OPTIMAL STRATEGY FOR THE VARDI CASINO WITH INTEREST PAYMENTS

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Abstract. A gambler starts with a fortune $f < 1$ and plays in a Vardi casino with infinitely many tables indexed by their odds $r \geq 0$. In addition, all tables return the same expected winnings $c < 0$ per dollar and a discount factor is applied after each round. We determine the optimal probability to reach fortune one, as well as an optimal strategy, different from bold play for fortunes larger than a critical value depending exclusively on c and the discount factor $1 + a$. The general result is computed explicitly for some relevant special cases. The question whether bold play is an optimal strategy is discussed for various choices of the parameters.

1. Introduction

The main result of the paper is Theorem 1 which finds the optimal probability $P_0(f)$ of reaching wealth at least one (non-extinction) when we start with wealth $0 \leq f \leq 1$ in a casino with a continuum range of odds $r \in \mathcal{R}$, at tables indexed by $r \geq 0$, with expected winnings per dollar at every table equal to $c \in (-1, 0)$ and inflation (or interest) rate $a > 0$. More precisely, a gambler starts off with wealth f and is allowed to stake at any table of the casino an amount s with restrictions $0 \leq s \leq f$ and $f + rs \leq 1 + a$. The latter condition can be ignored because as we show it is never violated for any optimizing strategy. Tables are indexed by their odds r, meaning that a stake s at table r is lost with probability $1 - w$ and, with probability w, returns rs if the gambler wins. In this paper, every time a game is played, the current wealth is discounted by a factor $(1 + a)^{-1}$, accounting for inflation (or interest rate). It is probably true that the optimal strategy is unique for $a > 0$ but for $a = 0$, this is false.

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What we mean by a *Dubins* (r, c) casino is the casino with only one table paying odds r and with expected payoff $c < 0$ on a dollar bet (also known as *subfair*). Dubins and Savage [6] consider the more general case where the casino has several tables, but they do not seem to have considered the casino proposed by Vardi [13] where a table, T_r, is available for every odds r and c is fixed and has the same negative value on all the tables. Such a casino will be called thereafter a *Vardi casino*, with or without interest, according to whether $a > 0$ or $a = 0$. This terminology was introduced in [12].

The expected payoff c is equal to $(+1)rw+(-1)(1-w) < 0$, which implies that throughout the paper $w = w(c, r) = (1 + c)/(1 + r)$. It thus provides an upper bound on any casino's optimal probability to reach fortune one if c is the largest expected return on any of the tables. It is shown by John Lou in a forthcoming thesis [9] that having all the additional tables and odds in the Vardi casino provides only a relatively small gain in the optimal probability to reach fortune one over that of the Dubins casino, which seems quite surprising.

All tables are independent, and all games at each table are independent of each other. More formally, let $\Omega = \{-1, 1\}^{\mathcal{R}\times\mathbb{Z}_+}$ with the σ-field $\mathcal{F}$ generated by cylinder functions and $\mathcal{F}_n$ the sequences of outcomes for all tables up to time n. Since $\Omega = \otimes\Omega_r$, where $\Omega_r = \{\omega(r, \cdot) \,|\, \omega \in \Omega\}$ are the projections of Ω, we denote the infinite product of Bernoulli measures P_r on Ω_r assigning probability $w(c, r) = (1 + c)/(1 + r)$ to $+1$ and $1 - w(c, r)$ to -1 for all n, and then we set $\mathbb{P} = \otimes_{r\in\mathcal{R}} P_r$.

A gambling strategy, or simply a *strategy* S is a sequence of measurable functions

$$S_n(\omega, \cdot) : [0, 1] \longrightarrow [0, 1] \times \mathcal{R}\,, \quad S_n(\omega, f) = (s, r)\,, \quad n \geq 0\,, \quad \omega \in \Omega \tag{1.1}$$

adapted to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$, that assign a pair $(s, r) = (s_n(\omega, f), r_n(\omega, f)) = S_n(\omega, f)$ to every value $f \in [0, 1]$ current at time n, with the *only* restriction

$$0 \leq s_n(\omega, f) \leq f. \tag{1.2}$$

The set of strategies will be denoted by $\mathcal{S}$.

In other words, S tells the gambler how much he should bet and at which table, for a given fortune, at a given time. More general strategies than Markovian strategies do not provide additional probability to reach one. Precisely, we show that all optimal strategies discussed are simply functions of f, and not of ω and n. In other words, all optimal strategies are Markovian. In the following, we shall omit ω in $S_n(\omega, f)$, and by abuse of notation we shall make the convention to omit the subscript n whenever $S_n(f)$ depends only on f.

We assume $\mathcal{R} = [0, \infty)$, but it is interesting to put the present results in the context of various other choices of $\mathcal{R}$. The classical result of Dubins and Savage [6] (see also [11] for more background of the problem) showed that when $a = 0$ and $\mathcal{R} = \{r\}$, the optimal strategy is *bold play*, more precisely $S_b(f) = f$ when $f \leq (1 + r)^{-1}$ and $S_b(f) = (1 - f)/r$ when $f > (1 + r)^{-1}$ (use the maximum bet allowed at any time). The bold play conjecture dating back to Coolidge [5] - see also [4] for more comments on this - is not valid for a Vardi casino without inflation, as shown in [12], where the parameters are $a = 0$ and $\mathcal{R} = [0, \infty)$. Non-optimality of bold play in a one-table casino in the presence of inflation is proved in earlier work [1, 2, 8] and in a different setting in [7]. In a primitive subfair casino with one table, satisfying the condition $1/r \leq a \leq r$, with $r > 1$, [3] shows that bold play is not optimal, yet it is conjectured that when $r < 1$ it is. A recent result [4] proves the conjecture under the additional assumption that $w \leq 1/2$.

The present article sheds some light on the interplay between various parameters defining the casino, and when bold play is optimal. We give a complete answer to the problem in the case $a > 0$ and $\mathcal{R} = [0, \infty)$. As anticipated from the preceding discussion, a dichotomy between bold and non-bold play regimes emerges, depending on the choice of parameters (c, a). There exists a value $\hat{f}$ such that the optimal strategy is strict bold play ($s(f) = f$) for $f \leq \hat{f}$, and a more cautious policy is required for $f > \hat{f}$ (see (4.4) for a concrete case). Some parameter combinations like $a^2 > |c|$ from Section 2 have $\hat{f} = 1$, allowing bold play only, in some sense concealing the nature of the general problem. Another remarkable feature of the solution is the presence of a jump at $f = 1$, meaning that as soon as $a > 0$, the extinction probability is bounded away from zero even as $f \to 1$, showing that the gambler cannot beat the inflation, even under optimal play. An interesting question is what effect would a random inflation rate have on the gap.

We believe that the present constructive approach based on the variational formulas (3.4) and (3.7) from Theorem 1 can settle the other cases, when various subsets $\mathcal{R}$ of $[0, \infty)$ are adopted for the definition of the casino.

Let ϕ denote a continuous function on $[0, 1]$. For a given strategy S, let $\{X_n^S\}_{n \geq 0}$ be the discrete time stochastic process representing the fortunes at times $n = 0, 1, \ldots$ under strategy S. The chain starts at $X_0^S = f \in [0, 1]$, is adapted to $\{\mathcal{F}_n\}_{n \geq 0}$, and satisfies the recurrence

$$(1.3)\quad \mathbb{E}(\phi(X^S_{n+1}) \,|\, X^S_n = g) = w(c, r_n(g))\phi(\frac{g + r_n s_n(g)}{1+a}) + (1 - w(c, r_n(g)))\phi(\frac{g - s_n(g)}{1+a})\,,$$

when $g \in (0,1)$ and equal to $\phi(g)$ if $g = 0$ or $g = 1$. We note that X_n *is not Markovian* in general, as the strategy S may take into account the whole past.

The chain $\{X^S_n\}$ is bounded above and below and is a super-martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ (Theorem 1), as a consequence of the subfair nature of the casino. With probability one, the limit $\lim_{n\to\infty} X^S_n$ exists and is denoted by X^S_∞. Let P^S_f be the probability to achieve the value one before extinction, starting with the initial fortune f, while applying the strategy S, and define

$$(1.4)\qquad P_0(f) = \sup_{S\in\mathcal{S}} P^S_f\,, \qquad P^S_f = \mathbb{P}(X^S_\infty = 1 \,|\, X^S_0 = f)\,.$$

Our goal is to determine $P_0(f)$ for any $f \in [0,1]$ and to formulate *at least one* strategy to achieve it.

Strict bold play is the strategy consisting of staking all the gambler's fortune for any $f \in (0,1)$. Of course this strategy is never optimal if a is small since it is foolish to exceed fortune one.

It is intuitively clear that for large a the player (i) will be forced to bet *all his wealth* f for all $f \in (0,1)$ (strict bold play), and (ii) the optimal strategy has a gap at one, i.e. $P_0(1-) < 1$. This behavior contrasts with the non-interest rate setting $a = 0$, where $P_0(f) = 1 - (1-f)^{1+c}$, proved in [12]. Section 2 presents the special case when a drops below the critical value $\sqrt{|c|}$ that imposes strategy (i). While this is covered by the general result from Theorem 1, it is shown directly in Proposition 1.

Section 3 proves the general result. The optimal probability $P_0(f)$ is defined via a variational formula (3.4) for $R(f) = 1 - P_0(f)$, and is shown to be convex (3.7). The discontinuity at $f = 1$ is consistent with the fact that $P_0(f)$ is lower semi-continuous.

For general parameters (a,c), the case $a^2 < |c|$ is more complex than $a^2 \geq |c|$ since $P_0(f)$ is not given explicitly, even though it is fully computable, technically speaking, not just numerically. This is because the infimum in (3.4) is a finite-dimensional problem, the number (3.2) of parameters $k \leq n(a,c) < \infty$, a constant dependent on a and c but not on f. For a relevant particular choice of parameters (a,c) such that $k = 1$, Section 4 derives explicit expressions (4.4) for $P_0(f)$ and the discontinuity at one.

Finally, section 6 provides an upper bound (6.1) for $P_0(f)$, equal to a smooth perturbation of the result of [12], which corresponds to $a = 0$. This approximation does not present the discontinuity at $f = 1$, making it useful in the intermediate range above the bold play range yet away from one for pairs (a, c) with large $n(a, c)$.

The gambler with current fortune f is allowed to exceed the value one before the discount $(1+a)^{-1}$ is applied by choosing to bet s dollars on a table with $1 + a - f \le rs$. However, if his strategy is optimal, then the strategy with $r = (1 + a - f)/s$ would be optimal as well, since r does not matter when the player loses, according to (1.3).

In addition, given that the gambler will stop either when his fortune reaches zero or achieves $f = 1$, the strategies can be defined arbitrarily at $f = 0$ and $f = 1$. We adopt the natural choice $s_n(0) = s_n(1) = 0$, $n \ge 0$ and r can be taken arbitrarily for fortunes $f = 0$ or $f = 1$, since the gambler does not actually play the next game. Notice that in the absence of inflation $a = 0$, it would be enough to specify that $s = 0$, whereas when $a > 0$ even passively waiting a turn and not playing sets back the fortune to $f/(1 + a)$. *Without loss of generality*, we shall assume throughout the paper that any strategy S satisfies

$$(1.5) \qquad 0 \le s_n(f) \le f\,, \qquad r_n(f)s_n(f) \le 1 + a - f\,, \qquad s_n(0) = s_n(1) = 0\,,$$

for any $n \ge 0$. For simplicity, we shall use the notation $p = (1 + c)/(1 + a)$.

2. The case $a^2 \ge |c|$.

We prove that the optimal probability of survival (1.4) is achieved by strict bold play $S_{sb}(f) = (f, (1 + a - f)/f)$ for all $f \in (0, 1)$ and is equal to

$$(2.1) \qquad P_0(f) = \begin{cases} pf\,, & \text{if} \quad f < 1 \\ 1\,, & \text{if} \quad f = 1 \end{cases} .$$

Proposition 1. *The function $P_0(f)$ satisfies (2.2).*

Proof. We have to prove

$$(2.2) \qquad P_0\Big(\frac{f + rs}{1 + a}\Big)\, w(c, r) + P_0\Big(\frac{f - s}{1 + a}\Big)\, (1 - w(c, r)) \le P_0(f)$$

when $f < 1$, $f + rs \le (1 + a)$, $0 \le s \le f$ and $r \ge 0$. When $f = 1$ the inequality is trivial, since $s = 0$.

1) If $(f + rs)/(1 + a) < 1$, the inequality is equivalent to $sc \le af$ which is evident since $c < 0$.

2) If $(f+rs)/(1+a)=1$, the inequality becomes

$$w+p\left(\frac{f-s}{1+a}\right)(1-w)\leq pf\,, \quad w=\frac{1+c}{1+r}\,. \tag{2.3}$$

Fix $r\geq 0$ and regard (2.3) as an inequality in f. The restriction $1+a=f+rs$ makes sense only if $r\geq a$. In addition, we must have $f\geq(1+a)/(1+r)$. Moving all the terms to the right hand side of the equality, the expression obtained is linear in f. It is sufficient to verify the inequality at the endpoints. The case $f=(1+a)/(1+r)$ is easy to verify. The case $f=1$ is

$$\frac{1+c}{1+r}+\Big(\frac{1+c}{1+a}\Big)\Big(\frac{1+r}{r(1+a)}-\frac{1}{r}\Big)\Big(1-\frac{1+c}{1+r}\Big)\leq\frac{1+c}{1+a}\,, \tag{2.4}$$

which is equivalent to $(r-a)(ra-|c|)\geq 0$. The restrictions on the parameters make $r\geq a$, concluding the proof. □

Remark. The range of r is indeed arbitrarily close to a for bold play, by taking $f=s=1-\epsilon$ and $r=(1+a-f)/f$.

Proposition 2. *Strict bold play, that is, for any $0<f<1$, betting the full wealth f on the table with odds $r=(1+a-f)/f$, achieves the probability of survival $P_0(f)$ from (2.1).*

Proof. We see by conditioning on the outcome of the first game that for all $f\in(0,1)$, the probability $P(f)$ of reaching wealth one when starting with wealth f under strict bold play satisfies

$$P(f)=\frac{1+c}{1+r}\,P\left(\frac{1+r}{1+a}f\right)=\left[\frac{1+c}{1+(1+a-f)f^{-1}}\right]P(1)=P_0(f)\,.$$

□

3. The general case

In the following, we shall use the notation $\eta=(a+a^2)/(a+|c|)$.

Proposition 3. *Let $\hat{f}$ be the largest value of $f\leq 1$ for which (2.2) is satisfied for any admissible choice of s, r in the sense of (1.5). Then $\hat{f}=1$ when $a^2\geq|c|$, and $\hat{f}=\eta$ when $a^2<|c|$.*

Remark. The value r' is the critical value from (2.4) and (3.1). With the restrictions from (1.5), the critical values can be achieved, for example when $r'=|c|/a$ and $s'=\eta$, under bold play.

Proof. Proposition 1 proved that if $a^2 \geq |c|$, then (2.2) is satisfied for all f, up to f equal to one, showing that $\hat{f} = 1$ in this case.

If $a^2 < |c|$, following the steps of Proposition 1, we see that (2.2) is satisfied automatically when $(f + rs)/(1 + a) < 1$. When $f + rs = 1 + a$, (2.2) becomes (2.3). The restrictions (1.5) imply that $f \geq (1 + r)/(1 + a)$. The easiest way to check this is to plot (s, t) with all other parameters fixed and see that the domain $0 \leq s \leq f$, $f + rs \leq 1 + a$ has vertices $(0, 0)$, $(0, 1 + a)$ and $((1 + r)/(1 + a), (1 + r)/(1 + a))$. We re-write (2.3) as

$$\frac{ra - |c|}{r(1 + a)} \left(f - \frac{1 + a}{1 + r} \right) \geq 0\,. \tag{3.1}$$

If $f \leq \eta$, $1 + r' = (1 + a)/\eta \leq (1 + a)/f \leq 1 + r$, so (3.1) is satisfied, showing that $\hat{f} \geq \eta$. However, when $f > \eta$, there exist admissible r such that (3.1) is not satisfied, implying that $\hat{f} \leq \eta$. □

Propositions 1, 2 and 3 suggest that there must be two regimes of play, according to whether the current fortune f is above or below the critical value $\hat{f} = \eta$. Assuming that we start with $f > \eta$, we shall look at sequences of descending fortunes f_j obtained for consecutive unsuccessful bets. In general, the only restriction is $(1 + a)f_j = f_{j-1} - s_{j-1} \leq f_{j-1}$. Once the fortune drops below η, intuitively we know that the optimal strategy is bold play. Finally, the optimal strategy is obtained by optimizing over all scenarios (descending sequences) leading to a fortune below η. We formalize these ideas starting with a definition.

Definition 1. *Let $f \in [0, 1)$. A descending sequence of length $k + 1$ for f is a sequence f_j, $j = 0, 1, \ldots, k$ such that (i) $f_0 = f$, (ii) k=0 if $f \leq \eta$, (iii) $f_j \leq (1 + a)^{-1} f_{j-1}$, $j = 1, \ldots, k$, and (iv) $f_{k-1} > \eta$ while $f_k \leq \eta$. Such a sequence will be denoted $\{f\}$, the set of descending sequences by $D(f)$ and the set of descending sequences of length k by $D_k(f)$.*

A descending sequence has finite length for any f. The maximum admissible length k is bounded above by

$$n(a, c) = -\ln \eta / \ln(1 + a) + 1 = \frac{\ln(a + |c|) - \ln(a + a^2)}{\ln(1 + a)} + 1\,, \tag{3.2}$$

a constant depending exclusively on (a, c). We recall that $p = (1 + c)/(1 + a)$.

For every f and every $\{f\} \in D_k(f)$ we construct the function

$$R^{\{f\}}(f) = (1 - pf_k)\Pi_{j=0}^{k-1}\Big(1 - p\frac{f_j - (1 + a)f_{j+1}}{1 - f_{j+1}}\Big)\,, \tag{3.3}$$

with the convention $R^{\{f\}}(f) = (1-pf)$ for $f \le \eta$, which is consistent with Definition 1.

We notice that $0 \le R^{\{f\}}(f) \le 1$ and define the function $P_0(f)$ by $P_0(f) = 1$ for $f = 1$ and for $f < 1$

$$R(f) = \inf_k \inf_{\{f\} \in D_k(f)} R^{\{f\}}(f)\,, \qquad P_0(f) = 1 - R(f)\,. \tag{3.4}$$

The infimum is achieved at least for a certain $k = k(f)$ and a certain $\{f\} \in D_k(f)$ because $k \le n(a,c)$ has a finite range independent of f, the functions $R^{\{f\}}(f)$ of variable equal to the vector $\{f\}$ are continuous, and for each fixed k, the domain where $\{f\} = f_0, f_1, \dots, f_k$ is defined is compact (depending on f). For each f, we choose one of the minimizing sequences of (3.4) and denote it by $\{f\}^-$. Then evidently $R(f) = R^{\{f\}^-}(f)$.

Proposition 4. *The function $P_0(f)$ is convex in f.*

Proof. We have to show that $R(f)$ is concave. As defined in (3.4), the infimum is taken over a set depending on f itself; we shall write it in a form that shows clearly that $R(f)$ is the infimum of a family of linear functions over a set independent of f. For a pair (f_j, f_{j+1}), let r_j and γ_j be defined by the equality

$$\gamma_j = \frac{f_j - (1+a)f_{j+1}}{1 - f_{j+1}} = \frac{1+a}{1+r_j}\,.$$

We can interpret $f_{j+1} = (f_j - s_j)/(1+a)$ as the result of losing a bet s_j at the table r_j chosen such that a winning bet would have brought the fortune to exactly one, that is $(f_j + r_j s_j)/(1+a) = 1$ (bold play). Then

$$r_j = \frac{1+a-f_j}{s_j} \ge \frac{1+a-f_j}{f_j} \ge a\,.$$

With this in mind, the sequence γ_j defines a finite number of parameters in $(0,1]$, without other restrictions depending on f.

To ease the computation, let $\alpha_j = (1+a-\gamma_j)^{-1}$ and $\beta_j = \gamma_j \alpha_j$. Then

$$\begin{aligned} f_1 &= \alpha_0 f_0 - \beta_0 \\ &\cdots \\ f_{k-1} &= \alpha_{k-2} f_{k-2} - \beta_{k-2} \\ f_k &= \alpha_{k-1} f_{k-1} - \beta_{k-1} \end{aligned} \tag{3.5}$$

and

$$f_k = Af - B\,, \qquad A = \Pi_{j=0}^{k-1}\alpha_j\,, \quad B = \sum_{j=0}^{k-1}\beta_j \Pi_{i=j+1}^{k-1}\alpha_i\,. \tag{3.6}$$

Re-casting (3.4) in terms of the independent $\{r_j\}_{0\le j\le k-1}$, respectively $0<\gamma_j\le 1$, and f, we have

$$R(f) = \inf_k \inf_{\{\gamma\}\in(0,1]^k} \Big(1 + pB - pAf\Big)\,\Pi_{j=0}^{k-1}\Big(1 - p\gamma_j\Big)\,, \tag{3.7}$$

where the first factor is a linear function of f as in (3.6). This proves that $R(f)$ is concave (see [10]). $\square$

For every $\{f\}\in D(f)$, we define the sequence of bets $\{s\} = s_0, s_1, \ldots, s_k$ as follows, according to the sequence length $k(f)$: if $k(f)=1$, then $s_0 = f_0$, and if $k(f)>1$, $s_j = f_j - (1+a)f_{j+1}$, $j = 0,\ldots,k-1$, with $s_k = f_k$. We denote $\{s\}^-$ the sequence corresponding to $\{f\}^-$.

Definition 2. *Let S^- be a strategy defined as follows. When $f=0$ or $f=1$, we stop. When $0<f<1$, we have two alternatives.*

1) If $f\le\eta$ we bet $s=s(f)=f$ on the table with $r=r(f)=(1+a-f)/s(f)$; if we win, we have reached one and stop, and if we lose, we stop as well, since $f-s(f)=0$.

2) If $f>\eta$ we bet $s(f)=f_0-(1+a)f_1^-$ on the table with $r=r(f)=(1+a-f)/s(f)$; if we win, we stop, and if we lose, we proceed by betting $s(f_1^-)=s_1^-$ and so on, until we either win or reach $f_{k(f)}^-\le\eta$, when we go to 1).

Theorem 1. *The optimal probability of reaching one when we start with wealth $0\le f\le 1$ is equal to $P_0(f)=1-R(f)$ if $f<1$ and $P_0(f)=1$ if $f=1$, and is realized by the strategy S^-. In addition, $1-P_0(1-)\ge(1-p)^{n(a,c)+1}>0$, where $n(a,c)$ is the bound in (3.2).*

The theorem will be proved in three steps: Proposition 5 shows that strategy S^- realizes $P_0(f)=1-R(f)$, Proposition 6 shows that $P_0(f)$ is an upper bound for the probability to reach one, and finally we prove the lower bound for the discontinuity at one.

Proposition 5. *The probability of survival $P_f^{S^-}$ defined in (1.4), starting from $f<1$ and corresponding to the strategy S^- from Definition 2, is equal to $P_0(f)$.*

Proof. Let $R^-(f) = 1 - P_f^{S^-}$. Let $f = f_0$ and let f_1^- be the wealth in case of loss while applying S^- (in one step). By conditioning upon the events of winning/losing in the first play, the law of total probability and (1.3) applied to the chain $X_n^{S^-}$ give that $R^-(f)$ satisfies the recurrence

$$R^-(f_0) = \Big(1 - p\frac{f_0 - (1+a)f_1^-}{1 - f_1^-}\Big)R^-(f_1^-)\,. \tag{3.8}$$

To see this, we remember that

$$\frac{f_0 - (1+a)f_1^-}{1 - f_1^-} = \frac{1+a}{1+r_1^-}\,, \qquad r_1^- = r(f_0)\,,$$

according to Definition 2. Notice that in the case $f \le \eta$, (3.8) leads trivially to equality between $R^-(f)$ and $R(f)$, since $f_1^- = 0$. When $f > \eta$, relation (3.8) is satisfied by $R(f)$ once again, as seen in (3.3) applied to the optimizing sequence. We can re-iterate the same reasoning for f_2^-, f_3^-, ... and so on to see that $R^-(f) = R(f)$. Alternatively, if $\{f\}^-$ is an optimal descending sequence for f and f_1^- is the second term in the sequence, then the truncated sequence $\{f_1^-\} = f_1^-, f_2^-, \dots, f_k^-$, that is, the same sequence shifted by one unit, is an optimal descending sequence for f_1^-. This fact is clear by construction. We have shown that $R^-(f) = R(f)$. $\square$

Proposition 6. *For any given (a,c) and any compatible set (f,s,r) in the sense of (1.5), $0 \le f < 1$, $0 \le s \le f$, $r \ge 0$, the function P_0 from (3.4) satisfies the inequality*

$$P_0(\frac{f+rs}{1+a})w + P_0(\frac{f-s}{1+a})(1-w) \le P_0(f)\,. \tag{3.9}$$

We note that $f = 1$ implies $s = 0$ and (3.9) is trivial.

Proof. By construction, the function $P_0(f)$ is convex, being the supremum over linear functions in f according to Proposition 4. As functions of s, both $P_0((f+rs)/(1+a))$ and $P_0((f-s)/(1+a))$ are convex, so the left-hand side of (3.9) is convex in s. The maximum can only be achieved at extreme values of s. Given the restrictions on s,

(1) if $f > (1+a)/(1+r)$, the extreme values are $s = 0$ and $s = (1+a-f)/r$, and

(2) if $f \le (1+a)/(1+r)$, the extreme values are $s = 0$ and $s = f$.

(i) Suppose $s = 0$, for both (1) and (2). If $f \le \eta$ we are back to (2.2), and the proof is the same. If $f > \eta$, then $f_0 = f$ and $f_1 = f/(1+a)$ can be seen as the first two admissible terms of a descending sequence $\{f\}$. By construction, $P_0(f_1) \le P_0(f_0)$.

(ii) Suppose $s = (1+a-f)/r$ in (1). This corresponds to bold play, that is, r is such that we reach one if we win. We have to prove that $R(f) \le R((f-s)/(1+a))(1-w)$. Writing $f_0 = f$ and $f_1 = (f-s)/(1+a)$, we have the equivalent inequality

$$R(f) \le \Big(1 - p\frac{f_0 - (1+a)f_1}{1-f_1}\Big)R(f_1)\,,$$

which is immediate by construction.

(iii) Suppose $s = f$ in (2). We reduced the problem to showing that $R(f)$ is less or equal to $R(f(1+r)/(1+a))w + (1-w)$ or equivalently,

$$P_0(\frac{1+r}{1+a}f)\,w \le P_0(f)$$

for all pairs (f, r) satisfying $0 \le r \le (1+a-f)/f = (1+a)/f - 1$ (since $s = f$). Re-write the desired inequality as

$$(1+c)P_0(\frac{1+r}{1+a}f) - (1+r)P_0(f) \le 0\,.$$

Let f be fixed. As a function of r, the left hand side is convex. The maximum is achieved at one of the endpoints. At $r = 0$ one has

$$P_0(f) \ge P_0(\frac{1}{1+a}f)(1-|c|)\,,$$

which is weaker that $P_0(f) \ge P_0(\frac{1}{1+a}f)$, or $R(f) \le R(\frac{1}{1+a}f)$. This inequality is true for any $f < 1$ by construction, adopting $f_0 = f$ and $f_1 = f/(1+a)$ as the first two terms of a descending sequence for f if $f > \eta$, and simply by direct verification when $f \le \eta$. Finally, at the upper end point $r = (1+a)/f - 1$, which implies that $f(1+r)/(1+a) = 1$, the inequality becomes $R(f) \le (1 - pf)$. This is true by construction by picking $f_0 = f$, $f_1 = 0$ as descending sequence. $\square$

Proof of Theorem 1. Denote $\{X_n^S\}_{n\ge0}$ the values at times $n = 0, 1, \ldots$ of the player's wealth under an admissible strategy $S \in \mathcal{S}$ defined in (1.3). We drop the superscript S since there is no possibility of confusion. In other words, at time n, the player chooses a stake and a table corresponding to the current value of its fortune X_n according to $\{(s_n(X_n), r_n(X_n))\}_{n\ge0}$ with the convention that X_n stays at zero (or one) once it has reached it for the first time. Moreover, since $\{X_n\}$ is bounded by one, it is easy to check that it is a super-martingale. The fact that P_0 is bounded and inequality (3.9) show that $P_0(X_n)$ is also a super-martingale. The limit X_∞ of X_n as $n \to \infty$ exists almost surely. Since $P_0(0) = 0$, $P_0(1) = 1$ and $\mathbf{1}_{[1,\infty)}(x) \le P_0(x)$, then

$$\mathbb{P}(X_\infty = 1 \,|\, X_0 = f) \le \mathbb{E}[P_0(X_\infty) \,|\, X_0 = f] \le \mathbb{E}[P_0(X_0) \,|\, X_0 = f] = P_0(f)\,.$$

Meanwhile, Proposition 5 shows that $\mathbb{P}(X_\infty = 1 \,|\, X_0 = f) \ge P_0(f)$, by applying strategy S^-.

Finally, it remains to show that $P_0(1-) < 1$, or equivalently that $R(1-) > 0$. Note that the product (3.3) has at most $k+1$ factors, each bounded below by $(1-p)$, and k is bounded above by a value depending on a and c only, which concludes the proof.

4. Explicit results when $k(f) \le 1$.

When $a^2 \le |c| \le 2a^2 + a^3$, we shall see that the descending sequence from Definition 1 has length $k \le 1$ and an explicit form of the optimum function $P_0(f)$ can be derived. In fact, this inequality between a and c is equivalent to having $k(f) \le 1$ for all f. Let $f_0 = f$ and $f_1 < f_0$ such that $f_1 \le (1+a)^{-1} f_0$. It is easy to see that $f_1 \le \eta$ if $2a^2 + a^3 \ge |c|$, for any initial f. On the other hand, let's assume that for any f, the second term $f_1 \le \eta$. We want to prove that $2a^2 + a^3 \ge |c|$. Since $s = f_0 - (1+a) f_1$, we introduce r, the table where we bet under $f + rs = 1 + a$ (bold play), and obtain,

$$f_1 = \frac{1+r}{r(1+a)} f_0 - \frac{1}{r}\,. \tag{4.1}$$

The condition is equivalent to

$$\frac{1+r}{r(1+a)} f - \frac{1}{r} \le \frac{a+a^2}{a+|c|}\,,$$

which reduces to

$$\frac{1}{r} \ge \frac{(1+a)(|c| - 2a^2 - a^3)}{a(a+|c|)(1+a)}\,, \tag{4.2}$$

satisfied by any r as long as

$$a^2 \le |c| \le 2a^2 + a^3\,. \tag{4.3}$$

The left hand side of the inequality is not required for the strategy, but was included to underscore the interval where the r is located.

Proposition 7. *If $a^2 < |c| \le 2a^2 + a^3$, then any descending sequence has $k(f) \le 1$ and the optimal probability of non-extinction is*

$$P_0(f) = \begin{cases} pf & \text{if} \quad f \le \eta \\ 1 - \left(\sqrt{|c|(1-p)} + p\sqrt{1+a-f}\right)^2 & \text{if} \quad \eta < f < 1 \\ 1 & \text{if} \quad f = 1 \end{cases} . \tag{4.4}$$

In addition, $1 - P_0(1-) = [\sqrt{|c|(1-p)} + p\sqrt{a}]^2 > 0$ and $P_0(f)$ has continuous derivative for $0 \le f < 1$.

Proof. We want to minimize $R(f) = 1 - P(f)$

$$R(f) = \inf_{f_1} \Big(1 - p\frac{f - (1+a)f_1}{1 - f_1}\Big)\Big(1 - pf_1\Big) = \inf_{f_1} U(f_1)\,, \tag{4.5}$$

where $0 \le f_1 \le (1+a)^{-1} f$.

The function $U(\cdot)$ in f_1 to be minimized is convex on the interval of interest $[0,1)$. We recall that strict bold play, when we bet $s = f$ for all f, corresponds to realizing the minimum at $f_1 = 0$ for all f, which we shall see is not the case. Re-writing,

$$U(x) = |c|(1 - px) + p(1-p)(1+a-f)(1-x)^{-1} + p^2(1+a-f)$$

so

$$U'(x) = p(1-p)(1+a-f)(1-x)^{-2} - p|c|\,,$$

and

$$U'(0) = p\frac{a + |c|}{1+a}\Big(\frac{a+a^2}{|c|+a} - f\Big)\,, \qquad U'(f/(1+a)) > 0\,.$$

As long as $f > \eta = (a+a^2)/(a+|c|)$, the derivative $U'(0) < 0$, showing that the minimizer $x = f_1^-$ is in $(0, f/(1+a))$. The exact value is

$$f_1^-(f) = 1 - \sqrt{\Big(1 - \frac{f}{1+a}\Big)\Big(1 + \frac{a}{|c|}\Big)}\,, \tag{4.6}$$

providing the exact strategy

$$s(f) = \begin{cases} f - (1+a)f_1^-(f)\,, & \text{if} \quad f > (a+a^2)/(a+|c|) \\ f\,, & \text{if} \quad f \le (a+a^2)/(a+|c|) \end{cases} . \tag{4.7}$$

It is easy to verify the value of the jump discontinuity at $f = 1$ and the equality of the one-sided limits at η. $\square$

5. **Calculations for** $k(f) = 2$.

To gain some insight in the computational difficulty of the case when the descending sequence used in the expressions (3.3)-(3.4) involves more than two terms, we investigate the simplest case when $k(f) = 2$ (the sequence length is at least three).

We have to evaluate

$$\inf_{f_1, f_2} \left(1 - p\frac{f - (1+a)f_1}{1 - f_1}\right)\left(1 - p\frac{f_1 - (1+a)f_2}{1 - f_2}\right)(1 - pf_2) \tag{5.1}$$

with $f_1 \le (1+a)^{-1}f$ and $f_2 \le (1+a)^{-1}f_1$. An alternative expression based on (3.7) is

$$\inf_{(\gamma_0,\gamma_1)\in(0,1]\times(0,1]} \left[\frac{(1+a-\gamma_0)(1+a-\gamma_1) + \gamma_0 - pf}{(1+a-\gamma_0)(1+a-\gamma_0)}\right](1 - p\gamma_0)(1 - p\gamma_1) \tag{5.2}$$

but we proceed with (5.1). We are interested in nontrivial (interior) critical points (f_1', f_2') of the function (5.1) in the variables (f_1, f_2), since the boundary cases correspond to $k \le 1$. We make the observation that if the function of $x \in [0, 1)$

$$x \quad \longrightarrow \quad \left(1 - p\frac{g - (1+a)x}{1 - x}\right)(\rho - x)\,, \quad \rho \ge 1\,, \tag{5.3}$$

has a nontrivial critical point $x' \in (0, 1)$, then that is

$$x' = 1 - \sqrt{\Big(\rho - 1\Big)\Big(\frac{1}{|c|} - 1\Big)\Big(1 - \frac{g}{1+a}\Big)}\,. \tag{5.4}$$

Fixing f_2 in (5.1) and applying (5.3)-(5.4) with $g = f$, $\rho = (1 + cf_2)p^{-1} \ge 1$, and $x = f_1$, and then again fixing f_1 in (5.1) and applying (5.3)-(5.4) with $g = f_1$, $\rho = p^{-1} \ge 1$ and $x = f_2$, we obtain the system

$$\begin{aligned} 1 - f_1' &= \sqrt{\Big(1 + \frac{a}{|c|} - (1+a)f_2'\Big)\Big(1 - \frac{f}{1+a}\Big)} \\ 1 - f_2' &= \sqrt{\Big(1 + \frac{a}{|c|}\Big)\Big(1 - \frac{f_1'}{1+a}\Big)}\,, \end{aligned}$$

equivalent to finding a real zero of a polynomial of degree four.

6. **An upper bound**

An upper bound for the probability of success is

$$P_1(f) = 1 - (1 - f)^{\frac{1+c}{1+a}}\,, \tag{6.1}$$

for any initial fortune $0 \le f \le 1$.

Remark. We note that $p = (1+c)/(1+a)$ is such that $0 < p < 1$, and any function $1-(1-f)^{p'}$ with $p' \geq p$ provides an upper bound.

Proposition 8. *Let $\{X_n^S\}_{n\geq 0}$ be the chain describing the evolution of the gambler's fortune, defined by (1.3), with initial value $X_0 = f$. Then, for any strategy $S \in \mathcal{S}$, the process $\{P_1(X_n)\}_{n\geq 0}$ is a super-martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$.*

Proof. Recall that the winning probability $w(c,r)$ is equal to $(1+c)/(1+r)$. The reasoning is identical to that leading to the upper bound from Theorem 1, obtained via Proposition 6. We have to prove the analogue of inequality (3.9) for the utility function (6.1), that is

$$P_1\Big(\frac{f+rs}{1+a}\Big)\, w(c,r) + P_1\Big(\frac{f-s}{1+a}\Big)\,(1-w(c,r)) \leq P_1(f) \tag{6.2}$$

for any $0 \leq f \leq 1$, $0 \leq s \leq f$, $f+rs \leq 1+a$ and $r \geq 0$. The inequality is equivalent to

$$\Big(1-\frac{f+rs}{1+a}\Big)^p \Big(\frac{1+c}{1+r}\Big) + \Big(1-\frac{f-s}{1+a}\Big)^p \Big(1-\frac{1+c}{1+r}\Big) \geq (1-f)^p\,. \tag{6.3}$$

We think of (6.3) as a function of (s,f) with fixed a and r.

For fixed f, the left hand side of (6.3) is a concave function of s, which shows that it is sufficient to check its values at the endpoints. We write

$$\psi(s) = \Big(1-\frac{f}{1+a}-\frac{r}{1+a}s\Big)^p w(c,r) + \Big(1-\frac{f}{1+a}+\frac{1}{1+a}s\Big)^p (1-w(c,r))$$

as

$$(c_1 - c_2 rs)^p w + (c_1 + c_2 s)^p (1-w)$$

with $c_1 = 1 - f/(1+a)$ and $c_2 = 1/(1+a)$. The derivative in s

$$p(c_1 - rc_2 s)^{p-1}(-c_2 r)w + p(c_1 + c_2 s)^{p-1} c_2 (1-w)$$

is decreasing, proving that $\psi(s)$ is concave.

It remains to verify the inequality (6.3) at the endpoints of the interval where s is compatible with (1.5). The restrictions on (s,f) impose the two cases: (1) when $a < r$ and (2) when $a \geq r$.

Case (1) is further split into (1.1) when $f \geq (1+a)/(1+r)$, implying that $0 \leq s \leq (1+a-f)/r$, and (1.2) when $f \leq (1+a)/(1+r)$, implying that $0 \leq s \leq f$.

Case (1.1), (1.2), (2) at $s=0$. The inequality is trivial in this case.

Case (1.1) We have to check (6.3) at $s = (1 + a - f)/r$, which is

$$\Big(1 - \frac{f - (1 + a - f)r^{-1}}{1 + a}\Big)^p (1 - w) \geq (1 - f)^p \,.$$

The inequality is trivially true for $f = 1$. Divide by the right hand side, and we have to show

$$(1 + \frac{1}{r})^p \Big(\frac{1 - \frac{f}{1+a}}{1 - f}\Big)^p (1 - w) \geq 1 \,.$$

The function in f is increasing, hence we need to prove the inequality at $f = (1+a)/(1+r)$,

$$(1 - \frac{1 + c}{1 + r}) \geq \Big(1 - \frac{1 + a}{1 + r}\Big)^p$$

for $r \geq 0$. This is a consequence of $(1 - px) \geq (1 - x)^p$ for all $x \geq 0$, with $x = 1/(1 + r)$ and $p = (1 + c)/(1 + a)$.

Case (1.2) We have to check (6.3) at $s = f$, i.e.

$$\Big(1 - (\frac{1 + r}{1 + a})f\Big)^p w + (1 - w) - (1 - f)^p \geq 0 \tag{6.4}$$

when $f \leq (1 + a)/(1 + r)$. One can see that the derivative of the function in f on the left hand side changes sign only once in the interval, from a positive to a negative value. This shows that the minima are to be found at $f = 0$ or $f = (1 + a)/(1 + r)$. It is sufficient to verify

$$(1 - \frac{1 + c}{1 + a}) \geq \Big(1 - \frac{1 + a}{1 + r}\Big)^p ,$$

true as shown above.

Case (2) In this case we have to check (6.3) at $s = f$, as in (1.2), which means to verify (6.4) on $0 \leq f \leq 1$, and again, with the same reasoning, the values at $f = 0$ and $f = 1$. At $f = 0$ we obtain zero and at $f = 1$ we have

$$\Big(1 - \frac{1 + r}{1 + a}\Big)^p w + (1 - w) > 0 \,.$$

$\square$

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References

[1] Chen, R. W. (1977). *Subfair primitive casino with a discount factor.* Z. Wahrscheinlichkeitsth. **39**, 167-174.

[2] Chen, R. W. (1978). *Subfair 'red-and-black' in the presence of inflation.* Z. Wahrscheinlichkeitsth. **42**, 293-301.

[3] Chen, R. W.; Shepp, L.A.; Zame, A. (2004). *A bold strategy is not always optimal in the presence of inflation.* J. Appl. Prob. **41**, 587-592.

[4] Chen, R.W.; Shepp, L. A.; Yao, Y.; Zhang, C. (2005). *On optimality of bold play for primitive casinos in the presence of inflation.* J. Appl. Prob. **42**, 121-137.

[5] Coolidge, J.L. (1909). *The gambler's ruin.* Ann. Math. (2) **10**, 181-192.

[6] Dubins, L.E.; Savage, L.J. (1965). *How To Gamble If You Must, Inequalities for Stochastic Processes.* McGraw-Hill, New York.

[7] Heath, D.C.; Pruitt W. E.; Sudderth, W. D. (1972). *Subfair red-and-black with a limit.* Proc. Amer. Math. Soc. **35**, 555-560.

[8] Klugman, S. (1977). *Discounted and rapid subfair red-and-black.* Ann. Statist. **5**, 734-745.

[9] John Lou, *Paper in preparation.*

[10] Rockafellar, R. T. *Convex analysis.* Princeton University Press, Princeton, NJ, 1997.

[11] Maitra, A. P.; Sudderth, W. D. *Discrete gambling and stochastic games.* Applications of Mathematics (New York), 32. Springer-Verlag, New York, 1996.

[12] Shepp, L. A. *Bold play and the optimal policy for Vardi's casino*, in "Random Walks, Sequential Analysis and Related Topics" Chao Agnes Hsiung, Zhiliang Ying and Cun-Hui Zhang Eds., World Scientific, Singapore, 2006.

[13] Vardi, Y. *Private communication.*

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