A uniqueness theorem for the AdS soliton

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The stability of physical systems depends on the existence of a state of least energy, or ground state. In gravity, this is guaranteed by the positive energy theorem. The proof employs spinor structure and can fail for certain spacetime topologies, such as those arising in non-supersymmetric Kaluza-Klein compactifications, which can decay to arbitrarily negative energy. The proof also fails for the topology of the adS soliton, a nonsingular Einstein spacetime with negative cosmological constant and negative mass-energy. Nonetheless, arguing from the adS/CFT correspondence, Horowitz and Myers proposed a new positive energy conjecture, stating that the adS soliton is the unique, stable ground state for its asymptotic class. We give a new general structure theorem for negative mass spacetimes and use it to prove uniqueness of the adS soliton. Our work relies on a novel exploitation of the special geometry of ground state spacetimes. It offers significant support for the new positive energy conjecture and adds to the body of rigorous results inspired by the adS/CFT correspondence.

The positive energy theorem [1,2] singles out Minkowski spacetime as the "ground state", or spacetime of lowest mass-energy, within the class of asymptotically flat spacetimes with local energy density ≥ 0 and without naked singularities. In the presence of a negative cosmological constant the appropriate ground state is anti-de Sitter spacetime [3]. These ground states are regular, globally static, supersymmetric, and of constant curvature. Moreover, it is known that Minkowski spacetime is the unique asymptotically flat, regular, stationary vacuum spacetime [4]. The analogous uniqueness result for the asymptotically globally adS case is proved in [5,6].

A simple scaling argument suggests that a ground state cannot have negative mass, for then it could be scaled to produce a state of even lesser mass [7]. Consider, then, the surprising properties of the adS soliton, first examined by Horowitz and Myers [8], which is a negative mass, globally static Einstein spacetime with cosmological constant $\Lambda < 0$. The metric in $n+1 \geq 4$ spacetime dimensions is

$$ds^2 = -r^2 dt^2 + \frac{1}{V(r)} dr^2 + V(r) d\phi^2 + r^2 \sum_{i=1}^{n-2} (dy^i)^2 \quad (1)$$

where $V(r) = \frac{r^2}{\ell^2}\left(1 - \frac{{r_0}^n}{r^n}\right)$, $\ell^2 = -\frac{n(n-1)}{2\Lambda}$, and r_0 is a constant. Regularity demands that ϕ be identified with period $\beta_0 = \frac{4\pi\ell^2}{n r_0}$. The periods of the y^i are arbitrary. The soliton is "asymptotically locally anti-de Sitter" with boundary at conformal infinity (scri) foliated by spacelike $(n-1)$-tori. The time slices of spacetime itself, when conformally completed, are topologically the product of an $(n-2)$-torus and a disk (a solid torus in $3+1$). The soliton spacetime is neither supersymmetric nor of constant curvature, but has minimal energy under small metric perturbations [8,9]. Remarkably, one cannot vary the soliton mass by simple scaling. To see why, note that rescaling the parameter $r_0 \to k r_0$ in (1) has the same effect as the coordinate transformation $(t, r, y^i, \phi) \mapsto (kt, k^{-1} r, k y^i, k\phi)$. Thus the new metric (with parameter $k r_0$) is *isometric* to the original one, provided the conformal boundary data are chosen to agree, so they must have the same physical mass.

Horowitz and Myers found that the negative mass of the adS soliton has a natural interpretation as the Casimir energy of a non-supersymmetric gauge theory on the conformal boundary. If a non-supersymmetric version of the adS/CFT conjecture holds [10], as is generally hoped, then this would indicate that the soliton is the lowest energy solution with these boundary conditions. This led them to postulate a new positive energy conjecture, that the soliton is the unique lowest mass solution for all spacetimes in its asymptotic class. The validity of this conjecture is thus an important test of the non-supersymmetric version of the adS/CFT correspondence. The conjecture is all the more remarkable because the soliton topology has certain circles that are not contractible at infinity but are contractible in the bulk. This leads to the failure of spinorial methods to produce a positive energy theorem here and is linked to a known instability in Kaluza-Klein theory [7,11].

As support for the new positive energy conjecture, we will give a uniqueness theorem for the adS soliton, singling it out as the only suitable ground state in the class of spacetimes with similar asymptotics. Our theorem is similar in spirit to [4,5], but relates to asymptotically "locally" adS spacetimes with Ricci flat conformal boundary. The proof is based on the fact that the soliton indeed shares one important geometric property with known ground states: its universal cover admits a foliation by totally geodesic null surfaces ruled by complete, achronal null geodesics called *null lines*. In other words, the spacetime admits non-focusing plane waves. A similar idea underlies the approach to mass positivity in [12], a significant difference being that the null lines we will construct here do not approach scri. Our construction of these null surfaces relies on results in [13].

Our arguments will be of necessity terse. Herein we give the flavour of the proof; details will appear in [14]. Our results considerably generalize a uniqueness result of [15]. Of related interest is a uniqueness theorem of Anderson for asymptotically hyperbolic Einstein metrics on 4-dimensional Riemannian manifolds [16].

Following the formalism of Chruściel and Simon [6], we consider static spacetimes of the form

$$M^{n+1} = \mathbb{R} \times \Sigma, \qquad g = -N^2 dt^2 \oplus h. \tag{2}$$

where (Σ, h, N) is *conformally compactifiable*. Thus we assume that Σ is the interior of a compact manifold with boundary $\widetilde{\Sigma} = \Sigma \cup \partial\widetilde{\Sigma}$ such that (a) $1/N$ extends to a smooth function $\widetilde{N}$ on $\widetilde{\Sigma}$, with $\widetilde{N} = 0$ and $d\widetilde{N} \neq 0$ along $\partial\widetilde{\Sigma}$ and (b) $N^{-2}h$ extends to a smooth Riemannian metric $\widetilde{h}$. (M, g) conformally embeds into $(\mathbb{R} \times \widetilde{\Sigma}, -dt^2 + \widetilde{h})$ and hence admits a natural timelike scri structure, making precise, in the static setting, what is meant by "asymptotically locally adS".

We assume that (Σ, h, N) obeys the static vacuum field equations,

$$R_{ab} = N^{-1}\nabla_a\nabla_b N + \frac{2\Lambda}{n-1} h_{ab}\,, \tag{3}$$

$$\Delta N = -\frac{2\Lambda}{n-1} N\,, \tag{4}$$

where $\Delta = \nabla^2$, ∇_a is the covariant derivative on (Σ, h_{ab}) and R_{ab} is its Ricci tensor. In terms of the rescaled metric $\widetilde{h}$, and associated $\widetilde{\Delta}, \widetilde{\nabla}_a$ and $\widetilde{R}_{ab}$,

$$\widetilde{R}_{ab} = \frac{-(n-1)}{\widetilde{N}} \widetilde{\nabla}_a \widetilde{\nabla}_b \widetilde{N}, \tag{5}$$

$$\widetilde{N}\,\widetilde{\Delta}\widetilde{N} = \frac{2\Lambda}{n-1} + n\widetilde{W}, \tag{6}$$

where $\widetilde{W} := \widetilde{h}^{ab}\widetilde{\nabla}_a\widetilde{N}\widetilde{\nabla}_b\widetilde{N} = N^{-2}h^{ab}\nabla_a N \nabla_b N$.

Recall that the rescaled metric $\widetilde{h} = N^{-2}h$, sometimes called the *Fermat (optical) metric*, has physical significance: geodesics of $(\widetilde{\Sigma}, \widetilde{h})$ correspond to the spatial paths of light rays in (M, g). That is, null geodesics in (M, g) project in the obvious way to geodesics in $(\widetilde{\Sigma}, \widetilde{h})$ (when suitably parametrized). Conversely, a geodesic γ in $\widetilde{\Sigma}$ (viewed as the slice $t = 0$) passing through a point $p \in \widetilde{\Sigma}$ lifts to a unique future directed null geodesic η passing through p. Moreover, if γ is a length minimizing geodesic segment in $(\widetilde{\Sigma}, \widetilde{h})$ then it lifts to an *achronal* null geodesic segment η. Thus, a line (inextendible geodesic, length minimizing on each segment) in $(\widetilde{\Sigma}, \widetilde{h})$ lifts to a null line (achronal inextendible null geodesic) in (M, g). This basic fact is used in an essential way in our arguments.

Our uniqueness result (Theorem 2) for the adS soliton is obtained as a consequence of a more general structure result (Theorem 1) which assumes a certain *convexity condition* near infinity. As discussed below, this convexity condition is related to the sign of the mass (or total energy) of the spacetime. In general, as follows from eq. (5) and the C^2 smoothness of $\widetilde{h}_{ab}$ at $\widetilde{N} = 0$, the conformal boundary $\partial\widetilde{\Sigma} = \{\widetilde{N} = 0\}$ is totally geodesic in $(\widetilde{\Sigma}, \widetilde{h})$. We say that (Σ, h, N) *satisfies condition (C)* if there exists a neighbourhood of scri in which each level surface $\widetilde{N} = c$ is *weakly convex* in $(\widetilde{\Sigma}, \widetilde{h})$, i.e., the second fundamental form with respect to the outward normal of the level surface is positive semi-definite. Equivalently, this condition requires the principal curvatures of each level surface sufficiently close to scri be nonnegative.

Theorem 1: *Consider a static spacetime as in (2) such that (i) (Σ, h, N) is conformally compactifiable, (ii) the static vacuum field equations hold, and (iii) condition (C) holds. Then the Riemannian universal cover $(\widetilde{\Sigma}^*, \widetilde{h}^*)$ of $(\widetilde{\Sigma}, \widetilde{h})$ splits isometrically as*

$$\widetilde{\Sigma}^* = \mathbb{R}^k \times \mathcal{W}, \qquad \widetilde{h}^* = h_E \oplus \sigma \tag{7}$$

where $(\mathbb{R}^k, h_E)$ is standard k-dimensional Euclidean space and $(\mathcal{W}, \sigma)$ is a compact *simply connected Riemannian manifold-with-boundary. The Riemannian universal cover (Σ^*, h^*) of (Σ, h) splits isometrically as a warped product of the form,*

$$\Sigma^* = \mathbb{R}^k \times \mathcal{W}_0, \qquad h^* = (N^{*2}h_E) \oplus \sigma_0\,, \tag{8}$$

where $N^* = N \circ \pi$ (π = covering map) depends only on $\mathcal{W}_0$, and $(\mathcal{W}_0, \sigma_0)$ is a complete simply connected Riemannian manifold such that $(\mathcal{W}_0, \sigma_0, N)$ is conformally compactifiable.

Theorem 1 is similar in spirit to a result of Cheeger and Gromoll [17] concerning the structure of compact Riemannian manifolds of nonnegative Ricci curvature. It implies a strong structure result for the fundamental group $\Pi_1(\widetilde{\Sigma})$, cf. [17].

Let us now consider how condition (C) relates to the sign of the mass. As we are using the conformal approach, the Ashtekar-Magnon [18] mass expression involving the electric part of the Weyl tensor is especially convenient. Consider a conformally compactifiable static spacetime (Σ, h, N), and view Σ as the slice $t = 0$. Setting $x = \widetilde{N}$, the metric $\widetilde{h}$ near the conformal boundary $x = 0$ may be written as

$$\widetilde{h} = \widetilde{W}^{-1}dx^2 + h_{AB}(x, x^C)dx^A dx^B\,, \tag{9}$$

where h_{AB} is the induced metric on $x = const$ surfaces. Let T^a denote the future-timelike unit normal to $\widetilde{\Sigma}$ in the rescaled spacetime metric, and on $\widetilde{\Sigma}$, let $n^a = -\sqrt{\widetilde{W}}\partial_x$ be the outward pointing unit normal field to the slices $x = const$ near the conformal boundary. The Weyl mass is then given, up to a positive constant, by $\int_{\partial\widetilde{\Sigma}} \mu d\widetilde{A}$, where $\widetilde{dA}$ is the volume element on $\partial\widetilde{\Sigma}$. The *mass aspect* μ, up to a positive constant, is given by

$$\mu = \lim_{x\to 0} \frac{\widetilde{E}_{ac}T^aT^c}{x^{n-2}} = \lim_{x\to 0} \frac{\widetilde{C}_{abcd}n^bn^dT^aT^c}{x^{n-2}}\,, \tag{10}$$

where $\widetilde{C}_{abcd}$ is the Weyl tensor (and $\widetilde{E}_{ac}$ its electric part) of the conformal spacetime metric $\widetilde{g} = N^{-2}g$.

In the static setting, the mass aspect μ can be directly related to the geometry of $(\widetilde{\Sigma},\widetilde{h})$. One finds, using the field equations, that, up to a positive constant, $\mu = -\partial^{n-2}\widetilde{R}/\partial x^{n-2}|_{x=0}$. (Chruściel and Simon [6] had previously identified, in the $3+1$ dimensional case, $-\partial\widetilde{R}/\partial x|_{x=0}$ as the mass aspect.)

Let $\widetilde{H}$ denote the mean curvature function of the slices $x = const$ with respect to the outward normal n^a; along each such slice, $\widetilde{H} = \widetilde{\nabla}_a n^a =$ the trace of the second fundamental form $=$ the sum of the principal curvatures. Using the field equations and the Gauss equation, one can show that $\widetilde{H}$ is related to μ by

$$(n-2)\sqrt{\widetilde{W}}\,\widetilde{H} = -\frac{x^{n-1}}{2(n-1)}\mu + \mathcal{O}(x^n) \tag{11}$$

when the conformal boundary has Ricci flat induced metric. In order to establish Equation (11), we must carry out a Fefferman-Graham type boundary analysis [19], in the gauge (9) relevant to our situation, and subject to the field equations (5) and (6). This analysis implies $\widetilde{R}|_{x=0} = 0$ and $\partial^\ell\widetilde{R}/\partial x^\ell|_{x=0} = 0$ for $1 \le \ell \le n-3$.

For Ricci flat conformal boundary, as is the case for the soliton, equation (11) implies that if the mass aspect μ is (pointwise) negative, the level surfaces $\widetilde{N} = c$ near conformal infinity are outwardly *mean convex* in $(\widetilde{\Sigma},\widetilde{h})$, i.e., have strictly positive mean curvature (and hence the sum of the principal curvatures is positive). In other words, if the mass aspect is negative (as it is for the adS soliton) then condition (C) holds *in the mean*. However, our proof of uniqueness of the adS soliton requires not just mean convexity, but (weak) convexity near infinity, and so we need to impose the following condition. We say that (Σ, h, N) *satisfies condition (S)* provided the principal curvatures of the level surfaces $\widetilde{N} = c$ near infinity are either all non-negative or all non-positive. This condition holds trivially in the adS soliton, since all but one of the principal curvatures vanish. It also holds in the Kottler spacetimes, regardless of the sign of the mass.

Theorem 2: *Consider a static spacetime as in (2) such that* (i) (Σ, h, N) *is conformally compactifiable,* (ii) *the static vacuum field equations hold, and* (iii) *condition (S) holds. Suppose in addition, that*

(a) *The boundary geometry of $(\widetilde{\Sigma},\widetilde{h})$ is the same as that of (1), i.e.,* $\partial\widetilde{\Sigma} = T^{n-2}\times S^1$, $\widetilde{h}|_{\partial\widetilde{\Sigma}} = d\phi^2 + \sum_{i=1}^{n-2}(dy^i)^2$, *with the same periods for ϕ and the y^i.*

(b) *The mass aspect μ of (Σ, h, N) is pointwise negative.*

(c) *Given the inclusion map $i : \partial\widetilde{\Sigma} \to \widetilde{\Sigma}$, the kernel of the induced homomorphism of fundamental groups,* $i_* : \Pi_1(\partial\widetilde{\Sigma}) \to \Pi_1(\widetilde{\Sigma})$, *is generated by the S^1 factor.*

Then the spacetime (2) determined by (Σ, h, N) is isometric to the adS soliton (1).

Assumption (a) is a natural boundary condition. Assumption (b), together with condition (S), guarantees that condition (C) of Theorem 1 holds. Assumption (c) asserts that the generator of the S^1 factor is contractible in $\widetilde{\Sigma}$, and moreover, that any loop in $\partial\widetilde{\Sigma}$ contractible in $\widetilde{\Sigma}$ is a multiple of the generator. As discussed in the proof, assumptions (a) and (c) together imply that $\Pi_1(\widetilde{\Sigma}) \approx \mathrm{Z}^{n-2}$. Were we to adopt the latter condition in lieu of assumption (c), then one could only conclude that (Σ, h, N) is *locally* isometric to the adS soliton (the universal covers will be isometric, however). For further discussion of this *discrete* nonuniqueness relevant to the adS soliton in $3+1$ dimensions, see [16]. We note that in $3+1$ dimensions, assumption (b) (when μ is constant), and the condition $\Pi_1(\widetilde{\Sigma}) \approx \mathrm{Z}^{n-2}$, hold automatically, cf. [6,20].

We now sketch the proofs of Theorems 1 and 2; details will appear in [14].

Sketch of the proof of Theorem 1. We proceed inductively, working in $(\widetilde{\Sigma}^*,\widetilde{h}^*)$. If $\widetilde{\Sigma}^*$ is compact then Theorem 1 holds with $k = 0$. So suppose $\widetilde{\Sigma}^*$ is noncompact. In this case there is a procedure for constructing a line in $(\widetilde{\Sigma}^*,\widetilde{h}^*)$. Fix a point p in the interior, and let $\{p_i\}$ be a sequence of points uniformly bounded away from $\partial\widetilde{\Sigma}^*$, such that the distance from p to p_i tends to infinity. For each i, p and p_i can be joined by a length minimizing geodesic segment γ_i which cannot meet the boundary (since it's totally geodesic). In fact, by condition (C), the segments γ_i must be uniformly bounded away from $\partial\widetilde{\Sigma}^*$. Each geodesic segment γ_i will have a midpoint r_i. Now since $\widetilde{\Sigma}$ is compact, it will have a compact fundamental domain D in $\widetilde{\Sigma}^*$. For each point in the covering space, there will be a covering space transformation mapping that point to a point in D. We therefore apply to each geodesic segment γ_i a covering space transformation that maps r_i into D. This produces a sequence of minimizing geodesic segments σ_i, still uniformly bounded away from $\partial\widetilde{\Sigma}^*$, whose lengths are unbounded in both directions. By standard compactness results, this sequence possesses a limit curve which is a complete, length minimizing geodesic, i.e., a line, in the interior of $\widetilde{\Sigma}^*$.

By the relationship between Fermat and null geodesics, this line lifts to a complete null line η in the physical covering spacetime (M^*, g^*), which is null geodesically complete and obeys the null energy condition, $R_{ab}X^aX^b \ge 0$ for all null vectors X^a. Then a null line will exist only under special circumstances. As proved in [13], η must be contained in a smooth achronal edgeless null hypersurface $\mathcal{H}$ which is *totally geodesic* (i.e., has vanishing expansion and shear). In a static spacetime this has further consequences. Since Σ^* (viewed as the slice $t = 0$) is totally geodesic, $\mathcal{H}$ meets Σ^* in a totally geodesic submanifold $\mathcal{W}$ of codimension one in Σ^*. But by moving $\mathcal{H}$ invariantly under the flow generated by $\partial/\partial t$, we see that spacetime is actually foliated by totally geodesic null hypersurfaces, and this gives rise to a foliation $\{\mathcal{W}_u\}$ of Σ^* by totally geodesic hypersurfaces in Σ^*. Moreover, it can be seen that this foliation is achieved by exponentiating out along the unit speed normal geodesics to $\mathcal{W}$ in the *Fermat* metric. The physical metric then takes the form,

$$h^* = N^{*2}(u, x^A)du^2 + h_{AB}(x)dx^A dx^B\,. \qquad (12)$$

Up to this point we have only used the null energy condition. Now using the field equations in a more explicit way, one can show $\partial N^*/\partial u = 0$, and hence the metric (12) is a genuine warped product. By multiplying (12) by $(N^*)^{-2}$, and showing that everything extends smoothly to the boundary, we conclude that $(\widetilde{\Sigma}^*, \widetilde{h}^*)$ is isometric to the Riemannian product $(\mathbb{R}\times\mathcal{W}, du^2 + d\widetilde{\sigma}^2)$, where $(\mathcal{W}, d\widetilde{\sigma}^2)$ is a complete Riemannian manifold-with-boundary. If $\mathcal{W}$ is compact then Theorem 1 holds with $k = 1$. If $\mathcal{W}$ is noncompact then one can carry out essentially the same procedure again to construct a line in $\mathcal{W}$, lift it to a new null line spatially orthogonal to the first, and split off another $\mathbb{R}$ factor. One can continue splitting off $\mathbb{R}$ factors until what remains is compact.

Sketch of the proof of Theorem 2. Let (Σ_0, h_0, N_0) denote the adS soliton associated with the boundary data in assumption (a), and let $(\widetilde{\Sigma}_0, \widetilde{h}_0, \widetilde{N}_0)$ denote the corresponding conformally compactified soliton. Assumption (b), condition (S) and equation (11) imply that condition (C) holds, and hence we can apply Theorem 1. Hence we can apply Theorem 1. As shown below, $\Pi_1(\widetilde{\Sigma}) \approx Z^{n-2}$, from which it follows that $k = n-2$. Thus, the universal cover $\widetilde{\Sigma}^*$ splits isometrically as $\mathbb{R}^{n-2}\times\mathcal{W}$, where $\mathcal{W}$ is diffeomorphic to a disk, and the metric on $\mathcal{W}$ is determined by the field equations (5), (6). By results on topological censorship [20], the homomorphism $i_* : \Pi_1(\partial\widetilde{\Sigma}) \to \Pi_1(\widetilde{\Sigma})$ is onto. But $\partial\widetilde{\Sigma} = A \times B$, where A is the $(n-2)$ torus and B is the circle of assumption (a), whence $\Pi_1(\partial\Sigma) \approx \Pi_1(A) \times \Pi_1(B)$. Since, by assumption (c), $\ker i_* = \Pi_1(B)$, it follows that $i_*|_{\Pi_1(A)} : \Pi_1(A) \to \Pi_1(\widetilde{\Sigma})$ is an isomorphism. This implies that the covering transformations of $\widetilde{\Sigma}^*$ are in one-to-one correspondence, via $i_*|_{\Pi_1(A)}$, to those of A^*, the universal covering space of A. Thus, $\widetilde{\Sigma} \simeq \widetilde{\Sigma}^*/\Pi_1(\widetilde{\Sigma}) \simeq (\mathbb{R}^{n-2} \times \mathcal{W})/\Pi_1(A) \simeq (A^*/\Pi_1(A)) \times \mathcal{W} \simeq A \times \mathcal{W}$, i.e., $\widetilde{\Sigma}$ is isometric to $T^{n-2} \times \mathcal{W}$, where T^{n-2} is the torus in assumption (a). Because of the product structure of $\widetilde{\Sigma}$, and the fact that $\widetilde{N}$ depends only on $\mathcal{W}$, the field equations (5) and (6) descend to the disk $\mathcal{W}$, and can be solved explicitly and uniquely, subject to the appropriate boundary conditions on $\partial\widetilde{\Sigma}$. The result is that $\widetilde{\Sigma}$ is isometric to $\widetilde{\Sigma}_0$ and $\widetilde{N} = \widetilde{N}_0$, from which we conclude that (Σ, h, N) is isometric to the adS soliton (Σ_0, h_0, N_0).

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