Appendix C: Original source translation

This is the author's translation of the initial sections of Leonhard Euler's, *Institutionum calculi integralis,* vol. I, St. Petersburg, 1768, posted in The Euler Archive. The punctuation and notation of the original were retained in this translation, which sometimes makes the translation seem awkward.

CHAPTER VII

ON

THE INTEGRATION OF DIFFERENTIAL EQUATIONS BY APPROXIMATION

Problem 85.

650.

Whenever presented a differential equation, find its complete integral very approximately.

Solution

The pair of variables x and y appear in a differential equation, and moreover this equation has the form $\frac{\partial y}{\partial x} = V$, the function V itself a function of x and y. We desire the complete integral, which is interpreted that as long as x is assigned a certain value $x = a$, the other variable y takes on a given value $y = b$. Therefore our primary goal is to find the value of y so that when x takes on a value that differs little from a, or we assume $x = a + \omega$, then we can find y. Since ω is a very small quantity, then the value of y itself differs minimally from b; so while x varies a little from a to $a+\omega$, one may consider the quantity V as a constant. When we specify $x = a$ and $y = b$ then $V = A$, and by virtue of the small change we have $\frac{\partial y}{\partial x} = A$, for that reason when integrating $y = b + A(x-a)$, a constant being added of course, so that when $x = a$ we have $y = b$. Therefore given the initial values $x = a$ and $y = b$, we obtain the approximate next values $x = a + \omega$ and $y = b + A\omega$, so that proceeding further in a similar way over the small interval, in the end arriving at values as distant as we would like from the earlier values. These operations can be placed for ease of viewing, displayed successively in the following manner.

Variable	successive values
x	$a,\ a',\ a'',\ a''',\ a^{IV},\ \ldots\ \ 'x,\ x$
y	$b,\ b',\ b'',\ b''',\ b^{IV},\ \ldots\ \ 'y,\ y$
V	$A,\ A',\ A'',\ A''',\ A^{IV},\ \ldots\ \ 'V,\ V$

Certainly from the given initial values $x = a$ and $y = b$, we have $V = A$, then for the second we have $b' = b + A(a' - a)$, the difference $a' - a$ as small as one pleases. From here in putting $x = a'$ and $y = b'$, we obtain $V = A'$, and from this we will obtain the third $b'' = b' + A'(a'' - a')$, when we put $x = a''$ and $y = b''$, we obtain $V = A''$. Now for the fourth, we have $b''' = b'' + A''(a''' - a'')$, from this, placing $x = a'''$ and $y = b'''$, we shall obtain $V = A'''$, thus we can progress to values as distant from the initial values as we wish. The first sequence of x values can be produced successively as desired, provided it is ascending or descending over very small intervals.

Corollary 1.

651. Therefore one at a time over very small intervals calculations are made in the same way, so the values, on which the next depend, are obtained. As values of x are done iteratively in this way one at a time, the corresponding values of y are obtained.

Corollary 2.

652. Where smaller intervals are taken, through which the values of x progress iteratively, so much the more accurate values are obtained one at a time. However the errors committed one at a time, even if they may be very small, accumulate because of the multitude.

Corollary 3.

653. Moreover errors in the calculations arise, because in the individual intervals the quantities x and y are seen to be constant, so we consider the function V as a constant. Therefore the more the value of V changes on the next interval, so much the more we are to fear larger errors.