Research synopsis

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My main area of interest is algebraic geometry and the way in which its interaction with physics, category theory and arithmetic is enriching our understanding of it. More specifically, I am working on derived categories (of coherent sheaves) on a variety (and other related categories like dg-categories and model categories). The basic philosophy is that the derived category can in fact be looked at as an invariant of the variety containing geometric information, instead as just an abstract tool or a language for convenient calculations and statements. Although this point of view is relatively new, it can be made to bear on very classical problems like existence of crepant resolutions, moduli of stable sheaves and vector bundles on Abelian varieties and the minimal model program. Also, many classical invariants such as K-theory and Hochschield homology/cohomology can be recovered from the derived category. At the same time, interaction with string theory has led to a whole new set of conjectural and speculative relationships with the Langlands program and special values of L-functions. In my future research I hope to achieve a better understanding of these categorical invariants and their geometric and arithmetic applications and related conjectures.

Categories as invariants

Although some times they are not stated in a form which will make it evident, there are many theorems in the literature that suggest that categories can be very useful invariants. Bondal and Orlov [5] proved a Torelli type theorem stating that a smooth Fano or anti-Fano variety is determined by it's derived category. I generalized this result to a relative setting to prove that a family of Fano or Anti-Fano varieties with smooth total space (fibers need not be smooth) over a fixed projective base is also determined by it's bounded derived category of coherent sheaves. The techniques used in this proof are different from the original proof of Bondal and Orlov and adapt well for extending the result to some other situations. It uses Orlov's theorem [28] on representing a functor as a Fourier-Mukai transform. So, combining these techniques with the results of Canonaco and Stellari [11] I have also proved a recovery result for twisted varieties. The recent work of Toën [31] on homotopy theory of dg-categories yields a very general Orlov type result and I hope to use the techniques in there to extend my results to singular varieties and families over more general base. An ideal generalization would be to associate a category $\mathcal{D}(f)$ to a morphism $f : X \to S$ so that when the base S is $\operatorname{Spec}(k)$, we recover the derived category of X and then to prove a Torelli theorem for this category. The most serious hurdle here perhaps is the lack of a good deformation theory for derived/triangulated categories which makes it hard to even guess what we want this category to look like. Another related problem is of studying the monodromy action on the derived category. Both of these are partly due to the fact that triangulated categories don't localize well. Namely, if we have a cover $\cup U_i \to X$ and we are given triangulated categories $\mathcal{T}_i$ for each U_i (e.g. $\mathcal{T}_i = \mathcal{D}^b(U_i)$)with restriction functors $U_i \to U_i \cap U_j$ satisfying sheaf theoretic compatibility conditions on the threefold intersection, we have no functorial way of constructing a triangulated category $\mathcal{T}(X)$. In fancier language this means that the functor $X \mapsto \mathcal{D}^b(X)$ is not a sheaf. These are hard problems because they do not fit in the 20th century tenet of proving local results using analysis or algebra and then gluing them together to get global ones. But some work has been done in this direction (Abramowich and Polishchuk [1], Lunts and Orlov [21, 22, 23]) and some monodromy calculations have been done by Kuznetsov [17, 18, 19, 20]. Another direction is looking at other categories (such as dg-categories) as better candidates for these type of problems. This is a very active field right now. While the question of attaching a category to a morphism in a meaningful way seems quite formidable to me, the answers to the other questions will be a useful and reassuring step towards this goal.

Another classical problem that I am trying to understand with the hope of relating it to categorical methods is the Brill-Noether conjecture. Roughly speaking, the question is "How many linear systems with degree d and dimension r exist on a general curve of genus g". A precise answer is known in terms of the Brill-Noether number. But the proofs available right now involve a tricky argument using K3 surfaces or by degeneration to a special curve, where as a more desirable one would be to lift such a line bundle step by step to higher and higher order neighborhoods to get an infinitesimal lift. An open problem for instance is to prove that if a curve with negative Brill-Noether number has a line bundle with r sections, then it can be lifted to a neighborhood of order at most r+1. Arbarello, Cornalba, Griffiths and Harris [2] did manage to prove the statement in this fashion for low degree curves in $\mathbb{P}^2$ and $\mathbb{P}^3$ but the sticking point in going to higher degrees and dimensions is that it becomes difficult to keep track of the choices of liftings made at each stage, each being unique only up to homotopy. Since the theory of model and dg-categories precisely aims at dealing with such choices in a consistent manner, we hope that a dg-categorical formulation of the problem will lead to a more conceptual proof.

Derived Category and Birational Geometry

Birational geometry is the study of properties of varieties that are invariant under birational transformations. Mori's minimal model program is an important credo in this field.

Two important geometric results that demonstrate the relevance of derived category to birational geometry are:

1 Theorem. (Bondal and Orlov) *If* $p : X \to Y$ *is such that* $Rp_*(\mathcal{O}_X) = \mathcal{O}_Y$*, then* $\mathcal{D}^b(Y)$ *embeds in* $\mathcal{D}^b(X)$ *as a right admisible subcategory. i.e. the inclusion functor* $Lp^* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ *is full and and has a right adjoint.*

Thus, we potentially have a new way of identifying a minimal model: It's the variety in it's birational class with the "smallest" derived category. This point is also supported by the following result:

2 Theorem. (Bridgeland) *Let X be a projective threefold with Gorenstein terminal singularies. Let $f : Y \to X$ be a crepant resolution of X. (So it only contracts finitely many curves). If $g : W \to X$ the flop of f, then $D(Y) \cong D(W)$.*

The theorem implies in particular that birational Calabi-Yau threefolds have equivalent derived categories and thus gives a new proof of the theorem (due to V.V.Batyrev) that birational Calabi-Yau threefolds have the same Hodge numbers.

There are many conjectures that support our philosophy that are still wide open. For instance, Bondal and Orlov have conjectured that any two birational Calabi-Yau varieties should have isomorphic derived categories. This result is known so far only in dimension ≤ 3 and it follows from Bridgeland's result. Can a similar conjecture be formulated for fibrations of Calabi-Yau varieties and proved using the techniques in my result ?

Perhaps more important than his result itself is the Moduli construction of Bridgeland, in particular the idea of a stability condition on a triangulated category [9]. This construction not only gives a good notion of stability of objects in the derived category needed for constructing moduli spaces of these objects but can be used to associate a complex manifold $Stab(\mathcal{T})$ ("the space of stability conditions") to an abstract triangulated category $\mathcal{T}$. This new complex manifold and it's cohomology etc. can then be thought of as invariants of $\mathcal{T}$. This work is directly inspired by the work of Douglas [12] on Stability of D-Branes which also relates (with "physics proofs") the Calabi-Yau monodromy to auto-equivalences of the derived category. It would be interesting to see how this physical intuition finally gets converted into rigorous proofs.

Derived categories and Mirror symmetry

Mirror symmetry in physics is a statement that says roughly that two models (called the A and B models) of the universe give equivalent theories up to an automorphism. The statement has been proved to a physicist's satisfaction. In mathematics however, it takes various highly elaborate forms. One such is as a conjectural equivalence between the derived category of coherent sheaves on a manifold X and the Fukaya category of Lagrangian submanifolds (with some more data of a line bundle with a flat connection) on it's mirror manifold Y. This "homological mirror symmetry" was conjectured by Kontsevich in [16] and has spurred a lot of activity since then. Although, on the face of it this conjecture seems very abstract, it has many (sometimes conjectural) geometric implications. A more classical statement formulated by Candelas is relating the number (suitably defined) of rational curves on X using the period integrals on Y. It was proved by Givental and another proof was given by Yau. Also, the moduli of complex structures on X corresponds to the moduli of Kähler structures on Y and the algebraic cycles on this moduli are supposed to be related to vanishing cycles on the other. These are very geometric statements and will facilitate calculations. Also, even though physicists were initially not interested in this homological statement, it has become important after the second string theory revolution and the "discovery" of branes.

Categories and Langland's program

The Langlands program is a statement relating representations of the Galois group of a number field to automorphic forms. By usual analogy between Riemann surfaces and number fields, we can obtain a geometric conjecture, which is an equivalence between the derived category of constructible sheaves on the stack Bun_G and the derived category of coherent sheaves on the stack of local systems on the curve X (where G is the Galois group of the function field of X). This geometric analog turns out to be intimately related to quantum field theory [13, 15] and the relationship gives new insights into the geometry itself. For example, it naturally extends to a ramified version of the geometric statement, which in some sense is the only interesting case from a number theoretic viewpoint. Also, abstract objects like Hecke eigensheaves and $\mathcal{D}$-modules seem to appear very naturally in physics, giving new insights. I would like to understand this interplay between the quantum field theory, geometry and number theory better.

Other reasons to study Derived categories

Although my main interest in studying categories is obtaining more classical geometric results, there are also other reasons to understanding them.

Firstly, there are certain phenomena that take place at the derived level that are not reflected in the classical geometry of the variety. For instance, we know many examples of pairs of varieties with isomorphic derived categories which suggests a a close relationship between them. But it has no direct classical geometric counterpart, although in many cases, this isomorphism can be used to get geometric results.

Secondly, the categorical approach to geometry is likely to generalize more easily to non-commutative spaces, although we do not have a good definition of them yet. Since these spaces do not have points in the classical sense, we can't do classical geometry on them. But the hope is that all the associated categories sheaves on it will still make sense. For example, we want to think of a non-commutative ring A as a ring of functions on a non-commutative space $Spec(A)$. Even though we don't have a pointwise description of $Spec(A)$, the category of coherent sheaves on it, i.e. of finitely generated modules on A makes sense.

Perhaps the strongest driving force for studying categories of sheaves and a big source of conjectures right now is string theory. The derived category of coherent sheaves on a variety naturally arises in string theory as the category of branes in a topological twist of the associated sigma model. Thus, in order to extract the information that seems to be contained in the conformal field theories associated to a variety one has to look at it in a categorical framework. For example, many duality statements in string theory can be mathematically interpreted as equivalences of derived categories. Thus, physics has become an important source of conjectures that algebraic geometry is currently far away from proving and categorical methods seem to play an important role in this direction.

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