Recovering a variety from it's derived category

Chaitanya Guttikar

Abstract

We prove that a fibration over a smooth scheme with projective Fano/Anti Fano fibers is determined by the derived category of the total space (A Torelli type result in the derived setting). This supports the basic philosophy that the derived category can in fact be looked at as an invariant of the variety containing geometric information, instead as just an abstract tool or a language for convenient calculations and statements.

Introduction

Derived categories have been an essential part of mathematics since Grothendieck introduced them. The main motivation at that time was to generalize the Serre duality theorem to a relative setting, allowing the morphism to be non-smooth. Verdier in his thesis extended the idea into an abstract duality theorem. Many statements involving spectral sequences and derived functors take a very simple form when expressed in the language of derived categories. Also, many complicated calculations and arguments can now be done more transparently using the abstract tools available for manipulating objects in the derived category.

Mukai discovered [26] that the derived category of an abelian variety A is isomorphic to the derived category of it's dual $\hat{A}$, and the isomorphism is in fact given by pulling back an object to $A \times \hat{A}$, then tensoring it with the Poincare bundle $\mathcal{P}$ and pushing it forward (every operation is in the derived sense). He used this fact to study moduli of sheaves and vector bundles on abelian varieties and also the closely related Kummer surfaces [27]. Another important discovery that followed was that the moduli of stable sheaves on a K3 surface X is another K3 surface M and these two surfaces have isomorphic derived categories, again the isomorphism being given by the universal family over $X \times M$. This led to a search for non-isomorphic varieties which have isomorphic derived categories.

In (1981), in their seminal paper on perverse sheaves [3], Beilinson, Bernstein and Deligne axiomatized the idea of a derived category and derived functors into the notion of triangulated categories and triangulated functors between them. They also introduced the notion of a t-structure on a triangulated category $\mathcal{T}$ and it's core $\mathcal{A}$, which is a abelian category, and they proved results describing the conditions on the t-structure under which $\mathcal{D}^b(\mathcal{A}) = \mathcal{T}$. So, the search for different varieties with the same derived categories is essentially the search for different t-structures on a given triangulated category.

Many important technical results were proved in the following two decades, perhaps the most important from our point of view being the

Orlov representation theorem: *Any full and faithful functor $\phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$, for X, Y smooth projective varieties, is given by an integral transform. i.e. $\exists\ E \in \mathcal{D}^b(X \times Y)$ such that $\phi(\mathcal{F}) = Rq_*(Lp^*(\mathcal{F}) \otimes^L E) := \psi_E(\mathcal{F})$, where $p, q : X \times Y \to X, Y$ are the projections. Moreover, such an E is unique up to an isomorphism.*

From the results on saturatedness proved in [7], it follows that all triangulated functors between derived categories of coherent sheaves have right adjoints. So the condition of existence of a right adjoint is superfluous.

But the point of view towards derived categories was still that of a tool or a language rather than an invariant.The first instance of a geometric result proved using the derived category was the McKay correspondence. The classical McKay correspondance is a statement giving bijection between the irreducible representations of a finite subgroup G of $SL(2, \mathbb{C})$ and the set of vertices of an extended Dynkin diagram of the type ADE. A geometric form of this correspondence was formulated and proved by Klein and was a statement about the K-theory of $\mathbb{C}^2/G$. The idea that this correspondence should lift to a statement about K-theory and to derived categories was apparently known to Verdier. But the first conjectural statement and some interesting calculations were done by M. Reid [30, 31]. The result was proved in a concrete form by Bridgeland, King and Reid [8] and states that if G is a finite group of automorphisms of a complex twofold or threefold M such that the canonical bundle ω_M is locally trivial as a G-sheaf, then the Hilbert scheme $Y = G - Hilb(M)$ parametrising G-clusters on M (these are essentially scheme theoretic G-orbits) is a crepant resolution for $X = M/G$ and there is a equivalence of derived categories of coherent sheaves on Y and G-equivariant coherent sheaves on M. (This identifies the K-theory of Y and G-equivariant K-theory of M, thus generalizing the classical statement of Klein). The important difference in this proof and the old proof of Klein is that Klein's result is mainly based on case by case analysis of classification of the singularities of $\mathbb{C}^2/G$. The proof of Bridgeland is unified and gives us the existence of a canonical crepant resolution for M/G, a fact not known previously. It has a much better hope for generalizing to higher dimensions, although there are known technical issues for doing that.

Two other important geometric results that emphasize our point of view are:

1 Theorem. (Bondal and Orlov) *If $p : X \to Y$ is such that $Rp_*(\mathcal{O}_X) = \mathcal{O}_Y$, then $\mathcal{D}^b(Y)$ embeds in $\mathcal{D}^b(X)$ as a right admisible subcategory. i.e. the inclusion functor $Lp^* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ is full and faithful and has a right adjoint.*

Thus, we potentially have a new way of identifying a minimal model: It's the variety in it's birational class with the "smallest" derived category. This point is also supported by the following result:

2 Theorem. (Bridgeland) *Let X be a projective threefold with Gorenstein terminal singularies. Let $f : Y \to X$ be a crepant resolution of X. (So it only contracts finitely many curves). If $g : W \to X$ the flop of f, then $D(Y) \cong D(W)$.*

The theorem implies in particular that birational Calabi-Yau threefolds have equivalent derived categories and thus gives a new proof of the theorem (due to V.V.Batyrev) that birational Calabi-Yau threefolds have the same Hodge numbers.

The next and major evidence and motivation for a change in view point towards derived categories and our main inspiration for this work is the paper of Bondal and Orlov [5] that proved that a smooth projective variety with ample canonical or anticanonical bundle is completely determined by it's derived category. More precisely:

3 Theorem. *If X and Y are smooth projective varieties such that ω_X is ample or anti-ample and if $\phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is a graded additive isomorphism, then $\exists\ f : X \to Y$ an isomorphism of varieties such that $\phi = Rf_*$. (upto twists by line bundles and shifts).*

Their proof is deceptively simple and beautifully geometric.The basic idea is to identify in the derived category $\mathcal{D}^b(X)$ the set of all skyscraper sheaves and their shifts just based on their cohomological properties. This gives us the variety X as a set. Then the line bundles on X are identified using their cohomological properties in relation to the skyscrapers. This gives us the Zariski topology on X as the closed sets can be identified as zeros of morphisms between the line bundles. Then the algebraic structure is determined by identifying the canonical ring, $\mathrm{R} = \bigoplus_n H^0(X, \omega_X^n)$ which, in case of ample or anti-ample ω_X, determines the variety.

There are many directions in which one might want to generalize this result, a singular version and a relative version being two of the choices. This work is about the relative version. Thus assume we are given a family $\pi : X \to S$ of Fano/anti-Fano varieties. Here is rough statement of our main result. We have omitted the technical assumptions about X, Y and S, the reasons will be clear shortly:

Relative Bondal Orlov Theorem: *Let $\pi_1 : X \to S$ and $\pi_2 : Y \to S$ be two families of Fano/Anti-Fano varieties over a scheme S. Let $\phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be an S-linear functor. If ϕ is an equivalence of categories, then $\exists\ f : X \to Y$ an S-linear isomorphism.*

If we try to imitate the argument of Bondal and Orlov, we immediately run into problems since the counterpart of the skyscraper sheaves in this case would be sections $\sigma : S \to X$, but these might not exist at all. There might be a way to complete the proof using local étale sections but they also exist only for smooth morphisms π.

Thus, we need to proceed along different lines than the proof of the Bondal Orlov discussed above. The motivation for this proof comes from looking closely at Bridgeland's proof of the theorem on flops. In it, the flop W is constructed as the fine moduli space of objects in $D(Y)$ (called "perverse point sheaves", although they are not really sheaves) and then the universal family $\mathcal{P}$, which is an object in $D(Y \times W)$, gives the isomorphism by the Fourier-Mukai transform $\psi_{\mathcal{P}} : D(Y) \to D(W)$. Moreover the morphism $g : W \to X$ is constructed by proving that the object $R(id \times f)_*(\mathcal{P}) \in D(X \times W)$ is in fact the structure sheaf $\mathcal{O}_{\text{gragh}(g)}$ of the graph of some morphism g (upto twisting by pull backs of line bundles). A lot of technical details enter this proof, the most important being the stability condition one needs to impose on the perverse point sheaves in order to enable us to construct the moduli using GIT. Bridgeland guesses the right condition and goes ahead to prove the existence of the moduli. In his later works [10], motivated by physics (mainly work of Douglas [13] on D-branes), he was able to define the notion of an abstract stability condition on objects in the derived category and proved that the his ad-hoc definition is in fact a special case of this new notion.

What we want to do is recast the proof of the Bondal Orlov result in such a way that X is reconstructed as a moduli space of it's skyscraper sheaves. Our life is a bit simpler since we already have an equivalence $\phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$. By the Orlov Theorem above, we get an object $E \in \mathcal{D}^b(X \times Y)$ such that $\phi = \psi_E$. The claim is that $Z = Supp(E) \subset X \times Y$ is mapped isomorphically onto X by the projection $p : X \times Y \to X$ (and similarly onto Y) which gives the required isomorphism of varieties.

In order to generalize this proof to relative situation, we need some version of the Orlov theorem. Suppose we have an object $E \in \mathcal{D}^b(X \times Y)$ which gives the functor. We can prove, using the proof for the single fiber case and some purely algebro-geometric results (viz. Any Generically geometrically finite flat morphism with connected fibers is an isomorphism) to conclude that $Z = Supp(E)$ maps isomorphically onto X and Y. But the morphism we obtain might not be S-linear as we want. The idea now is to prove that, in fact, $\bar{E} = E \otimes \mathcal{O}_{X\times_S Y}$ gives the same functor as E. This is achieved by proving first that the functors ψ_E and $\psi_{\bar{E}}$ agree on skyscraper sheaves, which form what is called a spanning class. Then we prove the abstract result that if two functors agree on a spanning class, then they are isomorphic. And since $Supp(\bar{E}) \subset X \times_S Y$, the projection is automatically an S-morphism.

Note that once we have the object $E \in \mathcal{D}^b(X \times Y)$ the rest of the proof does not add any new assumptions on X, Y or S. Thus, the generality in which we obtain the recovery result depends completely on the generality in which we can guarantee the existence of E. So, for example, if we just apply Orlov's result, we have the theorem we are proving for X and Y projective and smooth. We do not need the family to be smooth, only the total space. The recent work of Toën [32] proves that any triangulated functor between two unbounded derived categories of quasi-coherent sheaves is given by a quasi-coherent object on the product. We hope to prove a more general version of our result using this new development. Also, Canonaco and Stellari [12] have proved a version of the Orlov representation theorem for twisted varieties and our result immediately gives a recovery theorem in that case as well. As we understand the structure of functors between triangulated categories better, we will have more and more such Torrelli type results for derived categories, demonstrating it's role a strong and very useful geometric invariant.

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