RECURRENCE AND ERGODICITY FOR A CONTINUOUS AIMD MODEL

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Abstract. A scaled version of the general AIMD model of transmission control protocol (TCP) used in internet traffic congestion management leads to a Markov process $x(t)$ representing the time dependent data flow that moves forward with constant speed on the positive axis and jumps backwards to $\gamma x(t)$, $0 < \gamma < 1$ according to a Poisson clock whose rate $\alpha(x)$ depends on the interval swept in between jumps. Under very general condition that α is bounded above and away from zero, we show that the invariant measure is unique, has a bounded density, and the process is exponentially ergodic via the local Doeblin condition on general state spaces. When α is constant, an explicit formula for the invariant measure is provided together with estimates on the exponential moments of the first return time.

1. Introduction

Let (Ω, Σ, P) be a probability space and $\{x_t\}_{t\geq 0}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on Σ. In the following we study the time-homogeneous one particle process $\{x_t\}_{t\geq 0}$ with state space $(0,\infty)$ solving the martingale problem with generator $(B, \mathcal{D})$

$$B\phi(x) = \nabla\phi(x) + \alpha(x)(\phi(\gamma x) - \phi(x)), \qquad \phi \in \mathcal{D} = C_b^1((0,\infty)), \tag{1.1}$$

on the $C_b^0((0,\infty))$. Here $\gamma \in (0,1)$, $\alpha(x)$ is a measurable function (later on, α will be assumed bounded) and $C_b^k((0,\infty))$ is the space of functions with k continuous derivatives up to the boundary of $(0,\infty)$. This simple dynamics is the scaled version [13, 8, 12, 9] of an additive increase multiplicative decrease (AIMD) process modeling the traffic flow in internet congestion control [6, 7, 1, 9, 12]. Let τ_i', $i = 0, 1, \ldots$ be a non-decreasing sequence of random times representing the *loss events* (when packets of data are lost).

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Between two consecutive loss events (jumps) $\tau'_{i-1} \le t < \tau'_i$, the transmission rate (also known as *congestion window* or *cwnd*) $x(t)$ increases at constant speed one. Once the volume $\int_{\tau'_{i-1}}^t \alpha(x(s))ds$ reaches a random quantity $\chi_i > 0$ as $t \uparrow \tau'_i$, the rate falls back to $\gamma x(\tau'_i-)$. This mechanism leads to a Markov process by adopting a sequence of i.i.d. exponential r.v. χ_i, $i = 1, 2, \ldots$ and repeating the construction for $\tau'_0 = 0$, $x(0) = x_0 > 0$, $\tau'_i = \inf\{t > \tau'_{i-1} \mid \int_{\tau'_{i-1}}^t \alpha(x(s))ds > \chi_i\}$, $i = 1, 2, \ldots$. We notice that if i is the first rank such that $\tau'_i = \infty$, as it may happen if $\alpha(x)$ vanishes, the process remains well defined, following a deterministic motion with speed one for all $t \ge \tau'_{i-1}$. On the other hand, if $\alpha(x)$ becomes arbitrarily large, there is the possibility that the process finishes in finite time.

This paper is focussed on recurrence and ergodic properties of the process. The simplest yet most enlightening case is when $\alpha(x)$ is constant, when Theorem 3 gives the explicit form of the invariant measure (4.5)-(4.6). Formulas (4.2)-(4.3) and (4.6) imply that (1) the convergence to the invariant measure is exponential and (2) that the convergence depends on the initial state x_0, as reflected in the constant factor $V(x)$ in (2.2), whereas the rate ρ is independent of x. Proposition 8 shows that the *return times* have a finite exponential moment.

It is intuitively clear that as α approaches zero, the time before a jump backwards becomes very large and the particle escapes, at constant speed, to infinity. Evidently, there is no proper equilibrium probability measure (the point at infinity is a cemetery). At the other extreme, when α can be arbitrarily large, the particle jumps backwards very often and may be trapped in a neighborhood of the origin. In this case the process becomes 'transient' and the invariant measure becomes again trivial. Thus we assume that there exist positive constants α_0 and $||\alpha||$ such that

$$0 < \alpha_0 \le \alpha(x) \le ||\alpha|| < \infty \,, \qquad \forall x \in (0, \infty) \,. \tag{1.2}$$

Our setup is natural and concentrates on the convergence to equilibrium when the loss rate α is a function of x, in other words when the rate of marking/dropping is a function of the share of the bandwidth associated to the connection (see [2, 5] for the explicit formulas of the invariant measure when $\alpha(x)$ is a power of x). Theorem 1 is the main result of the paper, and, to our knowledge, the first rigorous AIMD result not based on the explicit computation of the invariant measure, thus valid for a general rate function $\alpha(x)$.

There is a natural relation $\rho \leq 1 - c$ between the constant c in the definition of a Doeblin set (Definition 1) and the geometric rate of convergence ρ from (2.2)(cf. [11, 4]). Theorem 1 provides a concrete estimate of the speed of convergence to equilibrium, which is essential in simulation and in justifying the *mean-value analysis* approach [13] which is used massively in the engineering literature.

In general, the analysis of continuous time Markov processes on general state spaces is much more delicate than in the countable state case and solid references are less readily available. We found useful to provide a brief review of the so called *local Doeblin theory* in Section 2, based mainly on [3] (for the continuous time case) and, to a lesser extent on [4, 10, 11].

Theorem 1. *If α satisfies (1.2), then the time homogeneous process with generator (1.1) has a unique invariant probability measure which is absolutely continuous with respect to the Lebesgue measure $l(dx)$, and is exponentially ergodic in the sense of (2.2).*

The proof of the theorem is postponed to subsection 3.3.

2. Local Doeblin theory for continuous time processes

We start with the introduction of the basic concepts relevant to the theory of continuous time Markov processes on general state spaces.

Let $\{x(t)\}_{t\geq 0}$ be a continuous time non - explosive Markov process on the state space $\mathcal{S}$ with Borel sets $\mathcal{B}(\mathcal{S})$. For a Borel set A, $\tau_A = \inf\{t \geq 0 \,|\, x(t) \in A\}$ is the first hitting time of A and if $x \in \mathcal{S}$

$$G(x, A) = E_x\Big[\int_0^\infty \mathbf{1}_A(x(t))dt\Big] \tag{2.1}$$

denotes the Green function associated to the process.

Definition 1. *(Local Doeblin condition and small sets) A Borel subset F in the state space $\mathcal{S}$ of the Markov process will be said*

(i) attractive, if $P_x(\tau_F < \infty) = 1$, for any $x \in \mathcal{S}$, and

(ii) small, if there exists a time $t > 0$, a probability measure $\nu_0(dx)$ concentrated on F, and a constant $c \in (0, 1)$ such that, for all $x \in F$ and all Borel sets B of $\mathcal{S}$, we have $P_x(x(t) \in B) \geq c\,\nu_0(B)$.

A set F satisfying (i)-(ii) is also called an *attractive Doeblin* set.

Definition 2. *Given ϕ a measure on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, a process is said ϕ - irreducible if for any Borel set B of $\mathcal{S}$ with $\phi(B) > 0$, then $G(x, B) > 0$ for any $x \in \mathcal{S}$. We also say that ϕ is an irreducibility measure. The process is said aperiodic if there exists a small set F with $\phi(F) > 0$ and a time $t_0 \geq 0$ such that $P_x(x(t) \in F) > 0$ for all $t \geq t_0$ and $x \in F$.*

We note that whenever there exists ϕ as above, there exists a maximal irreducibility measure ψ such that $\nu << \psi$ for any irreducibility measure ν. Hence aperiodicity can be defined directly in terms of ψ; at the same time, if we find a small set F with $\phi(F) > 0$ then automatically $\psi(F) > 0$.

Theorem 2 summarizes results from [3, 4, 10, 11]. Only [3] deals directly with the continuous time case.

For $\delta > 0$, define $\tau_F(\delta) = \inf\{t \geq \delta \,|\, x(t) \in F\}$. In the continuous time case, $\tau_F(\delta)$ replaces the *first time of return* to the set F from the discrete time setting.

Theorem 2. *Assume a Markov process is non-explosive, ψ - irreducible and aperiodic with an attractive Doeblin set F. If there exists $\delta > 0$, $\eta > 0$ such that $V(x) = E_x[\exp(\eta \tau_F(\delta))]$ is finite for all $x \in \mathcal{S}$ and $V(x)$ is uniformly bounded on F, then there exist a unique invariant probability measure $\mu(dx)$, constants $D > 0$ and $\rho \in (0, 1)$ such that for all $t \geq 0$ and $x \in \mathcal{S}$*

$$||P_x(x(t) \in \cdot) - \mu(\cdot)|| \leq DV(x)\rho^t \,, \tag{2.2}$$

where $|| \cdot ||$ denotes the total variation norm of a measure.

Remark. The existence of an attractive Doeblin set F implies that F is a *petite* set for the resolvent chain with transition probabilities $U_\lambda(x, dy) = \int_0^\infty \lambda e^{-\lambda t} P_x(x(t) \in dy)$, $\lambda > 0$. Then U_λ is Harris recurrent, implying that there exists an invariant measure $\mu(dx)$, not necessarily finite, for both the recurrent chain and $\{x_t\}_{t \geq 0}$. Positive recurrence, defined as $E_x[\tau(\delta)] < \infty$ for all $x \in \mathcal{S}$ is necessary and sufficient to show that $\mu(dx)$ is a probability measure. The condition $V(x) < \infty$ is much stronger, and implies exponential ergodicity.

Proof. The theorem is an immediate consequence of Theorems 6.2, 5.2 and 5.3 in [3] for the special function $f(x) \equiv 1$. □

3. **Proof of Theorem 1**

Step by step, we prove the necessary ingredients necessary to apply Theorem 2.

3.1. **Recurrence of the process.** In our case, $\mathcal{S} = (0,\infty)$, ϕ is the Lebesgue measure $l(dx)$ on $(0,\infty)$. In the proof of Theorem 1 (subsection 3.3) we show that $l(dx)$ is the maximal irreducibility measure ψ.

Proposition 1. *If $x \in (0,\infty)$, then for any $a' > 0$, $P_x(\tau_{(0,a')} < \infty) = P_x(\tau_{(0,a']} < \infty) = 1$. Moreover, for any open set A in $(0,\infty)$, we have $P_x(\tau_A < \infty) > 0$.*

Proof. Let $a' > 0$ and denote $\tau_0 = \tau_{(0,a')}$ (notice that it is a stopping time). Pick an arbitrary $\epsilon > 0$. Let $\chi_1, \chi_2, \dots$ be the i.i.d. holding times of the Poisson process driving $x(t)$, $w'_1, w'_2, \dots$ the actual holding times of the process and $\tau'_0 = 0$, $\tau'_1, \tau'_2, \dots$ be the actual jump times of $x(t)$. More precisely, for $j \geq 1$,

$$\tau'_j = \inf\{t > \tau'_{j-1} \,|\, \chi_j < \int_{\tau'_{j-1}}^{t} \alpha(x(s))ds\}\,, \qquad w'_j = \tau'_j - \tau'_{j-1}\,. \tag{3.1}$$

Part 1. We first show that $P_x(\tau_0 < \infty) = 1$ for all $x > 0$. By construction, (3.1) implies that $w'_i \leq \alpha_0^{-1}\chi_i$ with probability one. Right after exactly the n-th jump, a particle that started at x will be at

$$x(\tau'_n) = \gamma^n x + \sum_{k=0}^{n-1} \gamma^{k+1} w'_{n-k} \leq \gamma^n x + \alpha_0^{-1} \sum_{k=0}^{n-1} \gamma^{k+1} \chi_{n-k}\,. \tag{3.2}$$

It is straightforward to see that unless $x < a'$, we have that $x(\tau_0)$ coincides with the position $x(\tau'_n)$, exactly after a jump, for some $n \geq 1$. A coupling argument based on a process with constant $\alpha = \alpha_0$ driven by the same holding times $\{\chi_i\}_{i\geq 0}$, together with (3.2), shows that if the process with constant rate reaches $(0,a')$, then for sure $\{x(t)\}_{t\geq 0}$ reaches it even before. Proposition 8 (not dependent on the results in this section) concludes the argument.

Part 2. Without loss of generality, we assume that A is an open interval (a,b) with $a > 0$. Choose $a' \in (0,a)$. Let n be large enough to make $\gamma^n x$ small in comparison to a'. At the same time, we work with the event that the first n consecutive holding times w'_j are small, while w'_{n+1} is large enough to reach a. Then $\tau_A \leq \tau'_{n+1} < \infty$. More precisely, we use once again (3.2), the position at $t = \tau'_n$, after n consecutive holding times of length less than ϵ is

$$x(\tau'_n) = \gamma^n x + \sum_{k=0}^{n-1} \gamma^{k+1} w'_{n-k} \leq \gamma^n x + \epsilon\alpha_0^{-1}\gamma(1-\gamma)^{-1}\,. \tag{3.3}$$

We choose n and ϵ such that $\gamma^n x < a'/2$ and $\epsilon < (\frac{a'}{2})\alpha_0(1-\gamma)\gamma^{-1}$. Noticing that $\tau'_n \geq \tau_0$, we have

$$P_x(\tau_A < \infty) \geq P_x(\chi_1 \leq \epsilon,\, \chi_2 \leq \epsilon,\, \ldots \chi_n \leq \epsilon,\, w'_{n+1} > a) \geq (1-e^{-\epsilon})^n e^{-\alpha_0 a} > 0\,.$$

$\square$

Proposition 2. *For any Borel set A on $(0,\infty)$ with $l(A) > 0$ and any $x \in (0,\infty)$, we have $G(x,A) = \infty$.*

Remark. In continuous-time setting [3], l - irreducibility is defined as $G(x,A) > 0$ for any x and any A with $l(A) > 0$. Proposition 2 shows that $x(t)$ is l - recurrent, which means that $G(x,A) = \infty$. Intuitively, once $x(t)$ enters an open set, it will spend a positive time in the set with positive probability while waiting for the next jump.

Proof. Since $l(A) > 0$, there exists a compact set $K \subseteq A$ with $l(K) > 0$ and $0 < a < b < \infty$ such that $K \subseteq (a,b)$. Pick $a' \in (0,a)$. Starting with τ_0 defined in Proposition 1, for $i = 0,1,2,\ldots$ we set

$$\sigma_i = \inf\{t > \tau_i \,|\, x(t) \in (a,b)\}\,, \quad \tau_{i+1} = \inf\{t > \sigma_i \,|\, x(t) \in (0,a')\}\,. \tag{3.4}$$

We notice that $\tau_i - \tau_{i-1} > a - a' > 0$, bounded below uniformly in i, showing that $\lim_{n\to\infty} \tau_i = \infty$ with probability one. In view of the results from Proposition 1, the event that all $\tau_{i+1} - \sigma_i < \infty$, $i \geq 0$, has probability one. At the same time, $P(\sigma_i - \tau_i = \infty) \leq \Pi_{n>i} P(w'_n < a) = 0$. This shows that $\tau_{i+1} - \tau_i < \infty$, $i \geq 0$, has probability one.

We need to calculate

$$G(x,A) \geq E_x\Big[\int_{\tau_0}^{\infty} \mathbf{1}_A(x(t))dt\Big] \geq E_x\Big[\sum_{i=0}^{\infty}\int_{\tau_i}^{\tau_{i+1}} \mathbf{1}_A(x(t))dt\Big]\,. \tag{3.5}$$

Applying the strong Markov property to (3.5), it is enough to show that

$$\inf_{y\in(0,a')} E_y\Big[\int_0^{\tau_1} \mathbf{1}_A(x(t))dt\Big] > 0\,. \tag{3.6}$$

For χ an exponential random variable with mean value one, independent of the process, define

$$w' = \inf\{t > 0 \,|\, \chi < \int_0^t \alpha(x(s))ds\}\,,$$

the first jump time of $x(t)$. The rates $\alpha(x)$ are bounded above by $||\alpha|| \in (0, \infty)$ and $||\alpha|| w' \geq \chi$. Whenever $w' > b - y$ we have $\int_0^{\tau_1} \mathbf{1}_A(x(t))dt > l(K)$. Then, if $\chi > ||\alpha|| b$ we have that $w' > b$, which implies

$$E_y\Big[\int_0^{\tau_1} \mathbf{1}_A(x(t))dt\Big] \geq l(K) P_y\Big(\chi > ||\alpha|| b\Big) = l(K) \exp\{-||\alpha||\, b\} > 0\,,$$

where the lower bound is independent of y. □

Denote $A(y) = \int_0^y \alpha(x')dx'$ and recall the condition (1.2). Then A is Lipschitz continuous (upper bound on $\alpha(x)$) and strictly increasing, hence has an inverse A^{-1} which is also Lipschitz continuous (lower bound on $\alpha(x)$) and increasing

$$\alpha_0(y'' - y') \leq A(y'') - A(y') \leq ||\alpha||(y'' - y')\,, \quad y'' \geq y' > 0\,. \tag{3.7}$$

The Lipschitz constant for A is $||\alpha||$ and for A^{-1} it is α_0.

Proposition 3. *The probability distribution of $x(t)$ has a density for any $t > 0$.*

Proof. To prove the absolute continuity with respect to the Lebesgue measure, in view of (3.2), we have the formula

$$x(t) = x(\tau_n') + t - \tau_n' = t + \gamma^n x - \sum_{k=0}^{n-1} (1 - \gamma^{k+1}) w'_{n-k}\,, \tag{3.8}$$

when there are exactly n jumps before time $t > 0$ where $n = 0, 1, \ldots$. Due to continuity, equation (3.1) gives

$$\chi_j = \int_{\tau'_{j-1}}^{\tau'_j} \alpha(x(s))ds = \int_{x(\tau'_{j-1})}^{x(\tau'_j)} \alpha(y)dy\,, \qquad x(s) = x(\tau'_{j-1}) + s - \tau'_{j-1} \tag{3.9}$$

when $\tau'_{j-1} \leq s < \tau'_j$. Then

$$\chi_j = A(x(\tau'_{j-1}) + \tau'_j - \tau'_{j-1}) - A(x(\tau'_{j-1}))\,. \tag{3.10}$$

Following the calculation in (3.3) we see that $x(\tau'_j)$ are linear functions of the random variables w'_k, $1 \leq k \leq j$. The system is lower diagonal and (3.7) shows it is invertible. This, together with (3.1) and (3.2), proves that w'_j, $1 \leq j \leq n$ have a continuous joint distribution depending on the joint density of the vector $(\chi_1, \ldots, \chi_n)$, which demonstrates our claim. □

3.2. **Ergodic properties.** The next Proposition identifies a class of attractive small sets.

Proposition 4. *Any set $F = (0, a']$ with $a' > 0$ is an attractive small set. More precisely, $P_x(\tau_F < \infty) = 1$ for all $x \in (0, \infty)$ and there exists $t > 0$ and an interval I with $I \cap F \neq \emptyset$ such that the probability density of $P_x(x(t) \in dy)$ is bounded away from zero on I, uniformly in $x \in F$.*

Remark. In fact, for any t with $t/a' \in (0, \gamma^{-1})$, there exists $n \geq 1$ such that $[\gamma^n a' + \gamma t, t) \cap F \neq \emptyset$ and I can be chosen as a closed subinterval. The definition of a small set does allow that I, and implicitly ν_0, in this case the uniform probability measure on I, be dependent of t.

Proof. We shall use the same notations for the holding times and jump times as in the proof of Propositions 1, 2 and Theorem 1. In the proof of Proposition 1 we showed that $P_x(\tau_{(0,a')} < \infty) = 1$ for all $x \in (0, \infty)$. Having $\tau_{(0,a']} \leq \tau_{(0,a')}$, condition (i) in Definition 1 is satisfied for any set $(0, a']$.

Fix $t > 0$ such that $t/a' \in (\gamma^n(1-\gamma)^{-1}, (1-\gamma^n)\gamma^{-1})$. The interval is properly defined if $\gamma^n < 1 - \gamma$, which is true for sufficiently large n. By construction, there exist two numbers $t_2 > t_1 > 0$ such that $t_1 \geq \gamma^n a' + \gamma t$ and $t_2 < t$ (the second inequality has to be strict) and $I = [t_1, t_2] \cap F \neq \emptyset$. A possible choice is $t_1 = \gamma^n a' + \gamma t$ and $t_2 = t - \epsilon$ with small ϵ. Consider $q_1 < q_2$ in I.

Pick a point $x \in F = (0, a']$. We recall that $\tau'_n = \sum_{j=1}^n w'_j$ is the n-th jump (3.1).

A lower bound of the probability that $x(t)$ falls in the interval $(q_1, q_2]$ is

$$P_x\Big(q_1 < x(t) \leq q_2\Big) \geq P_x\Big(q_1 < x(t) \leq q_2\,,\, \tau'_n \leq t < \tau'_{n+1}\Big) \tag{3.11}$$

by intersection with the event that there were exactly n jumps up to time t. Writing $x(t) = x(\tau'_n) + (t - \tau'_n)$, using formula (3.2), the event on the right hand side of (3.11) is the intersection of

$$G_n = \Big\{q_1 < \gamma^n x + \sum_{j=1}^n \gamma^{n+1-j} w'_j + (t - \sum_{j=1}^n w'_j) \leq q_2\Big\} \tag{3.12}$$

and $R_n = \{\sum_{j=1}^n w'_j \leq t < \sum_{j=1}^{n+1} w'_j\}$. The choice of the interval I makes the inequality $\sum_{j=1}^n w'_j \leq t$ redundant. To see that, we re-write the left hand side of (3.12)

$$t > \sum_{j=1}^n (1 - \gamma^{n+1-j}) w'_j + q_1 - \gamma^n x \geq (1-\gamma) \sum_{j=1}^n w'_j + (\gamma^n a' + \gamma t) - \gamma^n a'$$

and simplify. The remaining condition $t < \sum_{j=1}^{n+1} w'_j$, written as $w'_{n+1} > t - \sum_{j=1}^{n} w'_j$, is equivalent to

$$A(x(\tau'_n) + t - \sum_{j=1}^{n} w'_j) - A(x(\tau'_n)) < \chi_{n+1} \, .$$

Denote

$$Y_n = \sum_{j=1}^{n} (1 - \gamma^{n+1-j}) w'_j \, . \tag{3.13}$$

We note that the random variables Y_j, $j = 1, 2, \ldots$ also depend on the initial point x. Inductively, from (3.8) and (3.10), one can see that the vector $w' = (w'_1, \ldots, w'_n)$, and all τ'_i, $1 \le i \le n$ are given as deterministic functions of $\chi = (\chi_1, \ldots, \chi_n)$. This implies that χ_{n+1} is independent of $w' = (w'_1, \ldots, w'_n)$.

Let's assume that $w' = (w'_1, \ldots, w'_n)$ and Y_n have density functions $\rho(w')$ respectively $g_n(y_n)$. On the interval I, the lower bound (3.11) can be written as

$$\int_{G_n} \exp\Big\{ - \Big(A(x(\tau'_n) + t - \sum_{j=1}^{n} w'_j) - A(x(\tau'_n)) \Big) \Big\} \rho(w') \, dw' \tag{3.14}$$

$$\ge \exp(-||\alpha|| t) \int_{G_n} \rho(w') \, dw' = \exp(-||\alpha|| t) \int_{-q_2 + \gamma^n x + t}^{-q_1 + \gamma^n x + t} g_n(y_n) \, dy_n \, . \tag{3.15}$$

Due to the choice of t_1 and t_2, $-q_2 + \gamma^n x + t > t - t_2 > 0$ and $-q_1 + \gamma^n x + t \le t(1 - \gamma)$. We also note that the two endpoint bounds do not depend on x. In addition, $t < a'/\gamma$ so $e^{-||\alpha|| t} \ge e^{-||\alpha|| a' \gamma^{-1}}$. If $d_n = \inf_{y_n \in [t - t_2, t(1-\gamma)]} g_n(y_n)$, then

$$P_x \Big(q_1 < x(t) \le q_2 \Big) \ge e^{-||\alpha|| a' \gamma^{-1}} d_n (q_2 - q_1) \, . \tag{3.16}$$

We know that $x(t)$ has a density from Proposition 3. Inequality (3.16) implies that the density is bounded below by d_n over the interval I.

It remains to prove that $g_n(y_n)$ exists and that $d_n > 0$ does not depend on x, which is done in Proposition 5. $\square$

The proof of Proposition 5 needs the following lemma.

Lemma 1. *Let V be a d - dimensional random variable and W be a one dimensional nonnegative random variable, with joint density $f(v, w)$ having the property that for any $b > 0$, there exists a positive constant $k_1 = k_1(b)$ such that $f(w | V = v) \ge k_1$ for all $v \in \mathbb{R}^d$*

and $w \in [0,b]$. Let $F(v)$ be a nonnegative measurable function. Then the density function of $Y = F(V) + W$ satisfies $f_Y(y) \geq k_1 P(F(V) \leq y)$ for all $y \in [0,b]$.

Proof. First we fix $b > 0$. Then for any $y \in [0,b]$,

$$f_Y(y) = \int_{F(v)\leq y} f(v, y - F(v))dv =$$

$$\int_{F(v)\leq y} f(y - F(v)|V = v) f_V(v)dv \geq k_1 \int_{F(v)\leq y} f_V(v)dv \geq k_1 P(F(V) \leq y)\,,$$

where we used the fact that both $F(v)$ and W are nonnegative implies that $y - F(v)$ belongs to $[0,b]$. ☐

Proposition 5. *The random variables Y_n defined in (3.13) have density $g_n(y_n)$ with the property that for any $b > 0$, there exists a constant $c_n(b) > 0$ such that $g_n(y_n) \geq c_n(b) y_n^{n-1}$ on $[0,b]$, with $c_n(b)$ independent of the initial state x of the process.*

Proof. We proceed by induction over n. The verification step for Y_1 is immediate since $A(x + w_1') - A(x) = \chi_1$ and the density function of w_1' satisfies $f_{w_1'}(w) \geq \alpha_0 \exp(-||\alpha||w)$ by the inversion formula for densities; we can take $c_1(b) = (1-\gamma)^{-1}\alpha_0 \exp(-||\alpha||b(1-\gamma)^{-1})$.

Assuming the statement is true for $n-1$, we prove it for n. We write $Y_n = F(V) + W$ with $d = n-1$, $V_j = w_j'$, $1 \leq j \leq n-1$, $F(V) = \sum_{j=1}^{n-1}(1 - \gamma^{n+1-j})w_j'$ and $W = (1-\gamma)w_n'$. The pair (V, W) satisfies Lemma 1 with $Y = Y_n$. If $x(\tau_j')$ (3.2) is the location of the process after the j - th jump, then $A(x(\tau_{n-1}') + w_n') - A(x(\tau_{n-1}')) = \chi_n$, thus

$$w_n' = A^{-1}(A(x(\tau_{n-1}')) + \chi_n) - x(\tau_{n-1}')\,. \tag{3.17}$$

We see that w_n' is defined in terms of $x(\tau_{n-1}') = \gamma^{n-1}x + \sum_{j=1}^{n-1}\gamma^{n-j}w_j'$. The conditional density $f(w|V = v)$ of w_n' given V is the conditional density $f(w|x_0)$ of w_n' given x_0 for the corresponding x_0 in the role of $x(\tau_{n-1}')$. This is bounded below by a constant independent of x, and we can take it equal to $c_1(b)$. Set $k_1 = c_1(b)$, with k_1 from Lemma 1. Because $(1+\gamma)Y_{n-1} \geq F(V)$, by the induction hypothesis, we obtain that

$$g_n(y_n) \geq c_1(b)P(F(V) \leq y_n) \geq c_1(b)P(Y_{n-1} \leq (1+\gamma)^{-1}y_n)$$

$$\geq c_1(b)c_{n-1}(b)\int_0^{(1+\gamma)^{-1}y_n} u^{n-2}du \geq \frac{c_1(b)c_{n-1}(b)}{(n-1)(1+\gamma)^{n-1}}y_n^{n-1}\,.$$

The proposition is proved with $c_n(b) = (n-1)^{-1}(1+\gamma)^{1-n}c_1(b)c_{n-1}(b)$. ☐

Proposition 6. *There exists $t_0 \geq 0$ and F a small set with $l(F) > 0$ such that $P_x(x(t) \in F) > 0$ for all $t \geq t_0$ and all $x \in F$ and hence the process $x(t)$ is l - aperiodic.*

Proof. Let $F = (0, a']$ as in Proposition 4. Let $t > 0$ be arbitrary. We shall construct the event

$$A_t = \{\gamma(a' + w_1') < ca'\} \cap \left(\cap_{i=2}^{N} \{\epsilon_1 < w_i' < \epsilon_2\} \right) \tag{3.18}$$

where c is a number in $(\gamma, 1)$, $\epsilon_2 = a'(1-c)$, $\epsilon_1 \in (0, \frac{\alpha_0}{||\alpha||}\epsilon_2)$ and $N = [t/\epsilon_1] + 2$. Set $t_0 = (c\gamma^{-1} - 1)a'$. Under A_t, the choice of t_0 ensures that the first jump occurs before t_0 and brings $x(t)$ below $ca' < a'$. The choice of the constant c ensures that both $\gamma(ca' + w_i') < ca'$ (the process returns to a point in $(0, ca')$ after each jump) and $ca' + w_i' < a'$ (the process will not exceed a') for the next $N - 1$ steps. The lower bound ϵ_1 ensures that when $x(t)$ starts at $x(0) = x \leq a'$ there can be at most N jumps in the time interval $[0, t]$ and that $x(t)$ stays in $(0, a']$ on $[t_0, t]$, hence $x(t) \in F$.

Since $||\alpha|| w_i' \geq \chi_i \geq \alpha_0 w_i'$, the event A_t includes the event

$$A_t^\chi = \{\chi_1 < \alpha_0 t_0\} \cap \left(\cap_{i=2}^{N} \{||\alpha||\epsilon_1 < \chi_i < \alpha_0\epsilon_2\} \right). \tag{3.19}$$

Then, for any $x \in F$, $P_x(x(t) \in F) \geq P_x(A_t^\chi) > 0$, proving that $x(t)$ is aperiodic. □

Proposition 7. *For $\delta > 0$, recall $\tau_F(\delta) = \inf\{t \geq \delta \,|\, x(t) \in F\}$ defined in Section 2. Then, there exists $\delta > 0$, $\theta < 0$ such that (i) $E_x[\exp(-\theta\tau_F(\delta))] < \infty$ for all $x \in (0, \infty)$ and (ii) there exists $M > 0$ such that $E_x[\exp(-\theta\tau_F(\delta))] \leq M < \infty$ for all $x \in F$.*

Proof. In the time interval $[0, \delta]$ the particle $x(t)$ cannot exceed the value $[x + \delta]$. We shall replace (i) and (ii) from the proposition with the stronger condition

$$\sup_{x \in (0,b]} E_x[\exp(-\theta\tau_{(0,a')})] < \infty\,, \qquad \forall b > 0\,. \tag{3.20}$$

We shall use the coupling with the process with constant rate α_0 as we did before, in the proof of Proposition 1, starting with equation (3.2). The first hitting time is either zero, if we start with $x \in (0, a')$, or will be one of the jump times τ_n'. Due to the bound (3.2), the first hitting time $\tau_{(0,a')}$ is bounded above by the first hitting time of the process with constant rate α_0. With the notations of Proposition 8, for $0 > \theta > \theta_0$, we have $\sup_{0 \leq x \leq b} E_x[\exp(-\theta\tau_{(0,a')})] < \infty$. □

3.3. **Proof of Theorem 1.** We first show that the Lebesgue measure $l(dx)$ is maximal (see Definition 2 and the discussion thereafter). Given that $l(dx)$ is an irreducibility measure from Proposition 2, it is sufficient to show that $P_x(x(t) \in dy)$ has a density. This is proven in Proposition 3. In that case, if $l(A) = 0$ implies that $P_x(x(t) \in A) = 0$ and so $G(x, A) = 0$. If ν is another irreducibility measure and $\nu(A) > 0$, we should have $G(x, A) > 0$, a contradiction; thus $\nu << l$.

Next, we shall use Theorem 2. The process is non-explosive since $\alpha(x)$ is bounded above and below away from zero. Proposition 2 proves l - recurrence, hence l - irreducibility. Proposition 4 identifies all intervals $(0, a']$ to be Doeblin attractive sets and Proposition 6 proves that $x(t)$ is aperiodic. Finally Proposition 7 shows that the conditions of Theorem 2 are satisfied for any $F = (0, a']$ with some $\delta > 0$ and $\eta > 0$.

4. The case α constant

In the present discussion we assume a constant $\alpha(t, x) = \alpha > 0$ and the process with generator (1.1).

4.1. **Explicit formula for the invariant measure when α is constant.** For many choices of $\alpha(t, x)$, both $z(t)$ approaches an equilibrium value z_{eq} as $t \to \infty$ and the jump rate $\alpha(t, z(t))$ becomes asymptotically equal to a constant α, showing that for large time t, the evolution of the model coincides with the case $\alpha = \textit{constant}$. The steady state equation is

$$\langle \nabla\phi + \alpha(\phi(\gamma x) - \phi(x))\,,\, \mu\rangle = 0\,, \quad \alpha = \alpha(z_{eq})\,, \qquad \mu(dx) \in M_1((0, \infty))\,, \tag{4.1}$$

where $\phi \in C_b^2((0, \infty))$. An alternative way to identify the steady state is by setting $\tilde{B}w(x) = \gamma^{-1}w(\gamma^{-1}x) - \alpha^{-1}\nabla w(x)$, and then a steady state is a fixed point of the operator $\tilde{B}$.

A heuristic approach, that offers insight in the way the equilibrium is approached by the process, is to look at the position Y_n of the Markov process *right before each jump.* A particle starting at $Y_0 = x_0 > 0$ drifts with constant velocity one in the positive direction. When an exponential clock with intensity α rings, it jumps to a position equal to γ times its current position. Let χ_n be a sequence of i.i.d. exponentials with intensity α. Let Y_n be the position right before the n th jump. Then

$$Y_1 = Y_0 + \chi_1\,, \quad Y_2 \;=\; \gamma Y_1 + \chi_2\,, \quad \ldots \quad Y_n = \gamma Y_{n-1} + \chi_n \tag{4.2}$$

yielding

$$Y_n = \gamma^{n-1} Y_0 + \sum_{j=1}^{n} \gamma^{n-j} \chi_j \tag{4.3}$$

according to (3.2). We calculate the limiting distribution of Y_n. The characteristic function is

$$E\Big[e^{i\xi Y_n}\Big] = e^{i\xi\gamma^{n-1}x_0}\, \Pi_{j=1}^{n}\Big(1 - (\frac{i\xi}{\alpha})\gamma^{n-j}\Big)^{-1} = e^{i\xi\gamma^{n-1}x_0}\, \Pi_{k=0}^{n-1}\Big(1 - (\frac{i\xi}{\alpha})\gamma^{k}\Big)^{-1} \tag{4.4}$$

with limit as $n \to \infty$ equal to

$$E\Big[e^{i\xi Y_\infty}\Big] = \Big[\Pi_{k=0}^{\infty}\Big(1 - i\xi(\frac{\gamma^k}{\alpha})\Big)\Big]^{-1}. \tag{4.5}$$

One can see from (4.5) that

$$Y_\infty \sim \sum_{k=0}^{\infty} \gamma^k \tilde{\chi}_k\,, \tag{4.6}$$

where $\tilde{\chi}_k$ are i.i.d. exponential with parameter α and $\sim$ indicates equivalence in probability law.

Theorem 3. *The steady state of the process, equal to the stationary solution of the process (1.1) for constant jump rate α, is the distribution of the random variable Y_∞ and has characteristic function (4.5). The distribution of Y_∞ has a bounded density function and for any open interval (a,b), $P_x(Y_\infty \in (a,b)) > 0$.*

Proof. We first show that the invariant measure is equal to the distribution of Y_∞ with characteristic function given in (4.5). An invariant measure $\mu(dy)$ of the process (1.1) satisfies $\int_0^\infty B\phi(x)\mu(dx) = 0$ (4.1) for any $\phi \in C_b((0,\infty))$. We shall show that the solution to (4.1) has the same characteristic function as (4.5). It is straightforward to extend (4.1) to bounded, complex functions with second derivative, and implicitly to functions of the form $x \to \exp(i\xi x)$. Any solution to the weak equation (4.1) would have characteristic function $\hat{\mu}(\xi)$ satisfying $\hat{\mu}(\xi\gamma) = (1 - i\xi\alpha^{-1})\hat{\mu}(\xi)$. The solution of this recurrence is exactly (4.5), as the characteristic function $\hat{\mu}(\xi)$ is continuous at $\xi = 0$ where it is equal to one.

For convenience, we recall that Y_∞ has the same distribution as $\sum_{k=0}^\infty \gamma^k \tilde{\chi}_k$, where $\tilde{\chi}_k$ are i.i.d. exponential with parameter α. Let $S_n = \sum_{k=0}^n \gamma^k \tilde{\chi}_k$ (notice that $S_n \sim Y_{n+1}$ modulo a constant) and denote the remainder $R_n = \sum_{k=n+1}^\infty \gamma^k \tilde{\chi}_k$. Let $\rho_n(y)$ be the density of S_n.

It is easy to show inductively that ρ_n is positive on $(0, \infty)$. Since the convolution of two bounded density functions is bounded by the minimum bound of the two and the density of an exponential distribution is bounded, we can see by induction that $\rho_n(y) \leq \alpha$. Since $S_n \Rightarrow Y_\infty$, for any open interval (a, b) in $(0, \infty)$, we have

$$P(Y_\infty \in (a, b)) \leq \liminf_{n\to\infty} P(S_n \in (a, b)) \leq \alpha(b - a)$$

which implies that $P(Y_\infty \in dy) = \rho_\infty(y)dy$ and $\rho_\infty(y) \leq \alpha$.

For any $m > n$, let the partial sum $R_{n,m}$ from the remainder term R_n be denoted as $R_{n,m} = \sum_{k=n+1}^{m} \gamma^k \tilde{\chi}_k$. Then the martingale decomposition on the sub-martingale $\{R_{n,m}\}_{m\geq n+1}$ provides $R_{n,m} = M_{n,m} + V_{n,m}$ where the martingale part is

$$M_{n,m} = \sum_{k=n+1}^{m} \gamma^k (\tilde{\chi}_k - \alpha^{-1})$$

and the remainder is $V_{n,m} = \alpha^{-1} \sum_{k=n+1}^{m} \gamma^k$. To control the martingale term, observe that for any $K > 0$ and $m > n$

$$P(\sup_{n+1\leq k\leq m} |M_{n,k}| > K) \leq \frac{EM_{n,m}^2}{K^2} \leq \frac{\gamma^{2(n+1)}}{K^2\alpha^2(1-\gamma^2)}\,,$$

by Doob's maximal martingale inequality. Hence

$$P(\sup_{n+1\leq k} |M_{n,k}| > K) = \lim_{m\to\infty} P(\sup_{n+1\leq k\leq m} |M_{n,k}| > K) \leq \frac{\gamma^{2(n+1)}}{K^2\alpha^2(1-\gamma^2)}\,. \tag{4.7}$$

Also note that $\lim_{m\to\infty} V_{n,m} = \frac{\gamma^{n+1}}{\alpha(1-\gamma)}$. From (4.7) for any small $\epsilon > 0$, there exists a large enough $N_0 \in \mathbb{N}$ to ensure both $P(\sup_{n+1\leq k} |M_{n,k}| > \epsilon/2) < 1/2$ and $\frac{\gamma^{n+1}}{\alpha(1-\gamma)} < \epsilon/2$ if $n \geq N_0$. For $n \geq N_0$, on the event $A_n = \{\sup_{n+1\leq k} |M_{n,k}| \leq \epsilon/2\}$, we have $0 \leq R_n < \epsilon$ and $P(A_n) > 1/2$. Since S_n has a strictly positive density function $\rho_n(y)$ on $(0, \infty)$, for any open interval $(a, b) \in (0, \infty)$ and for any small enough $\epsilon > 0$, the event $B_n = \{S_n \in (a, b - \epsilon)\}$ has positive probability and is independent of the event A_n (from the mutual independence of $\{\tilde{\chi}_k\}$). Therefore $P(Y_\infty = S_n + R_n \in (a, b)) \geq P(A_n \cap B_n) > P(B_n)/2 > 0$. □

For any $a' > 0$ and any $x > 0$, let $\tau_0 = \inf\{t > 0 \,|\, x(t) < a'\} = \tau_{(0,a')}$ and $u_\theta(x) = E_x[\exp(-\theta\tau_0)]$, for all θ where it is finite. Naturally the Laplace transform is finite for $\theta \geq 0$. We are interested in proving a bound for $\theta < 0$. In the following proposition we assume $\alpha = 1$ without any loss of generality.

Proposition 8. *There exists $\theta_0 < 0$ such that for all $x > 0$, the Laplace transform $u_\theta(x)$ is finite when $\theta > \theta_0$ and for any $b > 0$ and any $\theta > \theta_0$*

$$\sup_{0\le x\le b} u_\theta(x) < \infty\,. \tag{4.8}$$

Moreover, $u_\theta(x)$ satisfies the differential equation

$$u'_\theta(x) = (\theta+1)u_\theta(x) - u_\theta(\gamma x)\,, \qquad x \ge a' \tag{4.9}$$

with boundary conditions $u_\theta(x) = 1$ for $x \in (0, a')$.

Remark. The proposition trivially implies that $u_0(x) = P_x(\tau_0 < \infty) = 1$. We note that $u_\theta(a') \neq 1$.

Proof. The proof is based on the exponential decay of the tail $P_x(\tau_0 > t) \le \exp(-\lambda t)$ as $t \to \infty$ of the distribution on τ_0, where $\lambda = \lambda(a') > 0$ will be a uniform decay rate for all $x \in (0, \infty)$. First, if $\epsilon > 0$ is fixed,

$$P_x(\tau_0 > t) \le P_x(A_n^c) + P_x(A_n \cap \{\tau_0 > t\}) \tag{4.10}$$

where A_n is the event that there are at least $n = [t(1-\epsilon)]$ many jumps before time t. By the large deviations principle applied to n i.i.d. exponentials with rate one, there exists a constant $\lambda_0 > 0$ depending on ϵ but not on x such that $P_x(A_n^c) \le \exp(-\lambda_0 t)$. Recalling (3.3), we define the Markov chain

$$Z_n = \gamma^n b + \sum_{k=0}^{n-1} \gamma^{k+1}\chi_{n-k} \tag{4.11}$$

which is equal to $\gamma^{-1}Y_n$ from (4.2) with $Y_0 = b > x$ and represents the position of the particle $x(t)$ right after the n-th jump. Then

$$P_x(A_n \cap \{\tau_0 > t\}) \le P_x(Z_1 > a'\,, \dots, Z_n > a') \le \tag{4.12}$$

$$\Pi_{k=2}^n P_x(Z_k > a' \,|\, Z_{k-1} > a'\,, \dots, Z_1 > a') = \Pi_{k=2}^n P_x(Z_k > a' \,|\, Z_{k-1} > a')$$

by the Markov property. Using conditional probability

(4.13)

$$P_x(Z_k > a' \,|\, Z_{k-1} > a') = \Big(\int_{a'}^{\infty} P_x(\chi_k > \gamma^{-1}a' - z)P_x(Z_{k-1} \in dz)\Big)\Big(P_x(Z_{k-1} > a')\Big)^{-1}.$$

As $n \to \infty$, $Z_n \to \gamma^{-1} Y_\infty$ and we recall that Y_∞ has a density. The right hand side of (4.13) can be written as

$$(4.14) \qquad \Big(\int_{a'}^{\gamma^{-1}a'} e^{-(\gamma^{-1}a' - z)} F_{k-1}(dz) + \int_{\gamma^{-1}a'}^{\infty} F_{k-1}(dz) \Big) \Big(F_{k-1}((a', \infty)) \Big)^{-1}$$

for $F_{k-1}(dz) = P_x(Z_{k-1} \in dz)$, which converges as $k \to \infty$ to

$$(4.15) \qquad \Big(\int_{a'}^{\gamma^{-1}a'} e^{-(\gamma^{-1}a' - z)} F(dz) + \int_{\gamma^{-1}a'}^{\infty} F(dz) \Big) \Big(F((a', \infty)) \Big)^{-1}$$

where $F(dz) = P_x(\gamma^{-1} Y_\infty \in dz)$. We have used that $F_n \Rightarrow F$ implies that the distribution functions converge at all points of continuity (here this means all points). The constant (4.15) is strictly less than one since we know from Theorem 3 that $F((0, a')) > 0$. We have shown that for any $\lambda_1 > 0$ less or equal than the negative of the natural logarithm of (4.14) (depending on a' but independent of b), there is a rank $k_1 = k_1(a', b)$ (depending on a', b) such that (4.14) is dominated by $\exp(-\lambda_1)$ for all $k \geq k_1$. For $n \geq k_1$, the left hand side of (4.12) is less than $\exp(-\lambda_1(n - k_1))$. This shows that there exists $\lambda = \lambda(a') \leq \lambda_0 \wedge \lambda_1(a') > 0$ such that the right hand side of (4.10) is exponentially bounded

$$P_x(\tau_0 > t) \leq \exp(-\lambda_0 t) + \exp(-\lambda_1([t(1-\epsilon)] - k_1)) \leq C \exp(-\lambda t)\,, \quad t \geq 0\,,$$

where $C > 0$ depending on (a', b). This proves that the exponential moments are finite for all $\theta > \theta_0$, where we pick $\theta_0 := -\lambda$. In addition, $\sup_{0 \leq x \leq b} E_x[\exp(\eta \tau_0)] < \infty$ for any given $b > 0$.

To prove the equation (4.9), let χ_1 be the first exponential holding time of intensity one before the first jump. Then, if $x < a'$, we have $u_\theta(x) = 1$, and if $x \geq a'$ we can derive the relation

$$E_x[e^{-\theta \tau_0}] = \int_0^\infty E_x[e^{-\theta \tau_0} \mid \chi_1 = s]\, e^{-s}\, ds = \int_0^\infty E_{\gamma(x+s)}[e^{-\theta(s + \tau_0)}]\, e^{-s}\, ds$$

which reduces to

$$(4.16) \qquad u_\theta(x) = \int_0^\infty u_\theta(\gamma(x+s)) e^{-(\theta+1)s} ds\,, \qquad x \geq a'\,.$$

After a substitution $y = \gamma(x+s)$,

$$(4.17) \qquad u_\theta(x) e^{-(\theta+1)x} = \gamma^{-1} \int_{\gamma x}^\infty u_\theta(y) e^{-\gamma^{-1}(\theta+1)y} dy\,, \qquad x \geq a'\,.$$

In view of the bound (4.8), the integral on the right hand side of is finite when $\theta > \theta_0$. For such θ, the integrand is bounded, which implies that the left hand side of (4.17) is continuous. This, in turn, shows that $u_\theta(x)$ is differentiable in x, $x \geq a'$. After differentiating both sides of (4.17) and a simplification by a factor $e^{-(\theta+1)x}$, we have (4.9). Naturally, this is $(B - \theta)u_\theta = 0$, with B from (1.1) corresponding to $\alpha \equiv 1$. $\square$

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