{\pi_{u,v}}[b(\eta,\xi)(1-2\xi_0)]$. See [5] for details.

The method of proof for *Models 1* and *2* which are reversible, non-gradient systems is by now rather standard (in the sense that they have been worked out for the Ginzburg-Landau model [13] and the symmetric simple exclusion process [14]). These methods are all based on entropy and its rate of change. Fix $\varrho = (u,v) \in (0,1)\times(0,1)$ and let π_ϱ be a reference probability measure on the state space. If $\mu = f\pi_\varrho$ is any other probability measure on the state space we define its entropy as

$$H(f) = E^{\pi_\varrho}[f \log f].$$

If $\mu_t = f_t\pi_\varrho$ denotes the marginal distribution of our process with Dirichlet form $\mathcal{D}(f)$ then we have the general inequality

$$\frac{dH(f_t)}{dt} \le -\frac{1}{4}\mathcal{D}(f_t).$$

Changing to the macroscopic time scale $t = \epsilon^{-2}\mathbf{t}$ corresponds to multiplying the generator, or Dirichlet form by a factor ϵ^{-2}. Hence the initial entropy bound

$$H(f_0) \le K(\log 4)\epsilon^{-d} \tag{2.17}$$

with K a constant independent of ϵ produces the bound

$$\int_0^\infty \mathcal{D}(f_{\mathbf{t}})d\mathbf{t} \le K(\log 4)\epsilon^{2-d}.$$

The $\log 4$ is just the maximum entropy per site in a model with 4 possible values at each site (the constant K will take care of any arbitrary pair $\varrho_0 = (u_0,v_0)$, but we can assume $u = v = 1/2$ for this purpose). Now if $\gamma_1,\gamma_2 \ge \gamma > 0$ for each of models the Dirichlet form of the process dominates $\gamma\mathcal{D}^{(0)}$, the Dirichlet form of the symmetric simple exclusion. Hence we have the *entropy production bound*

$$\int_0^\infty \mathcal{D}^{(0)}(f_{\mathbf{t}})d\mathbf{t} \le \gamma^{-1}(\log 4)\epsilon^{2-d}. \tag{2.18}$$

From this bound follow the key estimates for non-gradient reversible systems. These will be described in Section 3 with references to the original proofs.

Model 3 is of non-gradient, non-reversible, mean zero type. For such models, a method was developed in Xu's thesis [15] , specifically applied to the mean zero asymmetric simple exclusion process. We did not have access to [15], but relied on notes of Varadhan's lectures on this topic at the Fields Institute [14]. Our proof follows their ideas very closely. The main ingredient, which allows the extension of the standard reversible machinery in these types of non-reversible systems, is the following *strong sector condition.*

LEMMA 2.5. *There exists a constant $C > 0$ such that for any $\pi = \pi_\varrho$, $\varrho \in [0,1] \times [0,1]$, $f, g \in \mathcal{F}$, and any of our three models,*

$$\left| \int f L g d\pi \right| \leq C \sqrt{\mathcal{D}(f)} \sqrt{\mathcal{D}(g)}. \tag{2.19}$$

Proof: For models 1 and 2 the result is immediate from the reversibility. We prove it for *Model 3* with $\gamma_1 = \gamma_2 = 0$. It then extends immediately to non-negative γ_1, γ_2. We rewrite the generator as $L = \sum_{x \sim y} L_{x,y}$ where

$$L_{x,y} g(\eta, \xi) = \frac{1}{2}[(\xi_x + \xi_y)(g(\eta, \xi^{x,y}) - g(\eta, \xi)) + (\eta_x \xi_x + \eta_y \xi_y)(g(\eta^{x,y}, \xi) - g(\eta, \xi^{x,y}))].$$

Recall the Dirichlet form $\mathcal{D}(f) = \sum_{x \sim y} \mathcal{D}_{x,y}(f)$ from (2.5). We write $E[f L_{x,y} g] = A + B$ where

$$A = \frac{1}{2} E\left[(\xi_x + \xi_y)(g(\eta, \xi^{x,y}) - g(\eta, \xi)) f(\eta, \xi)\right], \tag{2.20}$$

$$B = \frac{1}{2} E\left[(\eta_x \xi_x + \eta_y \xi_y)(g(\eta^{x,y}, \xi) - g(\eta, \xi^{x,y})) f(\eta, \xi)\right]. \tag{2.21}$$

Applying the exchange operator $\xi \mapsto \xi^{x,y}$ to A and resumming we obtain

$$A = -\frac{1}{4} E\left[(\xi_x + \xi_y)(g(\eta, \xi^{x,y}) - g(\eta, \xi))(f(\eta, \xi^{x,y}) - f(\eta, \xi))\right].$$

Applying $\eta \mapsto \eta^{x,y}$ and $\xi \mapsto \xi^{x,y}$ simultaneously in B we obtain

$$B = -\frac{1}{4} E\left[(\eta_x \xi_x + \eta_y \xi_y)(g(\eta, \xi^{x,y}) - g(\eta^{x,y}, \xi))(f(\eta^{x,y}, \xi^{x,y}) - f(\eta, \xi))\right]. \tag{2.22}$$

We write $B = B_1 + B_2 + B_3 + B_4$ where

$$\begin{aligned}
B_1 &= \frac{1}{4} E\left[(\eta_x \xi_x + \eta_y \xi_y)(g(\eta^{x,y}, \xi) - g(\eta, \xi))(f(\eta^{x,y}, \xi^{x,y}) - f(\eta, \xi^{x,y}))\right], \\
B_2 &= \frac{1}{4} E\left[(1 - \eta_x \eta_y)(\eta_x \xi_x + \eta_y \xi_y)(g(\eta^{x,y}, \xi) - g(\eta, \xi))(f(\eta, \xi^{x,y}) - f(\eta, \xi))\right], \\
B_3 &= \frac{1}{4} E\left[(1 - \eta_x \eta_y)(\eta_x \xi_x + \eta_y \xi_y)(g(\eta, \xi) - g(\eta, \xi^{x,y}))(f(\eta^{x,y}, \xi^{x,y}) - f(\eta, \xi^{x,y}))\right] \\
B_4 &= \frac{1}{4} E\left[(\eta_x \xi_x + \eta_y \xi_y)(g(\eta, \xi) - g(\eta, \xi^{x,y}))(f(\eta, \xi^{x,y}) - f(\eta, \xi))\right].
\end{aligned}$$

Notice that in B_2 and B_3 we have slipped in the term $1 - \eta_x \eta_y$ which vanishes when the lattice gradients vanish, but otherwise is 1. Now we have $(\eta_x \xi_x + \eta_y \xi_y) \leq (\xi_x + \xi_y)$ and therefore by Schwartz's inequality

$$|B_1| \leq \sqrt{\mathcal{D}_{x,y}(f)} \sqrt{\mathcal{D}_{x,y}(g)}.$$

For B_2 and B_3 note that

$$(1-\eta_x\eta_y)(\eta_x\xi_x+\eta_y\xi_y) \le (\xi_x+\xi_y) \wedge ((1-\eta_x)+(1-\eta_y))\,.$$

Again by Schwartz's inequality

$$|B_2+B_3| \le 4\sqrt{\mathcal{D}_{x,y}(f)}\sqrt{\mathcal{D}_{x,y}(g)}.$$

From $(\eta_x\xi_x+\eta_{x+e}\xi_{x+e}) = -(1-\eta_x)\xi_x - (1-\eta_{x+e})\xi_{x+e} + (\xi_x+\xi_{x+e})$,

$$B_4 + A = -\frac{1}{4}E\left[((1-\eta_x)\xi_x+(1-\eta_y)\xi_y)(g(\eta,\xi)-g(\eta,\xi^{x,y}))(f(\eta,\xi^{x,y})-f(\eta,\xi))\right].$$

Since $((1-\eta_x)\xi_x+(1-\eta_y)\xi_y) \le (1-\eta_x)+(1-\eta_y)$, Schwartz's inequality gives

$$|B_4+A| \le \sqrt{\mathcal{D}_{x,y}(f)}\sqrt{\mathcal{D}_{x,y}(g)}.$$

This proves that $E[fL_{x,y}g] \le 6\sqrt{\mathcal{D}_{x,y}(f)}\sqrt{\mathcal{D}_{x,y}(g)}$. Summing over nearest neighbour pairs x and y, an application of Schwarz's inequality completes the proof. ∎

3. Non-gradient systems. Let $\zeta = (\eta,\xi)$ be the vector valued occupancy number. For each $\epsilon > 0$ and initial distribution μ_0^ϵ our three models define Markov processes $\zeta(t)$ with state space $X_\epsilon = (\{0,1\}\times\{0,1\})^{\mathbf{Z}^d/\epsilon\mathbf{Z}^d}$. We denote by P_ϵ the corresponding measure on $D([0,\infty);X_\epsilon)$, the space of right continuous paths with left limits, equipped with the topology of convergence at continuity points. We are primarily interested in the comportment of $\zeta_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t})d\mathbf{x}$. Let $M(\mathbf{T}^d)$ be the set of nonnegative measures on $\mathbf{T}^d$ with total mass bounded above by 1 and $\hat{P}_\epsilon$ denote the corresponding probability measure on $D([0,T];M(\mathbf{T}^d)\times M(\mathbf{T}^d))$.

In any such model we have

$$d\zeta_x(t) = \sum_{i=1}^d \Big(W_{x-e_i,x}(t) - W_{x,x+e_i}(t)\Big)dt + dM_x(t) \tag{3.1}$$

where $W_{x,x+e} = W_{x,x+e}(t)$, the (vector) rate of particle jumps from x to $x+e$, is a local function of the form $W_{x,x+e} = \tau_x W_{0,e_i} = W_x^i$, and the M_x are martingales. We use e_i for the vector of unit length in the positive i direction on the lattice. The precise form of W_{0,e_i} will be given later. Let ϕ be a smooth function on the torus taking values in $\mathbf{R}^2$. We have

$$\begin{aligned}\int_{\mathbf{T}^d}\Big(\zeta_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}) - \zeta_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(0)\Big)\phi(\mathbf{x})d\mathbf{x} = \\ \int_0^{\mathbf{t}}\int_{\mathbf{T}^d}\nabla_\epsilon\phi(\mathbf{x})\epsilon^{-1}W_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{s})d\mathbf{x}d\mathbf{s} + M_\phi(\mathbf{t})\end{aligned} \tag{3.2}$$

where $(\nabla_\epsilon\phi)(\mathbf{x}) = \epsilon^{-1}[\phi(\mathbf{x}+\epsilon e_i) - \phi(\mathbf{x}] = \nabla\phi(\mathbf{x}) + O(\epsilon)$ and M_ϕ is a martingale with variance

$$E[(M_\phi(\mathbf{t}))^2] = \epsilon^d\int_0^{\mathbf{t}}\int_{\mathbf{T}^d}|\nabla_\epsilon\phi|^2(\mathbf{x})\sigma^2_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{s})d\mathbf{x}d\mathbf{s} \tag{3.3}$$

where σ_x is a (bounded) local function specific to the model. Hence the martingale term is of order $\epsilon^{d/2}$ and is negligible in the limit. The problem is therefore to show that as $\epsilon \to 0$,

$$\epsilon^{-1}W_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}) \rightharpoonup D(\varrho)\nabla\varrho \tag{3.4}$$

where $\varrho = (u, v)$ is the weak limit of $\zeta_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t})$, and $D = D(\varrho)$ is the diffusion matrix specific to the model. The symbol $\rightharpoonup$ is used to denote weak convergence. In other words (3.4) means that for any smooth $\phi(\mathbf{x}, \mathbf{t})$,

$$\int_0^{\mathbf{t}}\int_{\mathbf{T}^d} \phi(\mathbf{x},\mathbf{s})\epsilon^{-1}W_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{s})d\mathbf{x}d\mathbf{s} \to \int_0^{\mathbf{t}}\int_{\mathbf{T}^d} \phi(\mathbf{x},\mathbf{s})D(\varrho(\mathbf{x},\mathbf{s}))\nabla\varrho(\mathbf{x},\mathbf{s})d\mathbf{x}d\mathbf{s}$$

in probability.

At this point it helps to know what is $W_{0,e}$, the current, in each specific model. In *Model 1* it is

$$W^{(1)}_{0,e} = ((\gamma_1 + \frac{\xi_0+\xi_e}{2})(\eta_0-\eta_e), \gamma_2(\xi_0-\xi_e))\,, \tag{3.5}$$

in *Model 2* it is

$$W^{(2)}_{0,e} = ((\gamma_1 + \frac{\xi_0+\xi_e}{2})(\eta_0-\eta_e), (\gamma_2 + 1 - \frac{\eta_0+\eta_e}{2})(\xi_0-\xi_e)) \tag{3.6}$$

and in *Model 3* the current is the pair $W^{(3)}_{0,e} = (W^{(3),\eta}_{0,e}, W^{(3),\xi}_{0,e})$ where

$$\begin{aligned} W^{(3),\eta}_{0,e} &= (\gamma_1+\xi_0)\eta_0(1-\eta_e) - (\gamma_1+\xi_e)\eta_e(1-\eta_0)\,, \\ W^{(3),\xi}_{0,e} &= (\gamma_2+1-\eta_0)\xi_0(1-\xi_e) - (\gamma_2+1-\eta_e)\xi_e(1-\xi_0)\,. \end{aligned} \tag{3.7}$$

We shall denote the current generically as $W_{0,e}$ unless we need to differentiate between the three models. Remember that the first coordinate is the current of the η particles and the second is the current of the ξ particles. For some of the terms above a special simplification occurs, for example, even in *Model 3* there are some terms in the current of the form $\eta_0\xi_0 - \eta_e\xi_e$. Since it is a difference of a shift $\tau_e h$ of a function h with itself, called a *gradient*, a summation by parts reduces the key term on the right hand side of (3.2) to

$$\int_0^t\int_{\mathbf{T}^d} \Delta\phi(\mathbf{x})h_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{s})d\mathbf{x}d\mathbf{s}\,.$$

The difficult ϵ^{-1} is absorbed into the gradient on the test function through an integration by parts, and the much easier problem is now to show that

$$h_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}) \rightharpoonup \bar{h}(\varrho) \tag{3.8}$$

where $\bar{h}(\varrho) = E^{\pi_\varrho}[h]$. A system whose currents are of this form is called a *gradient system* (see [14] for a discussion of the question).

Notice that all three systems we are studying are of *non-gradient* type. So we have to prove (3.4). One way to do it might be to generate a microscopic variable which we knew converged to $D(\varrho)\nabla\varrho$ and then show that the difference between it and the field $\epsilon^{-1}W_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t})$ converges weakly to zero.

The simplest candidate is the following. Let ℓ be a positive integer and let $\bar{\zeta}^\ell_x$ denote the average value of ζ on a box Λ_ℓ of side length ℓ around site x and let

$$\Xi^\ell_x = \Big(\sum_{j=1}^d a_{ij}(\bar{\zeta}^\ell_x)(\bar{\zeta}^\ell_{x+e_j} - \bar{\zeta}^\ell_x)\Big)_{1\le i\le 2d}$$

where $D(\varrho) = (a_{ij}(\varrho))_{1\le i,j\le d}$ is the diffusion coefficient in the model. Then $\epsilon^{-1}\Xi^{\ell}_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t})$ is our natural candidate. On the other hand, for a given $\delta > 0$ it is an easy computation using Ito's formula to show that if L is the generator of the process and $g(\varrho,\zeta)$ is any function continuous in the density ϱ and depending only locally on ζ, then the field $\epsilon^{-1}Lg(\bar\zeta^{\delta\epsilon^{-1}}_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}, \tau_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}\zeta)(\epsilon^{-2}\mathbf{t})$ converges weakly to 0. Here $\bar\zeta^{\delta\epsilon^{-1}}_x$ is just the empirical density on a box of side length $\delta\epsilon^{-1}$ around x (the intermediate scale between the micro- and macroscopic levels). Hence we can replace our simple candidate by a linear combination of *gradient-type* terms plus a negligible part

$$\Xi^{\ell,g}_x = \Big(\sum_{j=1}^{d} a_{ij}(\bar\zeta^{\ell}_x)(\bar\zeta^{\ell}_{x+e_j} - \bar\zeta^{\ell}_x) + \tau_x Lg(\bar\zeta^{\ell},\zeta)\Big)_{1\le i\le 2d} \tag{3.9}$$

with coefficients a_{ij} dependent on $\varrho = (u,v)$ which determine the diffusion matrix $D(\varrho) = (a_{ij}(\varrho))_{1\le i,j\le d}$ uniquely. The problem can now be reduced to the following three lemmas.

LEMMA 3.1. *There exists a sequence g_n of local functions such that*

$$\epsilon^{-1}\Big[W_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}) - \Xi^{\ell,g_n}_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t})\Big] \rightharpoonup 0 \tag{3.10}$$

in P_ϵ probability, as $\epsilon\to 0$ followed by $\ell\to\infty$ and $n\to\infty$.

LEMMA 3.2. *The sequence of probability measures $\hat P_\epsilon$, as defined in (2.6), is relatively compact, and every limit point $\hat P$ is concentrated on absolutely continuous paths with marginal densities $\varrho(\mathbf{t},\mathbf{x})$ satisfying*

$$E^{\hat P}\left[\int_0^T\int (|\nabla\varrho(\mathbf{t},\mathbf{x})|^2 d\mathbf{x}d\mathbf{t}\right] < \infty. \tag{3.11}$$

We recall the definition of the Hilbert space $\mathcal{H}^{(0)}_{-1,\varrho}$ from (2.8).

LEMMA 3.3. *Let $\tilde P_{\epsilon,\ell}$ denote the joint distribution of the fields*

$$(\zeta_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}), \Xi^{\ell}_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}))$$

as elements of $\mathcal{H}^{(0)}_{-1,\varrho}$. The sequence is tight and any limit measure is concentrated on fields of the form $(\varrho, D(\varrho)\nabla\varrho)$.

Suppose we have a functional $F_{\epsilon,K}$ depending on ϵ and some additional parameters which we denote by K and we want to show that $\lim_K \lim_{\epsilon\to 0} E^{P_\epsilon}[F_{\epsilon,K}] = 0$. We now recall the standard machinery which reduces such problems to eigenvalue estimates. Recall that Q_ϵ denotes the equilibrium process, with initial distribution $\pi_{1/2,1/2}$ and that we have the entropy bound $H(P_\epsilon/Q_\epsilon) \le (\log 4)\epsilon^{-d}$ (see (2.17)).

LEMMA 3.4. *Suppose that P_ϵ and Q_ϵ are probability measures with*

$$H(P_\epsilon/Q_\epsilon) = \int \log\frac{dP_\epsilon}{dQ_\epsilon}dP_\epsilon \le C\epsilon^{-d}.$$

If for any $\lambda > 0$,

$$\lim_K \limsup_{\epsilon\to 0} \epsilon^d \log E^{Q_\epsilon}\left[\exp\{\lambda\epsilon^{-d}F_{\epsilon,K}\}\right] \le 0 \tag{3.12}$$

then

$$\lim_K \limsup_{\epsilon\to 0} E^{P_\epsilon}[F_{\epsilon,K}] = 0. \tag{3.13}$$

Proof: This follows from the entropy inequality

$$E^{P_\epsilon}[F] \le \log E^{Q_\epsilon}[\exp F] + H(P_\epsilon/Q_\epsilon). \tag{3.14}$$

□

LEMMA 3.5. *Let Q be a Markov process $\zeta(s)$, $s \ge 0$ with generator L which is in equilibrium with invariant measure μ. Let $\mathcal{D}$ denote the corresponding Dirichlet form $\mathcal{D}(f) = -E_\mu[fLf]$. Let $V(s,\zeta)$ be bounded. Then*

$$E^Q\left[\exp\left\{\int_0^t V(s,\zeta(s))ds\right\}\right] \le \exp\{\int_0^t \lambda(V(s))ds\} \tag{3.15}$$

where $\lambda(V)$ is the principal eigenvalue of $S+V$, $S=(L+L^*)/2$, given by the Raliegh-Ritz formula

$$\lambda(V) = \sup_{f\ge 0,\ \int f d\mu = 1}\left\{\int V f d\mu - \mathcal{D}(\sqrt{f})\right\}. \tag{3.16}$$

Proof: By the Feynman-Kac formula, $u(t,\zeta) = E_\zeta[\exp\{\int_0^t V(t-s,\zeta(s))ds\}]$ solves the equation $\partial_t u = [A+V]u$ with $u(0,\zeta) = 1$. Hence

$$\frac{d}{dt}\int u^2 d\mu = 2\left\{\int V u^2 - \mathcal{D}(u)\right\} \le 2\lambda(V)\int u^2 d\mu. \tag{3.17}$$

Therefore

$$\begin{aligned} E^Q[\exp\{\int_0^t V(t-s,\zeta(s))ds\}] &= \\ \int u(t)d\mu \le \sqrt{\int u^2(t)d\mu} &\le \exp\int_0^t \lambda(V(t-s,\zeta(s)))ds. \end{aligned} \tag{3.18}$$

□

In our applications $t = \epsilon^{-2}\mathbf{t}$ and hence after rescaling the variational formula becomes

$$\sup_{f\ge 0,\ \int f d\mu = 1}\left\{\epsilon^{-d}\int V f d\mu - \epsilon^{-2}\mathcal{D}(\sqrt{f})\right\}$$

so that we can restrict the variational problem to f with $\mathcal{D}_\epsilon(\sqrt{f}) \le C\epsilon^{2-d}$, which is the same as (2.17).

Since all of our Dirichlet forms have a lower bound in terms of the Dirichlet form $\mathcal{D}^{(0)}$ of symmetric simple exclusions, we can use $\mathcal{D}^{(0)}$ instead of the real Dirichlet form $\mathcal{D}$ in the variational problem to get an upper bound. Thus the key lemmas are reduced to eigenvalue problems for the generator of the symmetric simple exclusion process.

Next we state the standard one and two block estimates in our context (see Chapter 5 of [4] for a proof).

LEMMA 3.6. *Suppose f_ϵ is a sequence of densities of the particle system on $\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d$ with respect to invariant measures $\pi = \pi_{u,v}$ for some fixed $0 < u < 1$, $0 < v < 1$ and satisfying*

$$\mathcal{D}^0_\epsilon(\sqrt{f_\epsilon}) \leq C\epsilon^{2-d}.$$

Let g be a local function and $\bar{g}(\varrho) = E^{\pi_\varrho}[g]$. Then

$$\limsup_{\ell\to\infty}\limsup_{\epsilon\to 0} E^{f_\epsilon \pi}\left[Av_{x\in\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d}\left|Av_{|y-x|\leq \ell}\, g(\tau_x\zeta) - \bar{g}(\bar{\zeta}^\ell_x)\right|\right] = 0.$$

Let F be a continuous function on $[0,1]\times[0,1]$. Then

$$\lim_{\substack{\delta\to 0\\ \ell\to\infty}}\lim_{\epsilon\to 0} E^{f_\epsilon\pi}\left[Av_{x\in\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d}\left|Av_{|y-x|\leq\delta\epsilon^{-1}}F(\bar{\zeta}^{\delta\epsilon^{-1}}_y) - Av_{|y-x|\leq\ell}F(\bar{\zeta}^\ell_y)\right|\right] = 0.$$

Here Av denotes the average and $\bar{\zeta}^\ell_x = Av_{|y-x|\leq\ell}\zeta_y$, the average over y in a box of size ℓ.

Now we return to the key replacement which is Lemma 3.1 which in microscopic variables takes the form

$$\epsilon^{1+d/2}\int_0^{\epsilon^{-2}\mathbf{t}} \sum_{x\in\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d} \Omega_x(s)ds$$

where $\Omega_x(s) = \phi(\epsilon\mathbf{x},\epsilon^2\mathbf{s})[W_x - \Xi_x]$. Where for gradient systems the key replacement (3.8) is a local law of large numbers, which is proved in the one/two block estimates, for nongradient systems the key replacement is a local central limit theorem.

Let us make this more rigorous. For any vector local function $\mathbf{g}$ define

$$\Omega^{\ell,\mathbf{g}}_x = \phi(\epsilon\mathbf{x},\epsilon^2\mathbf{s})\tau_x[\frac{1}{(2\ell'+1)^d}\sum_{|y|\leq\ell'} W_y - D(\bar{\zeta}^\ell_0)(\bar{\zeta}^{\ell'}_e - \bar{\zeta}^{\ell'}_0) - \frac{1}{(2\ell'+1)^d}\sum_{|y|\leq\ell'}\tau_y L\mathbf{g}]$$

where $\ell' = \ell - |supp(g)|$ so that $\Omega^{\ell,\mathbf{g}}$ depends only on variables in a box of side length $2\ell+1$ about $0\in\mathbf{Z}^d$. Let $L^{(0)}_\ell$ denote the generator of the process where the η and ξ particles independently perform symmetric random walks with simple exclusion on a box of side length ℓ with reflecting boundary conditions. Let $E^{n,m,\ell}$ denote expectation with respect to the canonical measure $u^\ell_{n,m}$, the uniform distribution on configurations on this box with n particles of type η and m of type ξ. Since the system is ergodic when restricted to such a set of configurations and $\Omega^{\ell,\mathbf{g}}$ has mean 0, we can define a nonnegative definite matrix

$$\sigma^2_{n,m,\ell}(\mathbf{g}) = E^{\ell,n,m}[\Omega^{\ell,\mathbf{g}}(-L^{(0)}_\ell)^{-1}\Omega^{\ell,\mathbf{g}}]. \tag{3.19}$$

Let L be the generator of a Markov process X_t, $t\geq 0$, on a state space $\mathcal{S}$, reversible with respect to a probability measure μ and with Dirchlet form $\mathcal{D}(f) = -E_\mu[fLf]$. Given a function V on $\mathcal{S}$, let $\lambda(\epsilon V)$ be the principal eigenvalue of $L+\epsilon V$, as in (3.16). Let $m = E_\mu[V]$ and

$$\begin{aligned}\sigma^2(V) &= E_\mu[V(-L)^{-1}V]\\ &= \lim_{T\to\infty}\frac{1}{2}E_\mu\left[\left(\frac{1}{\sqrt{T}}\int_0^T V(X_t)dt\right)^2\right]\\ &= \sup_f\left\{2E_\mu[Vf] - \mathcal{D}(f)\right\}.\end{aligned}$$

We now make this more precise. Let γ be the spectral gap of L,

$$\gamma = \inf_f \{\mathcal{D}(f)/Var(f)\} \ .$$

The Rayleigh-Schrodinger perturbation series is

$$\lambda(\epsilon V) = \epsilon m + \epsilon^2 \sigma^2(V) + \cdots \ .$$

We are interested in the following result, which can be found in [13] and also in [4].

LEMMA 3.7. *Assume that L has a spectral gap $\gamma > 0$. Let V be bounded with $E_\mu[V] = 0$. Then*

$$0 \leq \lambda(\epsilon V) \leq \frac{\epsilon^2}{1 - 2\epsilon\gamma^{-1}\|V\|_\infty}\sigma^2(V).$$

Returning to the setup of our problem, recall the definition (3.19) of $\sigma^2_{n,m,\ell}(g)$.

LEMMA 3.8. *To prove Lemma 3.1 and Lemma 3.3, it suffices to prove that*

$$\inf_{\mathbf{g}} \limsup_{\ell\to\infty} \sup_{0\leq n,m\leq(2\ell+1)^d} \ell^d \sigma^2_{n,m,\ell}(g) = 0 \tag{3.20}$$

Proof: If ϕ is a smooth test function it is clear that

$$\int_0^{\mathbf{T}} \int_{\mathbf{T}^d} \phi(\mathbf{x}) W_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}) d\mathbf{x} d\mathbf{t} \quad \text{and} \quad \int_0^{\mathbf{T}} \int_{\mathbf{T}^d} \phi(\mathbf{x}) W^\ell_{\lfloor \epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t}) d\mathbf{x} d\mathbf{t}$$

will have the same limit where

$$W_0^\ell = \frac{1}{(2\ell'+1)^d} \sum_{|y|\leq \ell'} W_y.$$

Let

$$V_x^\ell = \epsilon^{-1} D(\bar\zeta_x^\ell)(\bar\zeta_{x+e}^{\ell'} - \bar\zeta_x^{\ell'}).$$

By an elementary re-summation $\sum_x \phi(\epsilon x)[V_x^{\epsilon^{-1}\delta} - V_x^\ell] = \epsilon^{-1}\sum_x B_x \nabla\zeta_x$, where $\nabla\zeta_x$ is the vector whose ith entry is $\zeta_{x+e_i} - \zeta_x$ and

$$B_x = Av_{|y-x|\leq \epsilon^{-1}\delta} D(\bar\zeta_y^{\epsilon^{-1}\delta})\varphi(\epsilon^{-1}y) - Av_{|y-x|\leq \ell'} D(\bar\zeta_y^\ell)\varphi(\epsilon^{-1}y).$$

There is a very simple integration by parts formula which says that for any function $f(\eta,\xi)$, $E^{\ell,n,m}[(\eta_{x+e} - \eta_x)f(\eta,\xi)] = -\frac{1}{2}E^{\ell,n,m}[(\eta_{x+e} - \eta_x)(f(\eta^{x,x+e},\xi) - f(\eta,\xi))]$ and analogously for ξ. Since B_x is invariant under the transformations $\eta \mapsto \eta^{x,x+e_i}$ and $\xi \mapsto \xi^{x,x+e_i}$ for any $c > 0$ there exists a $C < \infty$ so that

$$|E[B_x \nabla\zeta_x f]| \leq C E[|B_x|^2 f] + \epsilon^{-2}\frac{c}{2}\sum_{i=1}^d \mathcal{D}^{(0)}_{x,x+e_i}(\sqrt{f}). \tag{3.21}$$

Hence for any $c > 0$, f, and bounded ϕ_x

$$\begin{aligned}
&\epsilon^{d-1}E[\sum_x \phi_x(V_x^{\epsilon^{-1}\delta} - V_x^\ell)f] - c\epsilon^{d-2}\mathcal{D}^{(0)}(\sqrt{f}) \\
&\leq C\epsilon^d \sum_x E[|B_x|^2 f] - \frac{c}{2}\epsilon^{d-2}\mathcal{D}^{(0)}(\sqrt{f}).
\end{aligned}$$

From the continuity of $D(\varrho)$, this vanishes uniformly over densities f, in the limit $\epsilon \to 0$, followed by $\delta \to 0$, by the two block estimate.

Remark. In order to use the two-block estimate from above, one needs the continuity of the diffusive coefficient $D(u, v)$. We refer the reader to Theorem 5.8 from [4].

By applying Lemma 3.4 and Lemma 3.5, to prove Lemma 3.1 it suffices to verify that for any $\delta > 0$ and bounded ϕ_x

$$\inf_{\mathbf{g}} \limsup_{\ell\to\infty} \limsup_{\epsilon\to 0} \sup_f \{\epsilon^{d-1} E^{\pi_\ell}[\sum_x \Omega_x^{\ell,\mathbf{g}} f] - \delta\epsilon^{d-2}\mathcal{D}^{(0)}(\sqrt{f})\} \le 0. \tag{3.22}$$

The state space here is $(\{0,1\} \times \{0,1\})^{\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d}$ and the expectations are with respect to a product measure with some fixed $0 < u < 1$, $0 < v < 1$. If we denote $\mathcal{D}_\ell^{(0)} = \sum_{|x|\le\ell,|x+e|\le\ell,|e|=1} \mathcal{D}_{x,x+e}^{(0)}(\sqrt{f})$ where the expectation is with respect to the product measure $\pi_{u,v}^\ell$ on configurations on a box of side length $2\ell+1$, then we have $\sum_x \mathcal{D}_\ell^{(0)}(\sqrt{\tau_{-x} f}) \le L^d \mathcal{D}^{(0)}(\sqrt{f})$ where $L = 2\ell + 1$, and therefore

$$\begin{aligned} &\epsilon^{d-1} E[\sum_x \Omega_x^{\ell,\mathbf{g}} f] - \delta\epsilon^{d-2}\mathcal{D}^{(0)}(\sqrt{f}) \\ &\le \frac{\epsilon^{d-2}\delta}{L^d} \sum_x \sup_f \{\frac{L^d\epsilon}{\delta} E[\Omega_x^{\ell,\mathbf{g}} f] - \mathcal{D}_\ell^{(0)}(\sqrt{f})\} \end{aligned} \tag{3.23}$$

The expectation is with respect to $\pi_{u,v}^\ell$, but we could instead use the canonical measure $u_{n,m}^\ell$. Since the product measure is just a linear combination of the latter, if we prove it uniformly over n and m we have the result. Now by the previous lemma and the fact that the spectral gap of the exclusion process on a box of side length L is of order L^{-2},

$$\frac{\epsilon^{d-2}\delta}{L^d} \sum_x \sup_f \{\frac{L^d\epsilon}{\delta} E[\Omega_x^{\ell,\mathbf{g}} f] - \mathcal{D}_\ell^{(0)}(\sqrt{f})\} \le C\delta^{-1} L^d \sigma_{n,m,\ell}^2(g).$$

Letting $\ell \to \infty$ we obtain the desired result. □

The previous lemma reduces the proof of the hydrodynamic limit to the evaluation of the asymptotics of certain central limit theorem variances. We now describe how to make these computations. Note that $\Omega^{\ell,\mathbf{g}}$ is an average of shifts of local functions f of three types: 1. The current W; 2. The microscopic density gradients $\nabla\zeta$; 3. Incoherent fluctuations Lg. All three have the property that their expectation is zero with respect to any canonical measure on any box containing their support. They also satisfy the following integration by parts formulas with respect to any such measure: For any local h, and nearest neighbours x and y, $E[W_{x,y}h] = -\frac{1}{2}E[W_{x,y}(h(\eta^{x,y}, \xi^{x,y}) - h(\eta, \xi))]$, $E[(\zeta_y - \zeta_x)h] = -\frac{1}{2}E[(\zeta_y - \zeta_x)(h(\eta^{x,y}, \xi^{x,y}) - h(\eta, \xi))]$, and $E[L_{x,y}gh] = -\frac{1}{2}\mathcal{D}_{x,y}(g, h)$. In particular, each of the three functions f satisfies a bound

$$E[fh] \le C \sum_{|x-y|=1,|x|,|y|\le R} \mathcal{D}_{x,y}^{(0)}(h) \tag{3.24}$$

for some $C, R < \infty$, uniformly over boxes containing $|x|, |y| \le R$ and over the canonical measures on that box. The class of local functions f satisfying a bound of type (3.24) was denoted by $\mathcal{F}$ in (2.7). Note that this corresponds to local functions for which the asymptotic variance

$$(3.25)\quad \langle f,f\rangle^{(0)}_{-1,\varrho} = \lim_{\substack{\ell\to\infty \\ (\frac{n}{(2\ell+1)^d},\frac{m}{(2\ell+1)^d})\to\varrho}} \frac{1}{(2\ell)^d} E^{n,m,\ell}[\sum_{x\le\ell'}\tau_x f, (-L^{(0)}_\ell)^{-1}(\sum_{|x|\le\ell'}\tau_x f)],$$

is finite. For any g,h in $\mathcal{F}$ we can define $\langle g,h\rangle^{(0)}_{-1,\varrho}$ by polarisation, giving a semi-inner product on $\mathcal{F}$ and $||g||^{(0)}_{-1,\varrho} = (\langle g^2\rangle^{(0)}_{\varrho})^{\frac{1}{2}}$ becomes a semi-norm. Let $\mathcal{N} = \{g\in\mathcal{F} : ||g||^{(0)}_{-1,\varrho} = 0\}$. The completion of the quotient space $\mathcal{F}/\mathcal{N}$, denoted by $\mathcal{H}^{(0)}_{-1,\varrho}$, is thus a Hilbert space. The first part of the following result first appeared in [10]. A complete proof can be found in [4] so we will not prove it again here. The second part was first proved in a different context (mean zero asymmetric simple exclusion) by [15]. A nice review is [14] [Theorem A [Varadhan's Lecture 5, page 2, at Fields].

Theorem 3.9. *For each $\varrho = (u,v)\in(0,1)\times(0,1)$,*

1) *the closure of $L^{(0)}\mathcal{F}$ in $\mathcal{H}^{(0)}_{-1,\varrho}$ is a linear subspace of codimension $2d$ and the orthogonal subspace is provided by the span of $\nabla\zeta$.*

2) *the closure of $L^{(i)}\mathcal{F}$, $i=1,2,3$ in $\mathcal{H}^{(0)}_{-1,\varrho}$ is a linear subspace of codimension $2d$ and a complementary subspace is provided by the span of $\nabla\zeta$.*

Proof: We only prove 2). From 1), it suffices to prove the triviality of the kernel $\mathcal{K}$ of the orthogonal projection from $\overline{L^{(0)}\mathcal{F}}$ to $L\mathcal{F}$. Let $g\in\mathcal{K}$ and $\delta>0$. Since $g\in\overline{L^{(0)}\mathcal{F}}$ there is an $f\in\mathcal{F}$ with $\parallel g - L^{(0)}f\parallel^{(0)}_{-1,\varrho}\le\delta$. From the equivalence of the Dirichlet forms $\mathcal{D}^{(i)}$, $i=0,1,2,3$ we have $\parallel L^{(0)}f\parallel^{(0)}_{-1,\varrho}\le(\gamma^{-1}\langle L^{(0)}f, Lf\rangle^{(0)}_{-1,\varrho})^{1/2}$. Since $g\in\mathcal{K}$, $\langle L^{(0)}f,Lf\rangle^{(0)}_{-1,\varrho} = \langle L^{(0)}f-g,Lf\rangle^{(0)}_{-1,\varrho}\le\delta$. By Schwarz's inequality, $\langle L^{(0)}f-g,Lf\rangle^{(0)}_{-1,\varrho}\le\delta\parallel Lf\parallel^{(0)}_{-1,\varrho}$. Hence $\parallel L^{(0)}f\parallel^{(0)}_{-1,\varrho}\le\gamma^{-1}\delta\parallel Lf\parallel^{(0)}_{-1,\varrho}$. By the strong sector condition Lemma 2.5, $\parallel Lf\parallel^{(0)}_{-1,\varrho}\le C\parallel L^{(0)}f\parallel^{(0)}_{-1,\varrho}$. Letting $\delta\downarrow 0$, we have $\parallel g\parallel^{(0)}_{-1,\varrho}=0$. □

4. Tightness.. Hence the diffusion coefficient can be identified by the formula $W_{0,e_i} - D^{(i)}(\varrho)\nabla\zeta\in\overline{L^{(i)}\mathcal{F}}$ in $\mathcal{H}^{(0)}_{-1,\varrho}$. In the final section we derive more explicit expressions for D. It only remains to prove compactness of the density fields, Lemma 3.2. We start with a general lemma. For a pure jump function $x(\cdot)$ with a finite number of jumps, the polygonalization $\hat{x}(\cdot)$ is obtained by linearly interpolating between values at successive jumps.

Lemma 4.1. *Let $\{(Q_\epsilon,P_\epsilon)\}_{\epsilon>0}$ be probability measures on $D([0,T];\mathbf{R})$ which are supported on pure jump functions, such that for some $C_1,C_2<\infty$, $H(Q_\epsilon/P_\epsilon)\le C_1\epsilon^{-d}$. If, for any $0\le s<t\le T$, and any $\lambda>0$,*

$$(4.1)\qquad E^{P_\epsilon}\left[\exp\{\lambda\epsilon^{-d}(x(t)-x(s))\}\right]\le\exp\{C_2\epsilon^{-d}\lambda^2(t-s)\},$$

then there exists $C_3<\infty$ so that, for any $0<\delta\le T$,

$$\limsup_{\epsilon\to0} E^{Q_\epsilon}\left[\sup_{|t-s|<\delta,\,0\le s,t\le T}|\hat{x}(t)-\hat{x}(s)|\right]\le C_3\sqrt{\delta}\log\delta^{-1}.$$

Proof: The Garsia-Rodemich-Rumsey inequality [12] states that if $x(t)$ is a continuous function and $\psi(x)$ a strictly increasing function such that $\psi(0)=0$,

$\lim_{x\to\infty}\psi(x)=\infty$, if

$$B=\int_0^T\int_0^T \psi\left(|x(t)-x(s)|/\sqrt{|t-s|}\right)dsdt$$

then

$$\sup_{|t-s|<\delta\, 0\le s,t\le T}|x(t)-x(s)|\le 4\int_0^\delta \psi^{-1}\left(4Bu^{-2}\right)u^{-1/2}du.$$

Choosing $\psi(x)=\exp\{\epsilon^{-d}x\}-1$ one obtains after some computation that

$$4\int_0^\delta \psi^{-1}\left(4Bu^{-2}\right)u^{-1/2}du\le \epsilon^d C_4(\delta)(1+\log(4B+\delta^2)\vee 0)$$

where $C_4(\delta)=32\sqrt{\delta}\log\delta^{-1}$. Applying this to the polygonalization of $x(t)$,

$$\begin{aligned}&E^{P_\epsilon}\left[\exp\left\{\lambda\epsilon^{-d}\sup_{|t-s|<\delta,0\le s<t\le T}|\hat{x}(t)-\hat{x}(s)|\right\}\right]\\&\le E^{P_\epsilon}\left[\exp\{\lambda C_4(\delta)(1+\log(4B+\delta^2))\}\right]\end{aligned}$$

By choosing $\lambda=1/C_4(\delta)$, the right side is bounded by $C_5(T)\epsilon^{-d}$ for some $C_5(T)<\infty$ for each $T>0$, from (4.1). It only remains to apply the entropy inequality (3.14). ∎

LEMMA 4.2. *Let P_ϵ^{eq} be the process starting from equilibrium on $\mathbf{Z}^d/\epsilon^{-1}\mathbf{Z}^d$ and let $V_x=\tau_x V$ where V is any local function satisfying a bound of the form (3.24). Then there exists a constant $C<\infty$ such that for any smooth test function $\phi:[0,\mathbf{T}]\times\mathbf{T}^d\to\mathbf{R}$,*

$$\begin{aligned}&E^{P_\epsilon^{eq}}\left[\exp\left\{\epsilon^{-d}\int_{\mathbf{s}}^{\mathbf{t}}\int_{\mathbf{T}^d}\phi(\mathbf{u},\mathbf{x})V_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{u})d\mathbf{x}d\mathbf{u}\right\}\right]\\&\le\exp\{C\epsilon^{-d}\|\phi(\mathbf{u})\|^2_{L^2([\mathbf{s},\mathbf{t}]\times\mathbf{T}^d)}\}.\end{aligned}$$

Proof: By stationarity and Lemma 3.5 $\exp\{2(\mathbf{t}-\mathbf{s})\Lambda_\epsilon\}$ is an upper bound for the left hand side, where

$$\Lambda_\epsilon=\sup_{E^{\pi_\varrho}[f]=1,\ f\ge 0}\left\{\epsilon^{-(d+1)}\int_{\mathbf{T}^d}\phi(\mathbf{x})E^{\pi_\varrho}[V_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}f]d\mathbf{x}-\epsilon^{-2}\mathcal{D}(\sqrt{f})\right\}.$$

By (3.24) and $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ we obtain the result. ∎

THEOREM 4.3. *$\hat{P}_\epsilon$ is tight.*

Proof: By (3.1),

$$\begin{aligned}&\hat{P}_\epsilon\left(\sup_{0\le s<t\le T,\ |t-s|<\delta}\left|\int_{\mathbf{T}^d}[\zeta_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{t})-\zeta_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{s})]\phi(\mathbf{x})d\mathbf{x}\right|\ge 4\varepsilon\right)\\&\le\hat{P}_\epsilon\left(\sup_{0\le s<t\le T,|t-s|<\delta}\left|\int_{\mathbf{s}}^{\mathbf{t}}\int_{\mathbf{T}^d}\nabla_e\phi(\mathbf{x})\epsilon^{-1}W_{\lfloor\epsilon^{-1}\mathbf{x}\rfloor}(\epsilon^{-2}\mathbf{u})d\mathbf{x}d\mathbf{u}\right|\ge 2\varepsilon\right)\\&\quad+\hat{P}_\epsilon\left(\sup_{0\le s<t\le T}|M_\phi(\mathbf{t})-M_\phi(\mathbf{s})|\ge 2\varepsilon\right).\end{aligned}$$

The third term is of order ϵ^d by Doob's inequality. By the previous lemmas applied to the second term we obtain (3.12). By Lemma 3.4 this suffices. ∎

5. Diffusion Coefficient. In this section we derive formulae for the various diffusion coefficients. For any two vector functions $\mathbf{g}, \mathbf{h}$, we write

$$\langle \mathbf{g}, \mathbf{h} \rangle_{-1,\varrho} = (\langle g_i, h_j \rangle_{-1,\varrho})_{i,j}. \tag{5.1}$$

The current is given by $W = (W^\eta, W^\xi)$ where

$$W^\eta = (W^\eta_{0,e_1}, \dots, W^\eta_{0,e_d}), \qquad W^\xi = (W^\xi_{0,e_1}, \dots, W^\xi_{0,e_d}).$$

The diffusion coefficient in each of the three models $i = 1, 2, 3$ is defined by the equation $W^{(i)} - D^{(i)} \nabla\zeta \in \otimes\overline{L^{(i)}\mathcal{F}}$ as elements of $\mathcal{H}^{(0)}_{-1,\varrho}$. We can also define spaces $\mathcal{H}^{(i)}_{-1,\varrho}$, $i = 1, 2, 3$ by using the analogue of (3.25) as in (2.8) with

$$\langle f, f \rangle^{(i)}_{-1,\varrho} = \lim_{\ell\to\infty} \frac{1}{(2\ell)^d} E^{n,m,\ell}[\sum_{x \le \ell'} \tau_x f, (-L^{(i)}_\ell)^{-1}(\sum_{|x|\le\ell'} \tau_x f)].$$

Since the corresponding Dirichlet forms are equivalent, $\mathcal{H}^{(1)}_{-1,\varrho}$ and $\mathcal{H}^{(2)}_{-1,\varrho}$ are equivalent to $\mathcal{H}^{(0)}_{-1,\varrho}$. Hence we can solve $W^{(i)} - D^{(i)}\nabla\zeta \in \otimes\overline{L^{(i)}\mathcal{F}}$ in $\mathcal{H}^{(i)}_{-1,\varrho}$ for $i = 1, 2$. Model 1 is more straightforward, so we describe the details in the case of Model 2 and leave Model 1 to the reader.

Model 2. In *Model 2*, the two components of the current (3.6) read

$$W^{(2),\eta}_{0,e} = (\gamma_1 + \frac{\xi_0 + \xi_e}{2})(\eta_e - \eta_0), \quad W^{(2),\xi}_{0,e} = (\gamma_2 + 1 - \frac{\eta_0 + \eta_e}{2})(\xi_e - \xi_0).$$

For any local g and any $1 \le i \le d$, we can compute explicitly

$$\langle W^{(2),\eta}_{0,e_i}, L^{(2)} g \rangle^{(2)}_{-1,\varrho} = \frac{1}{2} E^{\pi_\varrho}[(\gamma_1 + \frac{\xi_{e_i} + \xi_0}{2})(\eta_0 - \eta_{e_i}) \nabla^\eta_{0,e_i} \sum_x \tau_x g], \tag{5.2}$$

$$\langle W^{(2),\xi}_{0,e_i}, L^{(2)} g \rangle^{(2)}_{-1,\varrho} = \frac{1}{2} E^{\pi_\varrho}[(\gamma_2 + 1 - \frac{\eta_{e_i} + \eta_0}{2})(\xi_0 - \xi_{e_i}) \nabla^\xi_{0,e_i} \sum_x \tau_x g], \tag{5.3}$$

$$\langle L^{(2)} g, L^{(2)} g \rangle^{(2)}_{-1,\varrho} = \frac{1}{2} \sum_{j=1}^d E\Big[(\gamma_1 + \frac{\xi_{e_j} + \xi_0}{2})(\nabla^\eta_{0,e_j} \sum_x \tau_x g)^2 \tag{5.4}$$

$$+(\gamma_2 + 1 - \frac{\eta_{e_j} + \eta_0}{2})(\nabla^\xi_{0,e_j} \sum_x \tau_x g)^2\Big], \tag{5.5}$$

$$\langle \nabla\zeta, L^{(2)} g \rangle^{(2)}_{-1,\varrho} = 0. \tag{5.6}$$

$$\langle W, W \rangle^{(2)}_{-1,\varrho} = \begin{pmatrix} (\gamma_1 + v) I_d & 0 \\ 0 & (\gamma_2 + 1 - u) I_d \end{pmatrix} \chi(\varrho) \tag{5.7}$$

$$\langle W, \nabla\zeta \rangle^{(2)}_{-1,\varrho} = \chi(\varrho). \tag{5.8}$$

THEOREM 5.1. 1). $D^{(2)}(\varrho)\langle \nabla\zeta, \nabla\zeta \rangle_{-1,\varrho} = \chi(\varrho)$.
2). *For any* $\mathbf{r} \in \mathbf{R}^d \times \mathbf{R}^d$,

$$\mathbf{r}\chi(\varrho) D^{(2)}(\varrho) \mathbf{r}' = \inf_g \sum_{i=1}^d \mathcal{D}_{0,e_i}(\mathbf{r} \sum_x x\zeta_x - \sum_x \tau_x g). \tag{5.9}$$

The infimum is over local functions g. Note that $\sum_x \tau_x g$ makes no sense alone, however since g is local, only finitely many terms in the sum are nonzero after applying the discrete gradients ∇_{0,e_i}.

Proof: Since $W^{(2)} - D^{(2)}(\varrho)\nabla\zeta$ is in the closure of $L\mathcal{F}$,

$$\langle W^{(2)} - D^{(2)}(\varrho)\nabla\zeta, \nabla\zeta\rangle_{-1,\varrho} = 0.$$

1) then follows from (5.8). 2) then follows from (3.9) and (5.2), (5.3) and (5.5). ☐

We still need to obtain the simpler formula (2.11) for *Model 2*. Note that if $\mathbf{r} = (r_1^1, \ldots, r_d^1, r_1^2, \ldots, r_d^2)$, then $\mathcal{D}_{0,e_i}(\mathbf{r}\sum_x x\zeta_x - \sum_x \tau_x g)$ is given explicitly by

$$\frac{1}{2}E[(\gamma_1 + \frac{\xi_{e_i} + \xi_0}{2})(r_i^1(\eta_{e_i} - \eta_0) - \nabla^\eta_{0,e_i}\sum_x \tau_x g)^2$$
$$+ (\gamma_2 + 1 - \frac{\eta_{e_i} + \eta_0}{2})(r_i^2(\xi_{e_i} - \xi_0) - \nabla^\xi_{0,e_i}\sum_x \tau_x g)^2].$$

Now for any constants a, b,

$$(a(\eta_e - \eta_0) - \nabla^\eta_{0,e}\sum_x \tau_x g)^2 = \left(\frac{a-b}{2}\right)^2 (\eta_e - \eta_0)^2$$
$$+ \left(\frac{a+b}{2}(\eta_e - \eta_0) - \nabla^\eta_{0,e}\sum_x \tau_x g\right)^2 + \frac{a^2 - b^2}{2}(\eta_e - \eta_0)\nabla^\eta_{0,e}\sum_x \tau_x g$$

$$(b(\xi_e - \xi_0) - \nabla^\xi_{0,e}\sum_x \tau_x g)^2 = \left(\frac{a-b}{2}\right)^2 (\xi_e - \xi_0)^2$$
$$+ \left(\frac{a+b}{2}(\xi_e - \xi_0) - \nabla^\xi_{0,e}\sum_x \tau_x g\right)^2 + \frac{b^2 - a^2}{2}(\xi_e - \xi_0)\nabla^\xi_{0,e}\sum_x \tau_x g$$

Now we claim that for any local function g,

$$E^{\pi_\varrho}[(\frac{\xi_e + \xi_0}{2})(\eta_e - \eta_0)\nabla^\eta_{0,e}\sum_x \tau_x g - (1 - \frac{\eta_e + \eta_0}{2})(\xi_e - \xi_0)\nabla^\xi_{0,e}\sum_x \tau_x g] = 0. \tag{5.10}$$

To prove it we transfer the $\nabla_{0,e}$ onto the $\nabla\zeta$ to obtain $E^{\pi_\varrho}\left[(\eta_e\xi_e - \eta_0\xi_0)\sum_{x\in\Lambda}\tau_x g\right]$ where the sum is over some large but finite box Λ. Now use the translation invariance of the measure to rewrite this as $E^{\pi_\varrho}\left[\sum_{x\in\Lambda}(\eta_{x+e}\xi_{x+e} - \eta_x\xi_x)g\right]$. The first term is a telescoping sum and we end up with $E^{\pi_\varrho}[fg]$ where f is mean zero and does not depend on variables ζ_x in a box A around the origin, while g depends only on ζ_x, $x \in A$. Since π_ϱ is a product measure, $E^{\pi_\varrho}[fg] = E^{\pi_\varrho}[f]\,E^{\pi_\varrho}[g] = 0$ which proves (5.10). [Then] (2.11) follows from this and the explicit form of $\mathcal{D}_{0,e_i}(\mathbf{r}\sum_x x\zeta_x - \sum_x \tau_x g)$ after a little computation.

Model 3. We now show the last part of Theorem 2.4.

Proof: First, we remind a general fact about matrices. Let L be an invertible matrix and $L_s = (L^* + L)/2$ its symmetrisation. One can check that $[(L^{-1})_s]^{-1} = L^* L_s^{-1} L$, or in variational form

$$< f, (-L)^{-1} f >= \sup_g \inf_h \{2 < f - Lh, g > - < h, Lh >\}.$$

In particular, taking $h = -g$ we have

$$< f, (-L)^{-1} f > \leq < f, (-L_s)^{-1} f > .$$

We will apply this in our particular situation where $L_\ell^{(2)} = (L_\ell^{(3)})_s$. Let $T_\ell = L_\ell^{(2)}(L_\ell^{(3)})^{-1}$. This makes sense for any mean zero function of configurations on $|x| \leq \ell$ with n particles of type η and m particles of type ξ, and $\langle T_\ell f, (-L_\ell^{(2)})^{-1} T_\ell f\rangle = \langle f(-L_\ell^{(3)})^{-1} f\rangle \leq \langle f(-L_\ell^{(2)})^{-1} f\rangle$, since on such hyperplanes the operators $L_l^{(i)}$ are invertible. Hence T_ℓ is bounded and therefore has a limit T defined on $\mathcal{H}_{-1,\varrho}^{(2)}$ which has the property that $T\overline{L^{(3)}\mathcal{F}} = \overline{L^{(2)}\mathcal{F}}$ and $TW^{(3)} = W^{(2)}$ and whose norm is bounded by 1. The diffusion coefficient is defined by $W^{(3)} - D^{(3)}(\varrho)\nabla\zeta \in \bigotimes_{i=1}^{2d} \overline{L^{(3)}\mathcal{F}}$ in $\mathcal{H}_{-1,\varrho}^{(2)}$. Applying T gives

$$W^{(2)} - D^{(3)}(\varrho)\mathbf{T}\nabla\zeta \in \bigotimes_{i=1}^{2d} \overline{L^{(2)}\mathcal{F}}, \tag{5.11}$$

which implies that $[D^{(3)}(\varrho)\mathbf{T} - D^{(2)}(\varrho)]\nabla\zeta \in \bigotimes_{i=1}^{2d} \overline{L^{(2)}\mathcal{F}}$ or

$$D^{(3)}(\varrho)\langle \mathbf{T}\nabla\zeta, \nabla\zeta\rangle_{-1,\varrho}^{(2)} = D^{(2)}(\varrho)\langle \nabla\zeta, \nabla\zeta\rangle_{-1,\varrho}^{(2)}. \tag{5.12}$$

For $\mathbf{r} \in \mathbf{R}^d \times \mathbf{R}^d$,

$$\begin{aligned}\langle \mathbf{T}\sum_i r_i \nabla\zeta_i, \sum_j r_j \nabla\zeta_j\rangle_{-1,\varrho}^{(2)} &= \sum_i \sum_j \langle \mathbf{T}\nabla\zeta_i, \nabla\zeta_j\rangle_{-1,\varrho}^{(2)} r_i r_j \\ &\leq \sum_i \sum_j \langle \nabla\zeta_i, \nabla\zeta_j\rangle_{-1,\varrho}^{(2)} r_i r_j\end{aligned}$$

due to the bound $\|\mathbf{T}\| \leq 1$. We also notice that the upper bound (in the sense of quadratic forms) is achieved if and only if $\mathbf{T}\nabla\zeta = \nabla\zeta$. This would imply that $\mathbf{T}\nabla\zeta \in \bigotimes_{i=1}^{2d} \overline{L^{(2)}\mathcal{F}}$. This leads to a contradiction, due to the property that $\langle \mathbf{T}\nabla\zeta, L^{(2)}g\rangle \neq 0$ (see [2], Section 5). This fact implies that $D_s^{(3)} \neq D^{(2)}$. □

Remark. The relation between the asymmetric and symmetric diffusion coefficients $D^{(3)}$ and $D^{(2)}$ is discussed in the context of (one-type particles) asymmetric simple exclusion in Section 5 of [8] and the references within. In that context $D^{(2)}$ is diagonal and becomes a multiple of the identity in the *isotropic* case, when the transition probabilities to the neighboring sites in the exclusion process are identical along all possible axes and not just direction-wise (which is the symmetric case). The two properties coincide in dimension $d = 1$. Only in this special situation one can derive that $D^{(3)} \geq const\, I$ in the sense of quadratic forms. The property is significant because it shows that the hydrodynamic limit exhibits diffusivity in excess of the one introduced by the random walk (Laplacian). Even though we are able to prove (5.12) we cannot derive that $[D^{(3)}]_{sym} \geq D^{(2)}$ except in *weak sense*, as in Theorem 2.4, meaning that there exist matrices Q (not necessarily symmetric) and V (symmetric) such that $D^{(3)}Q = D^{(2)}V$ and $V > Q_{sym}$.

The difficulties in our model come from two sources. First, we have two types of particles, which have distinct densities in equilibrium, henceforth the compressibility matrix $\chi(\varrho)$ (see Theorem 5.1) is diagonal but not proportional to the identity. Second, we do not have the option of proving the result in one dimension (like in the one-type particle models).

Under general conditions, without further knowledge of the properties of the matrices $Q = \langle \mathbf{T}\nabla\zeta, \nabla\zeta\rangle$ and $V = \langle \nabla\zeta, \nabla\zeta\rangle$, (5.12) *implies* $[D^{(3)}]_{sym} \geq D^{(2)}$ is false. In order to preserve the inequality sign between two matrices in the sense of quadratic forms we would need, for example, that the factors be commutative with the terms of the inequality, at a minimum that $QV = VQ$ in this case, which is not available to us.

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