Hodge structures from differential equations
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January 4, 2017

These are notes on a talk on the paper “Hodge structures from differential equations”. The goal is to discuss the method of computation of Hodge numbers of the total space of a Calabi-Yau fiber space coming from computations in their Picard-Fuchs equations.

1 Motivation

• The main goal, which has been, to some extent, completed, is to classify K3 surface fibered threefolds whose generic Picard lattice is of high rank. More specifically, threefolds which are Calabi-Yau.

Definition 1. Let $S$ be a K3 surface and $L$ a lattice of index $(1, n - 1)$. A primitive embedding $\iota : L \rightarrow \text{NS}(X)$ is called a lattice polarization of $S$ if the image of $\iota$ contains an ample class.

Definition 2. A family of K3 surfaces $X \rightarrow U$ is called $L$-polarized for some lattice $L$ if there’s some trivial local subsystem $L$ of $R\pi_*\mathbb{Z}$ so that $L$ restricted to a generic fiber is $L$ and the image of $L$ in $\text{NS}(X_t)$ contains an ample class.

• There are coarse moduli spaces of pairs $(X, \iota)$ of K3 surfaces polarized by a lattice $L$ which turn out to be arithmetic quotients of type IV symmetric domains (with divisors removed).

• The dimension of the space $\mathcal{M}_L$ is $20 - n$ where $n$ is the rank of the lattice $L$.

• K3 surfaces with polarization by a lattice of Picard rank 19 have moduli spaces which are Shimura curves (in some broad sense).

• If the Picard lattice is $M_n = E_8 \oplus E_8 \oplus H \oplus \langle -2n \rangle$, then the moduli space is the classical modular curve $\mathbb{H}/\Gamma_0(n)^+$.

• The theory of threefolds fibered by $M_n$-polarized K3 surfaces is therefore reminiscent of the theory of elliptic surfaces.

• In fact this theory is, in a way, easier in most cases.
Theorem 1 (DHNT 17). If \( n \neq 1 \), then there is a basic family of K3 surfaces \( \mathcal{X}_n \) so that every K3 surface fibered threefold with \( M_n \) polarization is birational to a pullback of \( \mathcal{X}_n \).

- For instance, the mirror quintic admits an \( M_2 \) polarized K3 surface fibration and is birational to a pullback of \( \mathcal{X}_2 \) along the map \( \lambda t^5/s^4(s-t) \).

- In particular, we have been interested in the question of classifying all Calabi-Yau threefolds which admit fibrations by K3 surfaces with Picard lattice \( M_n \) up to birational transformation.

Theorem 2 (DHNT 17). There exist non-rigid Calabi-Yau threefolds fibered by \( M_n \)-polarized K3 surfaces if and only if \( n = 1, \ldots, 9, 11 \).

- The numbers which appear in this classification are precisely the degrees of smooth rank 1 Fano threefolds. This has an explanation via mirror symmetry.

2 The Geometry of K3 fiber spaces

- Since we know what all K3 fibered threefolds with certain lattice polarization look like up to birational transformation, we can ask what sort of geometric characteristics of \( X \) can be determined from the map which characterizes them.

- We will work only with fibrations over \( \mathbb{P}^1 \) since we are ultimately only interested in Calabi-Yau varieties.

- Since studying isotrivial K3 fibered threefolds is essentially the study of automorphisms of K3 surfaces, we will focus on non-isotrivial K3 fibered threefolds.

Theorem 3. If \( Y \) is a threefold over \( \mathbb{P}^1 \) fibered by K3 surfaces which is not isotrivial, then

\[
h^1(Y) = 0, h^{1,1}(Y) = h^2(Y) = \sum (\rho_x - 1) + \rho_Y + 1
\]

where \( \rho_x \) is the number of irreducible components of \( f^{-1}(x) \), and \( \rho_Y \) is the rank of the \( \pi_1(C \setminus \Sigma, p) \) invariant part of \( H^2(f^{-1}(p), \mathbb{Q}) \).

- One can also compute the Hodge structure on \( H^3(X) \) of a K3 fibration over \( \mathbb{P}^1 \).

Theorem 4. If \( X, f \) and \( C = \mathbb{P}^1 \) are as in the discussion above with generic fiber a K3 surface, then we have that

\[
H^3(X, \mathbb{Q}) \cong H^1(C, R^2 f_* \mathbb{Q}_X) \oplus H.
\]

where \( H \) is some Hodge structure of weight 3 whose only nonzero Hodge numbers are \( h^{1,2} \) and \( h^{2,1} \) (these classes are supported on singular fibers). There is a natural Hodge structure on \( H^3(C, R^{i-1} f_* \mathbb{Q}) \) which make this into a decomposition of Hodge structures.
For $M_m$-polarized fibrations, the value of $\rho_X$ is always 19. The values of $\rho_x$ can be computed locally based on the map along which $X$ is a pullback by.

If the topology of the singular fibers of $X$ is nice enough (e.g. all components are rational and the fibers are normal crossings) then $H^3(X, \mathbb{Q}) \cong H^1(\mathbb{P}^1, R^2f_*\mathbb{Q}) \cong H^1(\mathbb{P}^1, j_*R\pi_*\mathbb{Q})$.

We’ve split our computations into two pieces:

1. Geometric – understand the geometry of singular fibers to determine classes in cohomology supported on them.

2. Variational – understand variation of Hodge structure. On a K3 surface, there are two things to look at – variation of algebraic classes, and variation of transcendental classes. The variation of the algebraic classes is finite, but the variation of transcendental data is a bit more difficult.

Our goal today is to discuss some very concrete computations regarding the variation of transcendental classes.

This can of course be generalized to different types of fiber space ...

3 The main theorem

The main theorem that we will employ is the following.

**Theorem 5.** If the generic Picard rank of the fibers of $X$ is 19, then the Hodge numbers of $H^1(\mathbb{P}^1, R^2f_*\mathbb{Q})$ can be deduced from the Picard-Fuchs differential equation.

This theorem has analogues for any fibration by Calabi-Yau varieties whose Hodge numbers are all 1.

4 Picard-Fuchs equations

Take the following data:

1. Hodge structure of $(1,\ldots,1)$-type over a curve $U = \mathbb{P}^1 \setminus \Sigma$ with underlying local system $\mathcal{H}$,

2. A meromorphic section $\omega$ of $\mathcal{F}^2$

The first piece is canonically associated to a rank 19 polarized family of K3 surfaces, since the VHS splits into a trivial part and its orthogonal complement which is of type $(1,1,1)$

for $\gamma$ a local section of $\mathcal{H}$ we get a multivalued meromorphic function

$$g_\gamma(t) = \langle \omega, \gamma \rangle$$
• The Picard-Fuchs equation of this VHS is the differential equation which annihilates all of these local sections. These functions form a local system which is a subsheaf of \( \mathcal{O}_U \).

• It is an ordinary differential which extends to a differential equation of rank 3 with at worst regular singularities on the compactification of \( U \).

• Such a DE can be written as

\[
 f_n(t)\delta^n + \cdots + f_2(t)\delta^2 + f_1(t)\delta + f_0(t)
\]

where \( \delta = t \frac{d}{dt} \). Here \( n \) is the rank of the underlying local system.

• The roots of the indicial equation

\[
 f_n(0)T^n + \cdots + f_2(0)T^2 + f_1(0)T + f_0(0) = 0
\]

are called the characteristic exponents of \( D \) at \( t = 0 \). These are denoted \( \mu_1^0, \mu_2^0, \mu_3^0 \) (listed in order of size). In general, the characteristic exponents at a point \( p \) will be denoted \( \mu_1^p, \mu_2^p \) and \( \mu_3^p \).

• The characteristic exponents of regular singular ODE can be used to compute the local monodromy action on its solutions.

5 The main theorem

• Zucker proves that if \( j: U \hookrightarrow \mathbb{P}^1 \) and \( \mathcal{H} \) is a local system on \( U \) bearing a variation of Hodge structure of weight \( d \), then \( H^1(\mathbb{P}^1, j_* \mathcal{H}) \) has a canonical pure Hodge structure of weight \( d + 1 \).

• In the case where \( \mathcal{H} = Rf_* \mathbb{Q} \) for \( f: X_0 \to U \) is some fibration, this is compatible with the mixed Hodge structure on \( H^d(X_0) \) by the Leray spectral sequence.

**Theorem 6** (DHT 17). Let \( X \) be a family of rank 19 polarized K3 surfaces over \( \mathbb{P}^1 \), and let \( D \) be the PFDE controlling its transcendental variation of Hodge structure

\[
 h^{3,0} = h^0(\mathbb{P}^1, \mathcal{O}(a_f - 4 - \frac{1}{2}a_{1/2} - \sum_{p \in \mathbb{P}^1} (|\mu^p_2| - |\mu^p_1| - 1)))
\]

Here \( a_f \) and \( a_{1/2} \) are numbers which count points in \( \mathbb{P}^1 \) around which local monodromy is of a specific kind.

• We can deduce the Hodge number \( h^{2,1} \) of \( H^1(\mathbb{P}^1, j_* R\pi_* \mathbb{Q}) \) easily from the rank formula for \( H^1(\mathbb{P}^1, j_* R\pi_* \mathbb{Q}) \).

• Note that our definition of the Picard-Fuchs equation is flexible – it does not depend on the choice of \( \omega \). The theorem above only sees differences in characteristic exponents which don’t depend on \( \omega \).
• **An example, The mirror quintic:**

The mirror quintic has a K3 surface fibration over $\mathbb{P}^1$ by K3 surfaces with seven singular fibers and two apparent singularities. The over $\infty$, its internal Picard-Fuchs equation has Riemann scheme

\[
\begin{array}{cccccc}
0 & \infty & p_1 & p_r & p_n \\
1/4 & 0 & 0 & 1 & 0 \\
1/2 & 0 & 1/2 & 2 & 2 \\
3/4 & 0 & 1 & 3 & 4 \\
\end{array}
\]

Then $a_{1/2} = 6, a_f = 1,$ and the sum

\[
\sum_{p \in \Sigma} (|\mu^p_2| - |\mu^p_1| - 1)) = -6
\]

Therefore, we have that

\[1 - 4 - 3 + 6 = 0\]

and $h^{3,0} = 0$ as expected.

• **Sketch of proof:** We know, following Zucker and del Angel, Müller-Stach, van Straten and Zuo that $h^{3,0}(j_*\mathbb{H})$ is equal to $h^1(\mathcal{E}^{0,d})$ where $\mathcal{E}^{i,d-i} = \mathcal{F}^{d-i}/\mathcal{F}^{d-i-1}$.

• We then compute the degree of this bundle using two facts: One is that the parabolic degree of $\deg_{par} \mathcal{E}^{i,i} = -\deg_{par} \mathcal{E}^{i,i}$.

• We use a computation of Eskin, Kontsevich, Möller and Zorich which computes the length of the cokernel of the Kodaira-Spencer maps $\theta_i : \mathcal{E}^{d-i,i} \to \mathcal{E}^{d-i-1,i+1}$ in terms of the characteristic exponents of $D$.

• There is a formula for $h^1(C, j_*\mathcal{H})$ for an arbitrary local system based on the local monodromy matrices of $\mathcal{H}$. Therefore, we can deduce all Hodge numbers of $H^1(C, j_*\mathcal{H})$ from just our differential equations.

• There’s a general technique which we may employ to compute the Hodge numbers of arbitrary $(1,1,1)$ type $\mathbb{R}$-VHS. The formulae are not so nice.

• More generally, we can compute the Hodge numbers of $H^1(\mathbb{P}^1, j_*\mathcal{H})$ for arbitrary $(1,1,\ldots,1)$ type $\mathbb{R}$-VHS using a combination of the results of EKMZ and dAMSvSZ.

6 **An application**

• We return to our original motivation for considering these questions. The family of K3 surfaces discussed in the previous example form the basic family $\mathcal{X}_2$. Therefore, every $M_2$-polarized K3 surface is birational to a pullback along the period map $g : C \to \mathbb{P}^1$.

• We can ask, conversely, along which rational maps $g : \mathbb{P}^1 \to \mathbb{P}^1$ can $g^*\mathcal{X}_2$ have a birational model which is Calabi-Yau. This can occur only when $h^{3,0}(\mathbb{P}^1, j_*g^*R^2f_*\mathbb{Q}) = 1$ and $C = \mathbb{P}^1$. 5
**Theorem 7.** The family $g^*X_2$ can be Calabi-Yau (if and) only if $g$ has two points over 0 and $g$ ramifies to order 1, 2 or 4 at these points or if there is only one point over 0 and $g$ ramifies to order 5, 6, 7, 8. If $X$ has smooth resolution, we can compute its Hodge numbers based on the map $g$.

- More generally, we are able to get a classification of all (non-rigid) Calabi-Yau three-folds fibered by $M_n$-polarized K3 surfaces and give a formula for their Hodge numbers.