MAT 111
Practice Test 3
Solutions
1. (25 points) Find the absolute maximum and minimum values of $f$ on the given interval.

(a) (15 points) $f(x) = x^3 - 1.5x^2 - 6x + 1$, $[-2, 4]

solution:

$f'(x) = 3x^2 - 3x - 6 = 3(x^2 - x - 2) = 3(x - 2)(x + 1)$

So, critical points are when $f'(x) = 0$. That is $x = 2$ and $x = -1$.

Evaluating $f$ at critical points and end points:

- $f(-2) = (-2)^3 - 1.5(-2)^2 - 6(-2) + 1 = -8 - 6 + 12 + 1 = 1$
- $f(-1) = -1 - 1.5 + 6 + 1 = 4.5$
- $f(2) = 8 - 6 - 12 + 1 = -9$
- $f(4) = 64 - 24 - 24 + 1 = 17$

So,

absolute minimum is $-9$ at $x = 2$

Absolute Maximum is $17$ at $x = 4$.

(b) (10 points) $f(x) = x + 2 \sin x$, $[-\pi, \pi]

Solution:

$f'(x) = 1 + 2 \cos x$, so critical points are when $1 + \cos x = 0$ that is when $

\cos x = -1/2$. In the interval $[-\pi, \pi]$, this happens for $x = 2\pi/3$ and $x = -2\pi/3$.

Evaluating $f$ at critical points and end points:

- $f(-\pi) = 1 + 2 \sin -\pi = 1 + 2(0) = 1$
- $f(-2\pi/3) = 1 + 2 \sin (-2\pi/3) = 1 + 2(-\sqrt{3}/2) = 1 - \sqrt{3}$
- $f(2\pi/3) = 1 + 2 \sin (2\pi/3) = 1 + 2(\sqrt{3}/2) = 1 + \sqrt{3}$
- $f(\pi) = 1 + 2 \sin \pi = 1 + 2(0) = 1$

So,

Absolute Maximum is $1 + \sqrt{3}$ at $x = 2\pi/3$

Absolute Minimum is $1 - \sqrt{3}$ at $x = -2\pi/3$
2. (25 points) Suppose $2 \leq f'(x) \leq 5$ for all values of $x$. Show that

$$8 \leq f(7) - f(3) \leq 20$$

Solution:
By mean value theorem

$$\frac{f(7) - f(3)}{7 - 3} = f'(c)$$

for some $c \in [3, 7]$. But it is given that $2 \leq f'(x) \leq 5$ for all $x$. So,

$$2 \leq f'(c) \leq 5$$

$$2 \leq \frac{f(7) - f(3)}{7 - 3} = f'(c) \leq 5$$

so, multiplying the inequality by 4, we bet

$$8 \leq f(7) - f(3) \leq 20$$
3. (35 points) A cone shaped martini glass is to be made so as to hold 27 cm$^3$ of cocktail. Find the height and radius of the glass so that the least amount of material is needed to make it.

Note: Volume of a cone is given by $V = \frac{1}{3} \pi r^2 h$ and surface area of cone is given by $A = \pi r \sqrt{r^2 + h^2}$.

Solution:
We want to minimize

$$A = \pi r \sqrt{r^2 + h^2}$$

And the constraint is

$$V = 27 = \frac{1}{3} \pi r^2 h$$

We can solve this in two ways:
Method 1: Implicit differentiation
Differentiating $A$ with respect to $r$, we get:

$$\frac{dA}{dr} = \pi \left[ \sqrt{r^2 + h^2} + r \frac{1}{2\sqrt{r^2 + h^2}} \left( 2r + 2h \frac{dh}{dr} \right) \right]$$

Simplifying we get

$$\frac{dA}{dr} = \pi \left[ \frac{r^2 + h^2 + r^2 + rh \frac{dh}{dr}}{\sqrt{r^2 + h^2}} \right]$$

We want to set $\frac{dA}{dr} = 0$ to find the minimum. But before doing that, let’s differentiate the constraint so that we can substitute for $\frac{dh}{dr}$ in the above expression. Differentiating the constraint, we get:

$$\frac{d}{dr}(27) = \frac{d}{dr} \left( \frac{1}{3} \pi r^2 h \right)$$

that is

$$0 = \frac{\pi}{3} \left( 2rh + r^2 \frac{dh}{dr} \right)$$

which on simplification gives

$$\frac{dh}{dr} = -\frac{2h}{r}$$

Substituting this in the expression for $\frac{dA}{dr}$ and setting it equal to zero, we get:

$$0 = \frac{dA}{dr} = \pi \left[ \frac{2r^2 + h^2 + rh \left( -\frac{2h}{r} \right)}{\sqrt{r^2 + h^2}} \right]$$
For this expression to be zero, the numerator of the ratio has to be zero. Simplifying this gives

\[ 0 = 2r^2 + h^2 - 2h^2 = 2r^2 - h^2 \]

That is \( h^2 = 2r^2 \) or \( r^2 = h^2/2 \). This gives \( r = \frac{h}{\sqrt{2}} \). Substituting in formula for volume, we get

\[ 27 = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi h^2 \frac{h}{2} = \frac{\pi h^3}{6} \]

So \( h^3 = 27 \left( \frac{6}{\pi} \right) \) or in other words

\[ h = 3\sqrt[3]{6 \pi} \]

And so

\[ r = \frac{h}{\sqrt{2}} = \frac{3}{\sqrt{2}} \sqrt[3]{6 \pi} \]

Method 2:

Explicit method. Write \( A \) as a function of only one variable. Substitute for \( h \) in terms of \( r \) using the constraint: \( h = \frac{(27)(3)}{\pi r^4} = \frac{3^4}{\pi r^4} \). So we have

\[ A = \pi r \sqrt{r^2 + \left( \frac{3^4}{\pi r^4} \right)^2} = \pi r \sqrt{r^2 + \left( \frac{3^4}{\pi r^4} \right)} \]

Differentiating with respect to \( r \) we get:

\[ \frac{dA}{dr} = \pi \left[ \sqrt{r^2 + \left( \frac{3^8}{\pi^2 r^4} \right)} + r \frac{1}{2 \sqrt{r^2 + \frac{3^8}{\pi^2 r^4}}} \left( 2r + \frac{3^8(-4)}{\pi^2 r^5} \right) \right] \]

simplifying and setting \( dA/dr = 0 \) we get

\[ 0 = \pi \left[ \frac{r^2 + \left( \frac{3^8}{\pi^2 r^4} \right)}{\sqrt{r^2 + \left( \frac{3^8}{\pi^2 r^4} \right)}} + r \left( r - \frac{2\cdot3^8}{\pi^2 r^5} \right) \right] \]

This ratio is zero only if the numerator is zero, so we get

\[ r^2 + \left( \frac{3^8}{\pi^2 r^4} \right) + r \left( r - \frac{2\cdot3^8}{\pi^2 r^5} \right) = 0 \]
\[ r^2 + \left( \frac{3^8}{\pi^2 r^4} \right) + r^2 - 2 \left( \frac{3^8}{\pi^2 r^4} \right) = 0 \]

\[ 2r^2 - \left( \frac{3^8}{\pi^2 r^4} \right) = 0 \]

\[ 2r^2 = \left( \frac{3^8}{\pi^2 r^4} \right) \]

\[ r^6 = \frac{3^8}{2\pi^2} \]

\[ r = \frac{3^8}{2^{1/6} \pi^{1/6}} \]

You can check that this is the same answer as we got by method 1, but looks much more complicated. Implicit differentiation is always easier.
4. (25 points) Find the area under the curve (i.e. evaluate the definite integral) in the
given interval for the following functions.

(a) (15 points) \( f(x) = 2 \cos 2x + \sin x \), \([-\pi/4, \pi/4]\).
Solution:
An anti-derivative for above \( f \) is given by
\[
F(x) = \sin (2x) - \cos x
\]
so the above definite integral is given by
\[
\int_{-\pi/4}^{\pi/4} (2 \cos 2x + \sin x) \, dx = F(\pi/4) - F(-\pi/4)
\]
\[
= [\sin (\pi/2) - \cos (\pi/4)] - [\sin (-\pi/2) - \cos (-\pi/4)]
\]
\[
= [1 - 1/\sqrt{2}] - [-1 - 1/\sqrt{2}] = 2
\]

(b) (10 points) \( f(x) = x^2 + \sec^2 x \), \([0, \pi/3]\).
Solution:
Again, we find the anti-derivative:
\[
F(x) = \frac{x^3}{3} + \tan x
\]
So, the given integral is given by
\[
\int_{0}^{\pi/3} (x^2 + \sec^2 x) \, dx = F(\pi/3) - F(0)
\]
\[
= \left[ \frac{(\pi/3)^3}{3} + \tan (\pi/3) \right] - \left[ \frac{0}{3} + \tan 0 \right] = \frac{\pi^3}{81} + \sqrt{3}
\]

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