

MAT 111  
Exam 2

**Solutions**

October 29, 2009

1. (10 points) The graph of the derivative  $f'(x)$  is given below. Draw the graph of  $f(x)$  and  $f''(x)$ .

2. (10 points) Find the equation of the tangent line to  $\sqrt{2x+2y} = 1+x^2y^2$  at the point  $(1,1)$ .

The equation is

$$y - y_0 = \frac{dy}{dx}(x - x_0)$$

So all we need is  $dy/dx$ . Differentiating the above equation implicitly, we get

$$\frac{1}{2\sqrt{2x+2y}}(2 + 2\frac{dy}{dx}) = 2xy^2 + 2y\frac{dy}{dx}x^2$$

So, substituting  $x = 1$  and  $y = 1$ , we get

$$\frac{1}{4}(2 + 2\frac{dy}{dx}) = 2 + 2\frac{dy}{dx}$$

solving, we get  $\frac{dy}{dx} = -1$ . So the equation of the tangent is

$$y - 1 = -1(x - 1)$$

OR

$$y = -x + 2$$

3. (30 points) Evaluate the following:

(a)  $\lim_{x \rightarrow 2} \frac{x^4 - 2x - 12}{x - 2}$

Solution:

Notice that if we let  $f(x) = x^4 - 2x$  then  $f(2) = 12$  and so the above limit is actually equal to

$$\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$$

which is precisely  $f'(2)$ . Thus, all we need to do is evaluate  $f'(x)$  and substitute 2.

$$f'(x) = 4x^3 - 2$$

and so

$$\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = f'(2) = 4 \cdot 2^3 - 2 = 30$$

. (For the PE class students, the same solution with  $f(x) = x^4 + 2x$  and  $f'(x) = 4x^3 + 2$  and limit = 34).

(b) Find  $f'(x)$  if  $f(x) = \frac{\sec 3x}{1 + \sin x}$  Solution: By Quotient Rule,

$$f'(x) = \frac{(1 + \sin x) \frac{d}{dx}(\sec 3x) - \sec 3x \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$$

By Chain Rule  $\frac{d}{dx}(\sec(3x)) = \sec(3x) \tan(3x) \frac{d}{dx}(3x) = 3 \sec(3x) \tan(3x)$ . So the final answer is:

$$\frac{3 \sec(3x) \tan(3x)(1 + \sin x) - \cos(x) \sec(3x)}{(1 + \sin x)^2}$$

(c) Find  $f'(x)$  if  $f(x) = \sin^2 \left( \frac{x\sqrt{x^3+1}}{\cos x + 2} \right)$ .

Solution: This is just a repeated application of chain rule, quotient rule and product rule: Outer most functions is square, next is sin and then there is a quotient to be differentiated.

$$f'(x) = 2 \sin \left( \frac{x\sqrt{x^3+1}}{\cos x + 2} \right) \frac{d}{dx} \left[ \sin \left( \frac{x\sqrt{x^3+1}}{\cos x + 2} \right) \right]$$

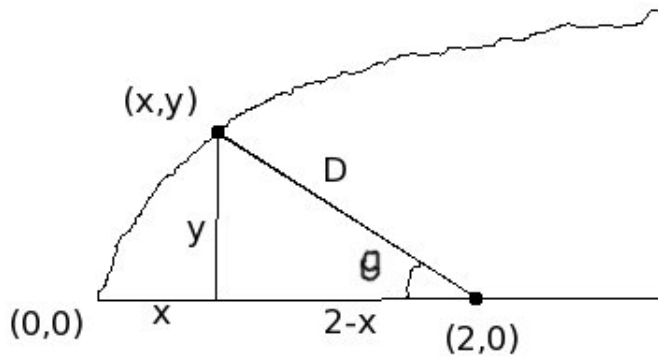
$$\begin{aligned}
&= 2 \sin \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \left[ \cos \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \right] \frac{d}{dx} \left\{ \frac{x\sqrt{x^3+1}}{\cos x+2} \right\} \\
&= 2 \sin \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \left[ \cos \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \right] \left\{ \frac{(\cos x+2) \frac{d}{dx}(x\sqrt{x^3+2}) - x\sqrt{x^3+2} \frac{d}{dx}(\cos x+2)}{(\cos x+2)^2} \right\} \\
&= 2 \sin \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \left[ \cos \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \right] \\
&\quad \cdot \left\{ \frac{(\cos x+2)(\sqrt{x^3+1} + x \frac{1}{2\sqrt{x^3+1}}(3x^2)) - x\sqrt{x^3+2}(-\sin x)}{(\cos x+2)^2} \right\} \\
&= 2 \sin \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \cos \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \left\{ \frac{(\cos x+2)(\sqrt{x^3+1} + \frac{3x^3}{2\sqrt{x^3+1}}) + x\sqrt{x^3+2} \sin x}{(\cos x+2)^2} \right\} \\
&= 2 \sin \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \cos \left( \frac{x\sqrt{x^3+1}}{\cos x+2} \right) \left\{ \frac{(\cos x+2)(\frac{5x^3+2}{2\sqrt{x^3+1}}) + x \sin x \sqrt{x^3+2}}{(\cos x+2)^2} \right\}
\end{aligned}$$

4. (25 points) Assuming that the earth is located at the point  $(2, 0)$ , a comet is observed to be moving along a parabolic path given by  $y = \sqrt{x}$  (when the distance is measured in units of million km). It is observed that the  $x$  co-ordinate of the comet is changing at 3000 km/s.

- How fast is the distance between earth and the comet changing, when the comet is at  $(1, 1)$ .
- A telescope on earth is constantly tracking the comet. How fast is the telescope rotating, when the comet is at  $(1, 1)$  ?

This problem is almost identical to Problem 20 from Section 2.8 which was on the homework.

Solution:



Part a):

The location of the comet is  $(x, y)$ , but since it moves on the curve  $y = \sqrt{x}$ , it's location is  $(x, \sqrt{x})$ . So, its distance  $D$  from earth's location  $(2, 0)$  is

$$D = \sqrt{(x - 2)^2 + (\sqrt{x} - 0)^2} = \sqrt{x^2 - 4x + 4 + x} = \sqrt{x^2 - 3x + 4}$$

Differentiating, we get

$$\frac{dD}{dt} = \frac{d}{dt} \sqrt{x^2 - 3x + 4} = \frac{1}{2\sqrt{x^2 - 3x + 4}} \cdot (2x - 3) \cdot \frac{dx}{dt}$$

Now, if distances are measured in units of "million km", then we have to convert the rate of change of  $x$  into that same unit. So,  $\frac{dx}{dt} = 3000/10^6$ . So, substituting  $x = 1$  and  $\frac{dx}{dt} = 3000/10^6$ , we get

$$\frac{dD}{dt} = \frac{1}{2\sqrt{1 - 3 + 4}} \cdot (2 - 3) \cdot 3000 \cdot 10^{-6} = \frac{-3000}{2\sqrt{2} \cdot 10^6} \text{ million kilometers/sec}$$

In other units:

$$\frac{dD}{dt} = \frac{-3000}{2\sqrt{2}} \text{ km/s}$$

Part b)

Now, we just have to relate the angle with quantities for which we do know rate of change. We have

$$\cos(\theta) = \frac{2-x}{D}$$

Now, we have two choices. We can do implicit differentiation, and substitute later (using  $dD/dt$  calculated in part a), or we can substitute for  $D$  in terms of  $x$  and then just differentiate. We will do the first method, since it's less messy. We get:

$$-\sin(\theta) \frac{d\theta}{dt} = \frac{D(-1) \frac{dx}{dt} - (2-x) \frac{dD}{dt}}{D^2}$$

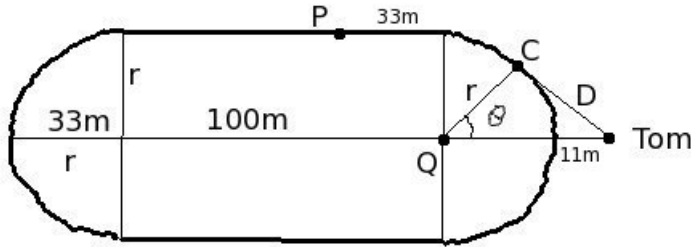
When the comet is at  $(1, 1)$ , we have  $D = \sqrt{2}$  and  $\theta = \pi/4$  so  $\sin \theta = 1/\sqrt{2}$  and  $\frac{dx}{dt} = 3000/10^6$ . We also know from part a) that at this point  $dD/dt = \frac{-3000}{2\sqrt{2} \cdot 10^6}$ . Substituting, we have

$$-\frac{1}{\sqrt{2}} \frac{d\theta}{dt} = \frac{-\sqrt{2} \frac{3000}{10^6} - (2-1) \frac{-3000}{2\sqrt{2} \cdot 10^6}}{(\sqrt{2})^2}$$

So, simplifying, we get

$$\frac{d\theta}{dt} = \frac{9}{4000} \text{ radians/s}$$

5. (25 points) A runner (Let's call him C) is running around a 400m running track (a real one this time), shown in the figure below, at 7m/s. His coach Tom is standing 11 meters away from the nearest point on the track as shown in the figure, recording the run on a handcam for analysis later (and for posting on youtube).



Assume that  $(100/\pi) \approx 33$  (for ease of calculation) and answer the following.

- How fast is the distance between Tom and C changing, when C is 33 meters away from Tom. (This is the rate at which the zoom on the camera needs to be adjusted to maintain a full frame close up of C).
- How fast should Tom be turning to keep C in frame, when C is at the point P shown.

This problem is really two problems that were discussed in class. Part a) is the same as Problem 37 in Section 2.7 and Part b) is the same as Problem 26 from the same section.

Solution: First we need to figure out some geometric information. According to the problem the total length of the track is 400m and the straight parts of the track are 100m each. So the remaining two semi-circular parts are also 100m each. So we have  $\pi r = 100$  and so  $r = 100/\pi \approx 33$ .

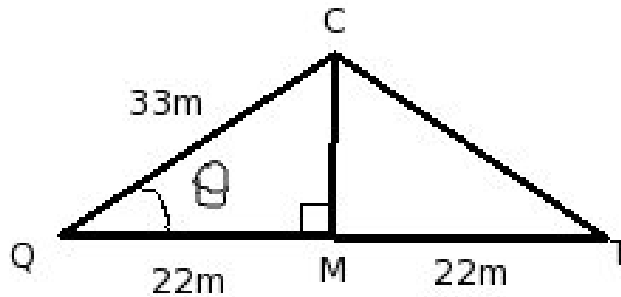
Part a): Since  $r = 33m$ , the distance between Tom (T for short) and the point Q which in the figure (the center of the semi-circle) is 44m. So, using the law of cosines for triangle TCQ, we can write

$$D^2 = 33^2 + 44^2 - 2 \cdot 33 \cdot 44 \cos \theta$$

So differentiating, we get

$$2D \frac{dD}{dt} = -2 \cdot 33 \cdot 44 (-\sin(\theta)) \frac{d\theta}{dt} = 2 \cdot 33 \cdot 44 \sin(\theta) \frac{d\theta}{dt}$$

Now, since C is running at 7m/s, we can calculate  $d\theta/dt$ . It takes him  $100/7$  seconds to run the semi-circle, which is  $\pi$  radians. So the angular speed is  $\frac{\pi}{100/7} = 7/33$  (again using the approximation  $100/\pi \approx 33$ ). Now, all we need to do is  $\sin \theta$  when  $D = 33$ , and we can just substitute. One brut force way is to calculate  $\cos \theta$  using the law of cosines again to find  $\cos \theta$  and then find  $\sin \theta$ . But when  $D = 33$ , the triangle  $CQT$  isosceles, so our life is much easier. If we drop a perpendicular from C onto line  $QT$ , it will bisect it into two equal parts  $QM$  and  $MT$  both of length 22m.



Thus, looking at the figure, we have  $CM = \sqrt{33^2 - 22^2} = 11\sqrt{5}$ . And so,

$$\sin \theta = \frac{CM}{CQ} = \frac{11\sqrt{5}}{33} = \frac{\sqrt{5}}{3}$$

. Now, we can substitute everything in the equation above, and we get:

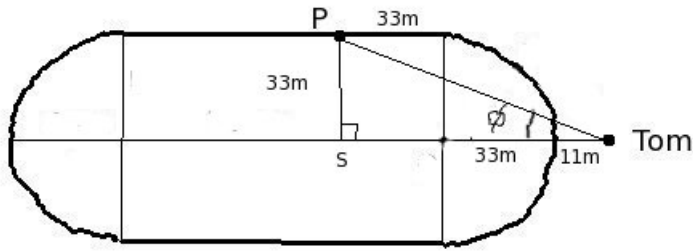
$$2 \cdot 33 \cdot \frac{dD}{dt} = 2 \cdot 33 \cdot 44 \cdot \frac{\sqrt{5}}{3} \cdot \frac{7}{33}$$

And this gives

$$\frac{dD}{dt} = \frac{28\sqrt{5}}{9}$$

Part b):

This part is much easier. We have the easy diagram shown below. We know that the distance  $TS$  is increasing at 7m/s and we want to find  $\frac{d\theta}{dt}$ .



We know that the distance  $TS$  is increasing at  $7\text{m/s}$  and we want to find  $\frac{d\theta}{dt}$ . But we have

$$\tan \theta = PS/ST$$

and so

$$ST = 33 \cot \theta$$

. Thus, differentiating we get:

$$\frac{d}{dt}ST = 33(-\csc^2 \theta) \frac{d\theta}{dt}$$

Now, from the diagram, when  $C$  is at  $P$ , we have  $ST = 33 + 33 + 11 = 77$  and so we have  $PT = \sqrt{33^2 + 77^2} = 11\sqrt{58}$ . So,  $\csc \theta = 11\sqrt{58}/33 = \sqrt{58}/3$ . Substituting in the above equation we get

$$7 = -33 \cdot \left(\frac{\sqrt{58}}{3}\right)^2 \cdot \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{-21}{11 \cdot 58} \text{ rad/s}$$

6. (10 points) Find the approximate value of  $\tan(51^\circ)$  using linear approximation at  $x_0 = 45^\circ$ .

In order to do calculus, we need convert everything into radians.  $45^\circ = \pi/4$  radians and  $51^\circ = 51\pi/180$  radians.

Now the linear approximation at  $x_0$  is

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

. In our case,  $x_0 = \pi/4$ ,  $f(x) = \tan(x)$  and so  $f'(x) = \sec^2(x)$ . So,  $f(x_0) = \tan \pi/4 = 1$  and  $f'(x_0) = \sec^2 \pi/4 = (\sqrt{2})^2 = 2$  and we have

$$L(x) = 1 + 2(x - \pi/4)$$

.

Substituting  $x = 51\pi/180$  we get

$$\tan(51^\circ) \approx 1 + 2\left(\frac{51\pi}{180} - \frac{\pi}{4}\right) = 1 + 2\left(\frac{51\pi}{180} - \frac{45\pi}{180}\right) = 1 + 2 \cdot \frac{6\pi}{180} = 1 + 2 \cdot \frac{\pi}{30}$$

So, if we approximate  $\pi \approx 3$ , we get  $\tan(51^\circ) \approx 1 + 2/10 = 1.2$ .