UNIQUENESS OF STATIC BLACK-HOLES WITHOUT ANALYTICITY

by

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Abstract. — We show that the hypothesis of analyticity in the uniqueness theory of vacuum, or electrovacuum, static black holes is not needed. More generally, we show that prehorizons covering a closed set cannot occur in well-behaved domains of outer communications.

1. Introduction

One of the hypotheses in the current theory of uniqueness of static vacuum black holes is that of analyticity. This is used to exclude null Killing orbits, equivalently to prove non-existence of *non-embedded degenerate prehorizons covering a closed set*, within the domain of outer-communications; see [5] for the details. The aim of this note is to show that analyticity is not needed to exclude such prehorizons, and therefore can be removed from the set of hypotheses of the classification theorems in the static case.

More generally, such prehorizons need to, and have been, excluded in dimension $n+1$ with $n-1$ commuting Killing vectors [4] without assuming analyticity. Our analysis here provides an alternative, simpler, approach to this issue for any stationary solutions satisfying the null energy condition, without the need to invoke more Killing vectors. (Note, however, that for solutions that are *not static*, all $n-1$ Killing vectors are used to prove that existence of a null Killing orbit implies existence of a prehorizon.)

More precisely, we consider asymptotically flat, or Kaluza-Klein (KK) asymptotically flat (in the sense of [6]) spacetimes, and show that (for definitions, see below and [5]):

Theorem 1.1. — *I^+-regular stationary domains of outer communication $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ satisfying the null energy condition do not contain prehorizons, the union of which is closed within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$.*

The reader is referred to [1] and references therein for progress towards removing the hypothesis of analyticity in the general stationary case.

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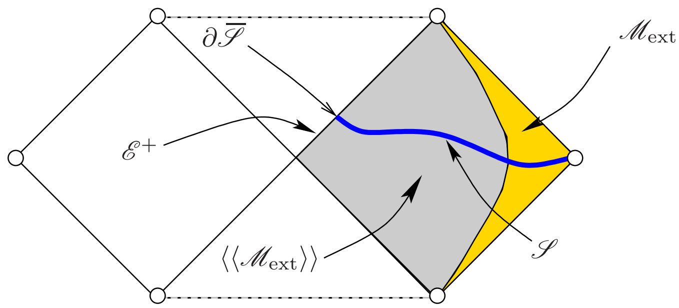

FIGURE 1. The hypersurface $\mathscr{S}$ from the definition of I^+–regularity.

2. The time of flight argument

For the convenience of the reader we recall some definitions from [4, 5]:

DEFINITION 2.1. — *Let $(\mathscr{M}, \mathfrak{g})$ be a space-time containing an asymptotically flat, or KK–asymptotically flat end $\mathscr{S}_{\rm ext}$, and let K be a stationary Killing vector field on $\mathscr{M}$. We will say that $(\mathscr{M}, \mathfrak{g}, K)$ is* I^+–regular *if K is complete, if the domain of outer communications $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and if $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ contains a spacelike, connected, acausal hypersurface $\mathscr{S} \supset \mathscr{S}_{\rm ext}$, the closure $\overline{\mathscr{S}}$ of which is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotic ends, such that the boundary $\partial\overline{\mathscr{S}} := \overline{\mathscr{S}} \setminus \mathscr{S}$ is a topological manifold satisfying*

$$\partial\overline{\mathscr{S}} \subset \mathscr{E}^+ := \partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \cap I^+(\mathscr{M}_{\rm ext}) \ , \tag{2.1}$$

with $\partial\overline{\mathscr{S}}$ meeting every generator of $\mathscr{E}^+$ precisely once. (See Figure 1.)

The definition appears to capture the essential ingredients required for a successful classification of vacuum [5] or electro-vacuum [13] black holes. Whether or not the definition is optimal from this point of view remains to be seen. In any case, one of its consequences is the *structure theorem* [4, 5], which in essence goes back to [11, Lemma 2], and which represents $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ globally as $\mathbb{R} \times \mathscr{S}$, with the Killing vector tangent to the $\mathbb{R}$ factor.

Another notion that is essential for the current work is:

DEFINITION 2.2. — *Let K be a Killing vector and set*

$$\mathscr{N}[K] := \{p \in \mathscr{M} \mid \mathfrak{g}(K,K)|_p = 0 \ , \ \ K|_p \neq 0\} \ . \tag{2.2}$$

Every connected, not necessarily embedded, null hypersurface $\mathscr{N}_0 \subset \mathscr{N}[K]$ to which K is tangent will be called a Killing prehorizon.

It follows from [5, Corollary 3.3 and Lemma 5.14] that in vacuum I^+–regular space-times which are static, or four-dimensional stationary and axisymmetric, or $(n+1)$–dimensional with $n-1$ commuting Killing vectors, the set covered by Killing prehorizons associated with a Killing vector field K is closed within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. This remains true for electrovacuum space-times in dimension $3 + 1$.

From what has been said, it should be clear that Theorem 1.1 follows immediately from:

THEOREM 2.3. — *Consider an asymptotically flat, or KK–asymptotically flat, globally hyperbolic domain of outer communications* $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$*, satisfying the null energy condition, diffeomorphic to* $\mathbb{R}\times\mathscr{S}$*, with the Killing vector, tangent to the* $\mathbb{R}$ *factor, approaching a time translation in the asymptotic region. If the region* $\mathscr{K}$ *defined as the union of Killing prehorizons forms a closed set within* $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$*, then*

$$\mathscr{K}=\emptyset\ .$$

PROOF OF THEOREM 2.3: Let $R\in\mathbb{R}$ be large enough so that the constant-time spheres lying on the timelike hypersurface

$$\mathscr{T}:=\mathbb{R}\times\{|\vec x|=R\}$$

are both *past* and *future inner trapped*, as defined in [6]. Let $\mathscr{C}$ denote the following class of causal curves:

$$\begin{aligned}\mathscr{C}\quad:=\quad&\{\gamma\mid\gamma:[0,1]\to\mathscr{M}\text{ is a causal curve which starts and ends}\\&\text{at }\mathscr{T}\text{, and meets }\mathscr{K}_0:=\mathscr{K}\cap(\{0\}\times\mathscr{S})\ \}\ .\end{aligned}$$

The *time of flight* τ_γ of γ is defined as

$$\tau_\gamma:=t(\gamma(1))-t(\gamma(0))\ ,$$

where t is the time-function associated with the decomposition $\mathscr{M}=\mathbb{R}\times\mathscr{S}$. We write $\mathscr{S}_\tau$ for $t^{-1}(\tau)\equiv\{\tau\}\times\mathscr{S}$.

Let $\mathring\tau$ denote the infimum of τ_γ over $\gamma\in\mathscr{C}$. We wish to show that if $\mathscr{K}_0$ is non-empty, then $\mathring\tau$ is attained on a smooth null geodesic $\mathring\gamma$, with a) initial and end points on $\mathscr{T}$, b) meeting $\mathscr{S}_0$ at $\mathscr{K}_0$, c) meeting $\mathscr{T}$ normally to the level sets of t.

In order to construct $\mathring\gamma$, let $\gamma_i\in\mathscr{C}$ be any sequence of causal curves such that $\tau_{\gamma_i}\to\mathring\tau$. Let γ be any causal curve in $\mathscr{C}$, then $0>t(\gamma_i(0))\geq-\tau_\gamma$ and $0<t(\gamma_i(1))\leq\tau_\gamma$ for i large enough. Hence for i large enough all the $\gamma_i(0)$'s belong to the compact set $[-\tau_\gamma,0]\times\{|\vec x|=R\}$; similarly the $\gamma_i(1)$'s belong to the compact set $[0,\tau_\gamma]\times\{|\vec x|=R\}$. By global hyperbolicity there exists an accumulation curve $\mathring\gamma$ of the γ_i's which is a C^0 causal curve.

Since $\mathscr{K}_0$ is closed in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, $\mathring\gamma$ meets $\mathscr{K}_0$ at some point $\mathring p$. It is standard that $\mathring\gamma\cap\{t<0\}$ is a smooth null geodesic, since otherwise $\mathring p$ would be timelike related to $\mathring\gamma(0)$, which would imply existence of a curve in $\mathscr{C}$ with time of flight less than $\mathring\tau$. Similarly, $\mathring\gamma\cap\{t>0\}$ is a smooth null geodesic.

Next, in a similar fashion (see [15, Lemma 50, p. 298]), $\mathring\gamma$ meets $\mathscr{T}_{t(\mathring\gamma(0))}$ and $\mathscr{T}_{t(\mathring\gamma(1))}$ orthogonally, where $\mathscr{T}_\tau:=\mathscr{S}_\tau\cap\mathscr{T}$.

We claim that $\mathring\gamma$ is also smooth at $\mathring p$. To see that, let $\mathring{\mathscr{K}}$ denote that leaf of $\mathscr{K}$ that passes through $\mathring p$. Then the portion of $\mathring\gamma$ that lies to the causal past of $\mathring p$ must meet $\mathring{\mathscr{K}}$ transversally: Otherwise $\mathring\gamma\cap J^-(\mathring p)$ would coincide with that portion of the null Killing orbit of K through $\mathring p$ that lies to the past of $\mathring p$, but those never reach $\mathscr{T}$, as K is null on $\mathscr{K}$ and timelike on $\mathscr{T}$. Similarly the portion of $\mathring\gamma$ that lies to the causal future of $\mathring p$ must meet $\mathring{\mathscr{K}}$ transversally. Suppose that the two geodesic segments forming $\mathring\gamma$ do not join smoothly at p. Then there exist arbitrary small deformations of $\mathring\gamma$ which produce a timelike curve with the same end points as $\mathring\gamma$, and hence the same time of flight. By transversality, and because there exists a small neighbourhood of $\mathring p$ in which the connected component of $\mathring{\mathscr{K}}$ passing through $\mathring p$ is a null embedded hypersurface, any such deformation, say $\hat\gamma$, will meet $\mathring{\mathscr{K}}$ at some point $\hat p$. Let ϕ_t denote the flow of K, then

$$\bar\gamma:=\phi_{-t(\hat p)}(\hat\gamma)$$

is a timelike curve in $\mathscr{C}$ which has the same time of flight as $\mathring{\gamma}$. Since $\bar{\gamma}$ is timelike, it can be deformed to a causal curve with shorter time of flight. This contradicts the definition of $\mathring{\tau}$, and hence proves a), b) and c).

Let $\tau_* = t(\mathring{\gamma}(0))$. We claim that d) $\mathring{\gamma}$ minimizes the time of flight amongst *all* nearby differentiable causal curves from $\mathscr{T}_{\tau_*}$ to $\mathscr{T}$. Indeed, by transversality of $\mathring{\gamma}$ to $\mathring{\mathscr{K}}$, there exists a neighbourhood $\mathscr{U}$ of $\mathring{\gamma}$ in the space of differentiable curves such that every curve γ in this neighbourhood intersects $\mathring{\mathscr{K}}$. Suppose, then, that there exists a causal curve $\gamma \in \mathscr{U}$ which starts at $\mathscr{T}_{\tau_*}$, ends at $\mathscr{T}$, and has time of flight smaller than $\mathring{\tau}$. Then γ intersects $\mathring{\mathscr{K}}$ at some p. But then $\phi_{-t(p)}(\gamma)$ is in $\mathscr{C}$ and has time of flight smaller than $\mathring{\tau}$, which contradicts the definition of $\mathring{\tau}$, whence d) holds.

This provides a contradiction to $\mathscr{K}$ being non-empty, as there are no causal curves with the property d) by [6, Proposition 3.3]. □

3. Non-rotating horizons and maximal hypersurfaces

In this section we provide an alternative simple argument to exclude prehorizons within the domain of outer communication, which applies to four-dimensional static vacuum space-times.

Let $(\mathscr{M}, \mathfrak{g})$ be an asymptotically flat, I^+-regular, vacuum space-time with a *hypersurface orthogonal* Killing vector K. By [8] all components of the future event horizon $\mathscr{E}^+$ are non-degenerate. We can therefore carry-out the construction of [16] if necessary to obtain that $\partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is the union of bifurcate Killing horizons. By [10], $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ contains a maximal Cauchy hypersurface $\mathscr{S}$. By [18] (compare the argument at the end of [5, Section 7.2]), $\mathscr{S}$ is totally geodesic. Decomposing K as $K = Nn + Y$, where n is the field of future-directed unit normals to $\mathscr{S}$, and where Y is tangent to $\mathscr{S}$, one finds from the Killing vector equations that

$$D_iY_j + D_jY_i = -2NK_{ij} \ .$$

But the right-hand-side vanishes, thus Y is a Killing vector of the metric γ induced on $\mathscr{S}$ by g. Now, Y is asymptotic to zero as one recedes to infinity in $\mathscr{M}_{\rm ext}$, hence $Y = 0$ by usual arguments, (see, e.g., the proof of [7, Proposition 2.1]). Next, N satisfies the equation

$$\Delta N = K^{ij}K_{ij}N \ , \tag{3.1}$$

vanishes on $\partial\mathscr{S}$, and is asymptotic to one as one recedes to infinity along the asymptotically flat region, and thus has no zeros by the strong maximum principle. Thus $K = Nn$ is timelike everywhere on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, and there are no prehorizons within $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$.

The above argument applies *verbatim* to higher dimensional vacuum metrics, as well as to four-dimensional electrovacuum metrics, for configurations where all horizons are non-degenerate. A proof of existence of maximal hypersurfaces with sufficiently controlled asymptotic behaviour near the degenerate horizons would extend this argument to the general case. In any case, the proof based on the time of flight covers more general situations.

4. Conclusions

Recall that a manifold $\widehat{\mathscr{S}}$ is said to be of *positive energy type* if there are no asymptotically flat complete Riemannian metrics on $\widehat{\mathscr{S}}$ with nonnegative scalar curvature and vanishing mass except perhaps for a flat one. This property has been proved so far for all n–dimensional manifolds $\widehat{\mathscr{S}}$ obtained by removing a finite number of points from a compact manifold of dimension $3 \le n \le 7$ [17], or under the hypothesis that $\widehat{\mathscr{S}}$ is a spin manifold of any dimension $n \ge 3$, and is expected to be true in general [2, 14].

Using results already established elsewhere [3, 5, 9, 12, 13] together with Theorem 1.1 one has:

THEOREM 4.1. — *Let $(\mathscr{M}, \mathfrak{g})$ be a vacuum $n+1$ dimensional space-time, $n \ge 3$, containing a spacelike, connected, acausal hypersurface $\mathscr{S}$, such that $\overline{\mathscr{S}}$ is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotically flat ends. Suppose that there exists on $\mathscr{M}$ a complete static Killing vector K, that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and that $\partial\overline{\mathscr{S}} \subset \mathscr{M} \setminus \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Let $\widehat{\mathscr{S}}$ denote the manifold obtained by doubling $\mathscr{S}$ across the non-degenerate components of its boundary and compactifying, in the doubled manifold, all asymptotically flat regions but one to a point. If $\widehat{\mathscr{S}}$ is of positive energy type, then $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Schwarzschild space-time.*

THEOREM 4.2. — *Under the remaining hypotheses of Theorem 4.1 with $n = 3$, suppose instead that $(\mathscr{M}, \mathfrak{g})$ is electrovacuum with the Maxwell field invariant under the flow of K. Then $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is isometric to the domain of outer communications of a Reissner-Nordström or a standard Majumdar-Papapetrou space-time.*

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