These are notes for a talk on perverse sheaves of categories and Sklyanin algebras.

Preface

Our goal here is to describe a situation in which perverse sheaves of categories (perverse schobers) arise naturally, and where they naturally give rise to things that have been studied classically. In general, we will work with dg categories with pretriangulated structure, and in particular, we will work in the homotopy category of small dg categories over $k$.

That being said, you probably won’t lose much if you imagine that everything is done in terms of triangulated categories, it’s just not technically kosher to do so.

Semiorthogonal decompositions

Here we work with triangulated $k$-linear categories.

- The basic idea that we want to start with is that of a semiorthogonal decomposition.

**Definition 1.** A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ if the embedding $\iota_\mathcal{A}$ has both right and left adjoints.

- **Example:** If we have an object $E$ in $\mathcal{T}$ satisfies $\text{Hom}(E, E) = k$ and $\text{Hom}(E, E[i]) = 0$ for $i \neq 0$ (such objects are called exceptional), then this defines a functor from $D^b(k)$ to $\mathcal{T}$ which sends a complex $K$ to the total complex of $E \otimes K$. This functor has adjoints $\text{RHom}(E, F)$ and $\text{RHom}(F, E)^\vee$.

- If we have an admissible subcategory, then we can break apart every object in $\mathcal{T}$ into $\text{Cone}(a \rightarrow b)$ for $a \in \mathcal{A}$ and $b \in \perp \mathcal{A}$, where $\perp \mathcal{A} = \{b \in \mathcal{T} : \text{Hom}(b, a) = 0 \text{ for all } a \in \mathcal{A}\}$ (which is also admissible).

- This gives a “semiorthogonal decomposition” of $\mathcal{T}$. This is denoted $(\mathcal{A}, \perp \mathcal{A})$.

- In general, we say that a triangulated category $\mathcal{T}$ has a semiorthogonal decomposition $\mathcal{A}_1, \ldots, \mathcal{A}_n$ if each $\mathcal{A}_i$ is an admissible subcategory and there’s a diagram decomposing each element of $\mathcal{T}$.

- The main example is of an exceptional collection of objects. For instance, a classical result of Beilinson says that $D^b(\text{coh } \mathbb{P}^n)$ has a semiorthogonal decomposition into categories generated by the objects $\mathcal{O}, \ldots, \mathcal{O}(n - 1)$.

- Most nice categories that we want to work with have an extra symmetry called the Serre functor.

**Definition 2.** A Serre functor is an exact autoequivalence of $\mathcal{T}$ so that $\text{Hom}(a, b) = \text{Hom}(b, S(a))^\vee$ for every pair $a, b$ in $\mathcal{T}$. 

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• **Example:** Serre duality tells us that if $\mathcal{T} = D^b(\text{coh} \ X)$ for a smooth projective variety, then $\otimes \omega_X[-\dim X]$ is a (the) Serre functor.

• There’s also another distinguished class of autoequivalences on a triangulated category, which are called **spherical twists** associated to spherical functors.

**Definition 3.** A triangulated functor $F : \mathcal{C} \to \mathcal{T}$ is called spherical if it has right and left adjoints $R$ and $L$ and the following conditions hold

1. The functor $T$ which sends an object $a \in \mathcal{T}$ to $\text{Cone}(RF(a) \xrightarrow{\alpha} a)$

is an isomorphism.

2. The natural transformation $R \to RFL \to CL[1]$

is an isomorphism of functors.

• The functor $C$ is the “cotwist” autoequivalence of $\mathcal{C}$, which is given by sending $c \in \mathcal{C}$ to $\text{Cone}(c \to FR(c))[-1]$.

**Definition 4.** We will say that a spherical functor is Calabi-Yau if the cotwist of $F$ is equal to the Serre functor of $\mathcal{C}$ up to shift.

• Standard examples of Calabi-Yau pairs come from geometry.

• If $X$ is a Fano variety, and $Y$ is a smooth section of $\omega_X^{-1}$, with embedding $\iota : Y \to X$, then $\iota^*$ is a spherical functor which is Calabi-Yau.

• If $Z$ is a subvariety of $X$ and $\omega_X|_Z = \mathcal{O}_Z$, then the embedding $\iota_*$ is a Calabi-Yau functor.

• The importance of Calabi-Yau spherical functors is that they provide a link between exceptional subcategories and spherical functors.

**Theorem 1 (Addington, Kuznetsov).** If $\mathcal{A}$ is an admissible subcategory of $\mathcal{T}$ with embedding $\iota$, and there is a Calabi-Yau spherical functor $F : \mathcal{T} \to \mathcal{C}$, then the functors $F\iota$ is a spherical Calabi-Yau pair.

• So if we have a category $\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ with $F : \mathcal{T} \to \mathcal{C}$ a CY spherical functor, then $F\iota : \mathcal{A} \to \mathcal{C}$ is a CY spherical functor.

• Addington and Aspinwall prove that in this situation, the spherical functor associated to $F$ is the product of the spherical functors associated to $F_i$.

• So get a decomposition of spherical functors coming from semiorthogonal decomposition.

• **Example:** if $E \subseteq \mathbb{P}^2$, then there’s a set of exceptional objects $\mathcal{O}, \mathcal{O}(1)$ and $\mathcal{O}(2)$ which give a semiorthogonal decomposition of $D^b(\text{coh} \ \mathbb{P}^2)$. The pullbacks $\iota^*\mathcal{O}(i)$ are spherical objects.

**Reconstruction**

• The main point is now that one can use the functors $F_i : \mathcal{A}_i \to \mathcal{C}$ to reconstruct the category $\mathcal{T}$.

• (This part technically uses dg extensions of triangulated categories).
There's a construction due to Kuznetsov and Lunts, which tells us that we can build categories with semiorthogonal decompositions from a pair $A, B$ of dg categories and an $A$-$B$ dg bimodule.

**Example:** If we have two functors $F : A \to C$ and $G : B \to C$, then we form the product $A \times_{F,G} B$ whose objects are triples $(a, b, \mu)$ where $a \in A, b \in B$ and $\mu : F(a) \to G(b)$.

**Theorem 2** (HK). Let $T = \langle A, A^\perp \rangle$, and let $F : T \to C$ be a CY spherical functor. Then there are spherical functors $F : A \to C$ and $H : A^\perp \to C$. The category $A \times_{G,H} A^\perp$ is isomorphic to $T$.

The claim that we made at the beginning of the section then follows by applying this recursively with a bit more detail.

There a slightly more “perverse” way of looking at this.

**Definition 5.** Let $C$ be a (pre)-triangulated (dg) category, then $S_n(C)$ is the category of objects which are diagrams ...

The case where $n = 2$ coincides with Drinfeld’s category $\mathcal{M}or(C)$.

The category $S_n(C)$ admits a whole ton of functors. We’ll distinguish a bunch. Let $f_i$ be the projection onto the $i^{th}$ object. The functor $f_{n+1}$ sends an object to the cone of $(i - 1)^{st}$ morphism.

This should be thought of diagrammatically...

**Theorem 3** (H-Katzarkov). The homotopy fiber product of the diagram

$$\coprod A_i \to \coprod C \leftarrow S_n(C)$$

is equal to the category $T$.

This should be thought of as the category of global sections of a constructible sheaf of categories on a cell complex.

Moreover, the cell complex on which this is a graph with natural cyclic ordering of edges leaving each vertex.

This is then constructible sheaf of categories on a ribbon graph. There’s a canonical embedding of this graph into a topological Riemann surface. In this case, the surface is just the disc.

Morally, this sheaf of categories on the skeleton should be coming from something on the disc itself.

What is the benefit of thinking about things this way?

Bondal and Polishchuk used a very similar construction in the early 90s to build noncommutative deformations of $\mathbb{P}^2$ (“Sklyanin algebras”).

They show the following: take an elliptic curve in $\mathbb{P}^2$. The bundles $\mathcal{O}, \mathcal{O}(1)$ and $\mathcal{O}(2)$ are spherical objects on $E$.

$D^b(\text{coh } \mathbb{P}^2)$ can be reconstructed as the fiber product of the spherical functors associated to $S_0, S_1$ and $S_2$.

If we take $\tau \in \text{Pic}^0(E)$, we get an automorphism of $E$.

For any such $\tau$, there are three new spherical objects, $S_0, S_1$ and $\tau^*S_2$. The categories of global sections of the fiber product associated to these spherical objects.
PERVERSE SCHOBERS

- The structure in the previous section is known as a perverse schober (according to Kapranov and Schectman, 2014).
- The perverse schober in their paper is meant to encode data of the Fukaya-Seidel category of a Lefschetz fibration.
- This can also be viewed as an analogue of perverse sheaves but with coefficients in (dg pretriangulated) categories rather than vector spaces.
- Perverse sheaves are a specific heart in the derived category of constructible sheaves on a manifold.
- The Riemann-Hilbert correspondence gives an equivalence between the category of regular holonomic D-modules on a complex manifold and the bounded derived category of constructible sheaves. Perverse sheaves are the image of regular holonomic D-modules under this correspondence.
- They can be characterized in terms of the support of cohomology sheaves.
- Example: Let $D$ be the unit disc in $\mathbb{R}^2$, and let $p = (0,0)$. There is a very simple linear algebraic/combinatorial description of perverse sheaves on $D$ which are local systems away from $(0,0)$.

**Theorem 4** (Galligo-Granger-Maisonobe, 1985). The category of perverse sheaves is equivalent to a full sub category of representations of the quiver

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\]

Which is given by pairs of vector spaces $V$ and $U$ equipped with a pair of maps $f : V \to U$ and $g : U \to V$ so that $\text{id}_V - g \cdot f$ is invertible.

- Another way to think about this (according to Galligo-Grainger-Maisonobe) is that there’s a real skeleton of the disc (the non-negative real line) and a functor

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Perv(D, p) \to \text{Constr}(K, p)
\]

which is fully faithful.
- Kapranov and Schectman (following Kontsevich) claim that the proper categorical version of this is that there’s a pair of categories connected by a spherical functor.
- A perverse schober (which should be some sort of analogue of perverse sheaves on a disc with coefficients in categories) on a disc with stratification given by a set of points $p_1, \ldots, p_k$ should be given by a constructible sheaf on the following skeleton.

**Theorem 5** (HK). Sklyanin algebras can be obtained by deforming the perverse schober associated to $\mathbb{P}^2$ in the natural way.

- (This corresponds to results of Auroux-Katzarkov-Orlov).
- There’s a natural way to get certain noncommutative deformations of $\mathbb{P}^3$ in a similar way, though the construction is less direct.
- If we try to play the same game, we don’t get much. A generic smooth anticanonical hypersurface in $\mathbb{P}^3$ does not admit continuous deformations.
- We take noncommutative deformations of singular hypersurfaces (quadric intersecting quadric, cubic intersecting hyperplane) so that the spherical objects deform with the hypersurface. This gives us a perverse schober.
• In fact we can extend this to a 2-dimensional perverse schober and show that there are natural deformations which give rise to known noncommutative deformations of $\mathbb{P}^3$.

**Theorem 6.** There’s a two dimensional perverse schober whose natural deformations recover the quartic Sklyanin algebra.