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Topological Censorship for Kaluza–Klein Space-Times

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Abstract. The standard topological censorship theorems require asymptotic hypotheses which are too restrictive for several situations of interest. In this paper we prove a version of topological censorship under significantly weaker conditions, compatible, e.g., with solutions with Kaluza–Klein asymptotic behavior. In particular we prove simple connectedness of the quotient of the domain of outer communications by the group of symmetries for models which are asymptotically flat, or asymptotically anti-de Sitter, in a Kaluza–Klein sense. This allows one, e.g., to define the twist potentials needed for the reduction of the field equations in uniqueness theorems. Finally, the methods used to prove the above are used to show that weakly trapped compact surfaces cannot be seen from Scri.

1. Introduction

A restriction on the topology of domains of outer communications is provided by the *topological censorship principle* [12], which says that causal curves originating from, and ending in a simply connected asymptotic region do not see any non-trivial topology, in the sense that they can be deformed to a curve entirely contained within the asymptotic region. The result is one of the key steps in the black holes uniqueness theorems (see, e.g., [7] and references therein). Precise statements to this effect have been established in the literature under various conditions [10,13,14,17,18,21]. Of particular relevance to our work is [14], where topological censorship is reduced to a *null convexity* condition of timelike boundaries. The first main result of this work is the proof that the conditions of [14] can be replaced by the considerably weaker hypothesis, that the timelike boundaries are inner future and past trapped, as defined below.

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The need for this generalisation arises when studying the topology of higher dimensional domains of outer communications invariant under isometry groups. Recall that for asymptotically flat stationary space-times, whatever the space-dimension $n \geq 3$, simple connectedness holds for globally hyperbolic domains of outer communications satisfying the null energy condition. Indeed, the analysis in [10,12–14], carried-out there in dimension $3+1$, is independent of dimensions. However, there exist significant higher dimensional solutions which are asymptotically flat *in a Kaluza–Klein sense* and which are *not* simply connected in general, as demonstrated by Schwarzschild $\times \mathbb{T}^m$ "black branes".

Now, whenever simple connectedness fails, the twist potentials characterising the Killing vectors might fail to exist, and the whole reduction process [2,4], that relies on the existence of those potentials, breaks down. Our next main result is the proof that the quotient space $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle / G_s$ remains simply connected for KK-asymptotically flat, or KK-asymptotically adS models, which is sufficient for existence of twist potentials under mild conditions on $\langle\langle \mathscr{M}_{\mathrm{ext}} \rangle\rangle / G_s$, and has some further significant applications in the study of the problem at hand, see [6]. Here a uniformity-in-time condition is assumed on the asymptotic decay of the metric, which will certainly be satisfied by stationary solutions.

It turns out that the methods here are well suited to address the following: it is a well established fact in general relativity that compact future trapped surfaces cannot be seen from infinity. The weakly trapped counterpart of this has often been used in the literature, without a satisfactory justification available.[1] Our last main result here is the proof that borderline invisibility does indeed hold under appropriate global hypotheses.

2. Preliminaries

All manifolds are assumed to be Hausdorff and paracompact. We use the signature $(-,+,\ldots,+)$, and all space-times have dimension greater than or equal to three.

2.1. Trapped Surfaces

Let $(\mathscr{M}, g)$ be a space-time, and consider a spacelike manifold $S \subset \mathscr{M}$ of co-dimension two. Assume that there exists a smooth unit spacelike vector field n normal to S. If S is a two-sided boundary of a set contained within a spacelike hypersurface $\mathscr{S}$, we shall always choose n to be the outwards directed normal tangent to $\mathscr{S}$; this justifies the name of *outwards normal* for n. If $S \subset \{r = R\}$ in a KK-asymptotically flat or adS space-time, as defined in Sect. 4 below, then the outwards normal is defined to be the one for which $n(r) > 0$.

At every point $p \in S$ there exists then a unique future directed null vector field n^+ normal to S such that $g(n, n^+) = 1$, which we shall call the *outwards future null normal* to S. The *inwards future null normal* n^- is defined by the requirement that n^- is null, future directed, with $g(n, n^-) = -1$.

[1] This fact has been first brought to our attention by David Maxwell.

We define the *null future inwards and outwards mean curvatures* $\theta^\pm$ *of* S as

$$\theta^\pm := \mathrm{tr}_\gamma(\nabla n^\pm), \tag{2.1}$$

where γ is the metric induced on S. In (2.1) the symbol $n^\pm$ should be understood as representing any extension of the null normals $n^\pm$ to a neighborhood of S, and the definition is independent of the extension chosen.

We shall say that S is *weakly outer future trapped* if $\theta^+ \leq 0$. The notion of *weakly inner future trapped* is defined by requiring $\theta^- \leq 0$. A similar notion of *weakly outer or inner past trapped* is defined by changing $\leq$ to $\geq$ in the defining inequalities above. We will say *outer future trapped* if $\theta^+ < 0$, etc. One can also think of such conditions as *mean null convexity conditions*.

Let $\mathscr{T}$ be a smooth timelike hypersurface in $\mathscr{M}$ with a globally defined smooth field n of unit normals to $\mathscr{T}$. We shall say that $\mathscr{T}$ is *weakly outer past trapped* with respect to a time function t if the level sets of t on $\mathscr{T}$ are weakly outer past trapped. A similar definition is used for the notion of *weakly outer future trapped* timelike hypersurfaces, etc.

2.2. Space-Times with Timelike Boundary

A space-time $(\mathscr{M}, g)$ with timelike boundary $\mathscr{T}$ will be said *globally hyperbolic* if $(\mathscr{M}, g)$ is strongly causal and if for all $p, q \in \mathscr{M}$ the sets $J^+(p) \cap J^-(q)$ are empty or compact. In this case a hypersurface $\mathscr{S}$ is said to be a *Cauchy surface* if $\mathscr{S}$ is met by every inextendible causal curve precisely once. A smooth function t is said to be a *Cauchy time function* if it ranges over $\mathbb{R}$, if ∇t is timelike past directed, and if all level sets are Cauchy surfaces.

As an example, let $\mathscr{T}$ be any sufficiently distant level set of the usual radial coordinate r in Schwarzschild space-time. Then $\mathscr{T}$ is both inner future and past trapped, see (4.6) below.

The causal theory of space-times with timelike boundary has been studied in detail in [27]. Many important results are shown to be valid in this context. For instance, chronological future and past sets are open and global hyperbolicity as defined above implies causal simplicity. The following basic property of Cauchy surfaces holds as well, and is stated for future reference.

Proposition 2.1. *If* $\mathscr{S}$ *is a Cauchy surface and* K *is a compact subset of* $\mathscr{M}$ *then* $J^+(K) \cap J^-(\mathscr{S})$ *and* $J^+(K) \cap \mathscr{S}$ *are compact.* □

3. Topological Censorship for Space-Times with Timelike Boundary

We have the following generalisation of [14, Theorem 1]:

Theorem 3.1. *Let* t *be a Cauchy time function on a space-time* $(\mathscr{M}, g)$ *with timelike boundary* $\mathscr{T} = \cup_{\alpha \in \Omega} \mathscr{T}_\alpha$*, and satisfying the null energy condition (NEC):*

$$R_{\mu\nu} X^\mu X^\nu \geq 0 \quad \text{for all null vectors } X^\mu. \tag{3.1}$$

Suppose that there exists a component $\mathscr{T}_1$ of $\mathscr{T}$ with compact level sets $t|_{\mathscr{T}_1}$ such that

$$\mathscr{T}_1 \text{ is weakly inner future trapped}$$

with respect to t. If

$$\text{all connected components } \mathscr{T}_\alpha,\ \alpha \neq 1, \text{ of } \mathscr{T} \text{ are inner past trapped}$$

with respect to t, then

$$J^+(\mathscr{T}_1) \cap J^-(\mathscr{T}_\alpha) = \emptyset \quad \textit{for} \quad \mathscr{T}_\alpha \neq \mathscr{T}_1.$$

Remarks 3.2. 1. Nothing is assumed about the nature of the index set Ω.

2. The condition that at least one of the defining inequalities is strict is necessary. Indeed, let $\mathscr{M}' = \mathbb{R} \times \underbrace{S^1 \times \cdots \times S^1}_{n\text{factors}}$ with a flat product metric, let t be a standard coordinate on the $\mathbb{R}$ factor, and let $\varphi \in [0, 2\pi]$ be a standard angular coordinate on the first S^1 factor, then $\mathscr{M} = \{0 \leq \varphi \leq \pi\} \subset \mathscr{M}'$ satisfies the hypotheses above except for the strictness condition, and does not satisfy the conclusion.

Let t be a time function on $\mathscr{M}$, and let $\gamma : [a, b] \to \mathscr{M}$ be a future directed causal curve. The *time of flight t_γ of* γ is defined as

$$t_\gamma = t(\gamma(b)) - t(\gamma(a)).$$

As a step in the proof of Theorem 3.1, we note:

Proposition 3.3. *Let $(\mathscr{M}, g)$ be a globally hyperbolic space-time satisfying the null energy condition containing a past inwards trapped hypersurface $\mathscr{T}$. Let $S \subset \mathscr{M}$ be future inwards weakly trapped. Then there are no future directed causal curves, starting inwardly at S, meeting $\mathscr{T}$ inwardly, and minimising, amongst nearby causal curves, the time of flight between S and $\mathscr{T}$.*

Proof. Suppose that the result is wrong, thus there exists a future directed causal curve $\gamma : [a, b] \to \mathscr{M}$ with $\gamma(a) \in S$, $\gamma(b) \in \mathscr{T}$, locally minimising the time of flight. Standard considerations show that γ is a null geodesic emanating orthogonally from S without S-conjugate points on $[a, b)$. In particular $\dot{J}^+(S)$ is a smooth null hypersurface near $\gamma([a, b))$. Let $t_+ = t(\gamma(b))$, set $S_+ = \{t = t_+\} \cap \mathscr{T}$, then $\dot{J}^-(S_+)$ is a smooth null hypersurface near S_+, that contains a segment $\gamma([b-\varepsilon, b])$. Moreover, $\dot{J}^-(S_+)$ lies to the causal past of $\dot{J}^+(S)$ close to γ since otherwise we could construct a timelike curve from S to S_+ close to γ, thus violating the minimisation character of γ. Since $\dot{\gamma}(a)$ is inwards pointing and $\dot{\gamma}(b)$ outwards pointing, the Raychaudhuri equation shows that the null divergence of $\dot{J}^+(S)$ along γ is non-positive whereas the null divergence of $\dot{J}^-(S_+)$ is positive near S_+. For points $\gamma(s)$, with $s \neq b$ but close to b, this contradicts the maximum principle for null hypersurfaces [15,16], establishing the result. $\square$

Example 3.4. An example to keep in mind is the following: let $p, q \in \mathbb{R}^{n+1}$ be two spatially separated points in Minkowski space-time $\mathbb{R}^{1,n}$. Let $S = \dot{J}^-(p) \cap \dot{J}^-(q)$. The null generators of $\dot{J}^-(p)$ and $\dot{J}^-(q)$ are converging, when followed to the future, and they meet S normally, which shows that S is an outwards and inwards future trapped (non-compact) submanifold of $\mathbb{R}^{1,n}$. Of course, in this case the choice of "inwards" and "outwards" is a pure matter of convention.

Let $n = 3$, choose $p = (2, 2, 0, 0)$, $q = (2, -2, 0, 0)$, let $\mathscr{T}$ be the timelike surface $\mathscr{T} = \{r = 1\}$, which is both future and past inwards trapped. The null achronal geodesic segments $\gamma_\pm(t) = (t, \pm t, 0, 0)$, $0 \le t < 2$, are in $\dot{J}^+(S)$ and, by symmetry considerations, *maximise* the time of flight between S and $\mathscr{T}$. Since $\dot{J}^+(S)$ lies below $\dot{J}^-(\{t = 0\} \cap \mathscr{T})$, the argument in the proof below, when applied to $\gamma_\pm$, does not lead to a contradiction. But the example shows that the existence of an achronal null geodesic segment between S and $\mathscr{T}$ is compatible with the hypotheses above. In particular "locally minimising" cannot be replaced by "extremising".

Proof of Theorem 3.1: Let S be a weakly inner trapped compact Cauchy surface of $\mathscr{T}_1$. Suppose there exists a causal curve c from S to a point p in a different component $\mathscr{T}_2$. Let $\mathscr{S}$ be the Cauchy surface $\{t = t(p)\}$ for M. We want to construct a fastest null geodesic from S to $\mathscr{T} \setminus \mathscr{T}_1$; for this we need to show that only finitely many components, $\mathscr{T}_a$, of $\mathscr{T} \setminus \mathscr{T}_1$ meet the set $A = J^+(S) \cap \mathscr{S}$, which is compact by Proposition 2.1. Suppose to the contrary, there are infinitely many of these components that meet A. Then we obtain an infinite sequence of points $\{x_n\}$ in A, each point in a different component. Since A is compact we can pass to a convergent subsequence, still called $\{x_n\}$, such that $x_n \to x \in A$. Since $\mathscr{T} \cap \mathscr{S}$ is closed, x is in $\mathscr{T}$. But this contradicts the half-neighborhood property of manifolds with boundary.

The time function t on M restricts to a time function on $\mathscr{T}$. By the observation in the preceding paragraph, the set $(\mathscr{T} \setminus \mathscr{T}_1) \cap J^+(S) \cap J^-(\mathscr{S})$ is compact and thus we can now minimize t on causal curves from S to $\mathscr{T} \setminus \mathscr{T}_1$ contained in the aforementioned compact set to obtain a fastest causal curve γ from S to $\cup_{a \neq 1} \mathscr{T}_a$. Since t has been minimized, γ meets $\mathscr{T}$ only at its endpoints, and hence must be a null geodesic. This contradicts Proposition 3.3, and establishes the result. See [27] for a more detailed exposition. $\square$

We now proceed to establish a general topological censorship result for globally hyperbolic space-times with timelike boundary.

Theorem 3.5. *Let $(\mathscr{M}, g)$ be a space-time with a connected timelike boundary $\mathscr{T}$. Let*

$$\langle\langle \mathscr{T} \rangle\rangle := I^+(\mathscr{T}) \cap I^-(\mathscr{T})$$

be the domain of communications of $\mathscr{T}$. Further assume $\langle\langle \mathscr{T} \rangle\rangle$ has a Cauchy time function such that the level sets $t|_{\mathscr{T}}$ are compact. If the NEC holds on $\langle\langle \mathscr{T} \rangle\rangle$, and if $\mathscr{T}$ is inner past trapped and weakly inner future trapped with respect to t, then *topological censorship holds, i.e., any causal curve included within* $\langle\langle \mathscr{T} \rangle\rangle$ *with end points on* $\mathscr{T}$ *can be deformed, keeping end points fixed, to a curve included in* $\mathscr{T}$.

Proof. First notice that the inclusion $j \colon \mathscr{T} \hookrightarrow \langle\langle \mathscr{T} \rangle\rangle$ induces a homomorphism of fundamental groups $j_* \colon \pi_1(\mathscr{T}) \to \pi_1(\langle\langle \mathscr{T} \rangle\rangle)$. Thus there exists a covering $\pi \colon M \to \langle\langle \mathscr{T} \rangle\rangle$ associated to the subgroup $j_*(\pi_1(\mathscr{T}))$ of $\pi_1(\langle\langle \mathscr{T} \rangle\rangle)$. This covering is characterized as the largest covering of $\langle\langle \mathscr{T} \rangle\rangle$ containing a homeomorphic copy $\mathscr{T}_0$ of $\mathscr{T}$, that is, $\pi|_{\mathscr{T}_0}$ is a homeomorphism onto $\mathscr{T}$ [19]. Furthermore, this covering has the property that the map $i_* \colon \pi_1(\mathscr{T}_0) \to \pi_1(M)$ induced by the inclusion $i \colon \mathscr{T}_0 \to M$ is surjective. Endowing M with the pullback metric $\pi^*(g)$ we get a globally hyperbolic space-time with timelike boundary $\pi^{-1}(\mathscr{T})$. Now, let $\gamma \colon [a, b] \to \langle\langle \mathscr{T} \rangle\rangle$ be a causal curve with endpoints in $\mathscr{T}$. Lift γ to $\gamma_0 \colon [a, b] \to M$ with $\gamma_0(a) \in \mathscr{T}_0$. By Theorem 3.1 we know that $\mathscr{T}_0$ can not communicate with any other component of $\pi^{-1}(\mathscr{T})$, hence $\gamma_0(b) \in \mathscr{T}_0$. As a consequence, γ_0 is homotopic to a curve in $\mathscr{T}_0$ and the result follows. □

As noted in [17], topological censorship can be viewed as the statement that *any* curve in $\langle\langle \mathscr{T} \rangle\rangle$ with endpoints in $\mathscr{T}$ is homotopic to a curve in $\mathscr{T}$, or equivalently that the map $j_* \colon \pi_1(\mathscr{T}) \to \pi_1(\langle\langle \mathscr{T} \rangle\rangle)$ is surjective. We reproduce here the argument for completeness.

Theorem 3.6. *With the same hypotheses as above, the map* $j_* \colon \pi_1(\mathscr{T}) \to \pi_1(\langle\langle \mathscr{T} \rangle\rangle)$ *induced by the inclusion* $j \colon \mathscr{T} \to \langle\langle \mathscr{T} \rangle\rangle$ *is surjective.*

Proof. Let $\pi \colon M \to \langle\langle \mathscr{T} \rangle\rangle$ be the universal cover of $\langle\langle \mathscr{T} \rangle\rangle$ and let $\{\mathcal{I}_\alpha\}$, $\alpha \in A$, be the collection of connected components of the timelike boundary $\pi^{-1}(\mathscr{T})$. Let us define $\langle\langle I \rangle\rangle_{\alpha,\beta} := I^+(\mathcal{I}_\alpha) \cap I^-(\mathcal{I}_\beta)$. We claim that the collection of sets $\langle\langle I \rangle\rangle_{\alpha,\beta}$ forms an open cover of M. Indeed, let $p \in M$; since $\pi(p) \in \langle\langle \mathscr{T} \rangle\rangle$ there exists a causal curve through $\pi(p)$ which starts and ends in $\mathscr{T}$. Then γ lifts to a causal curve through p which starts in some $\mathcal{I}_\alpha$ and ends in some $\mathcal{I}_\beta$, hence the result. Now, by Theorem 3.5 the sets $I^+(\mathcal{I}_\alpha) \cap I^-(\mathcal{I}_\beta)$ are empty if $\alpha \neq \beta$. It follows that the sets $\langle\langle I \rangle\rangle_{\alpha,\alpha}$ are pairwise disjoint, cover M, and since M is connected we conclude that $|A| = 1$ and hence $\pi^{-1}(\mathscr{T})$ is connected. The result now follows from the following topological result [17, Lemma 3.2]: □

Proposition 3.7. *Let* M *and* S *be topological manifolds,* $\imath \colon S \hookrightarrow M$ *an embedding and* $\pi \colon M^* \to M$ *the universal cover of* M*. If* $\pi^{-1}(S)$ *is connected then the induced group homomorphism* $\imath_* \colon \pi_1(S) \to \pi_1(M)$ *is surjective.*

4. Kaluza–Klein Asymptotics

In the sections that follow we shall apply Theorem 3.5 to obtain topological information about space-times with *Kaluza–Klein* asymptotics: we shall say that $\mathscr{S}_{\rm ext}$ is a *Kaluza–Klein asymptotic end*, or *asymptotic end* for short, if $\mathscr{S}_{\rm ext}$ is diffeomorphic to $\mathbb{R}_r \times N \times Q$, where N and Q are compact manifolds. The notation $\mathbb{R}_r$ is meant to convey the information that we denote by r the coordinate running along an $\mathbb{R}$ factor. Let $\mathring{m}_r$ be a family of Riemannian metrics parameterized by r, let $\mathring{k}$ be a fixed Riemannian metric on Q, let finally $\mathring{\lambda}$ and $\mathring{\nu}$ be two functions on $\mathbb{R}$, the *reference metric* $\mathring{g}$ on $\mathbb{R}_t \times \mathscr{S}_{\mathrm{ext}}$ is defined as

$$\mathring{g} = -e^{2\mathring{\lambda}(r)}\mathrm{d}t^2 + \underbrace{e^{-2\mathring{\nu}(r)}\mathrm{d}r^2 + \mathring{m}_r + \mathring{k}}_{=:\mathring{\gamma}}. \tag{4.1}$$

The reason for treating N and Q separately is that the metrics $\mathring{m}_r$ are allowed to depend on r (in the examples below we will actually have $\mathring{m}_r = r^2\mathring{m}$, for a fixed metric $\mathring{m}$), while $\mathring{k}$ is not. The manifold $\mathbb{R}_t \times \mathbb{R}_r \times N$ will be referred to as the *base manifold*, while Q can be thought of as the internal space of Kaluza–Klein theory (see, e.g., [11]).

To apply our previous results, we will need the hypothesis that the hypersurfaces

$$\mathscr{T}_R := \{r = R\}$$

are inner future and past trapped for the reference metric $\mathring{g}$. We define the *outwards* pointing $\mathring{g}$-normal to $\mathscr{T}_R$ to be $n := e^{\mathring{\nu}}\partial_r$, and the two null future normals $n_\pm$ to $\{t = \text{const},\ r = \text{const}'\}$ are given by $n_\pm = e^{-\mathring{\lambda}}\partial_t \pm n$. The requirement of "mean outwards null $\mathring{g}$-convexity" of $\mathscr{T}_R$ reads

$$\pm\,\mathring{\theta}_\pm = \frac{e^{\mathring{\nu}-\mathring{\lambda}}}{\sqrt{\det \mathring{m}_r}}\partial_r(\sqrt{\det \mathring{m}_r}\,e^{\mathring{\lambda}}) > 0. \tag{4.2}$$

We will be interested in metrics g which are asymptotic, as r goes to infinity, to metrics of the above form. The convergence of g to $\mathring{g}$ should be such that the positivity of $\pm\theta_\pm$ holds, for R large enough, uniformly over compact sets of the t variable. Two special cases seem to be of particular interest, with asymptotically flat, or asymptotically anti-de Sitter base metrics.

4.1. Asymptotically Flat Base Manifolds

A special case of the above arises when $\mathscr{S}_{\mathrm{ext}}$ is diffeomorphic to $\big(\mathbb{R}^n \setminus \overline{B}(R)\big) \times Q$, where $\overline{B}(R)$ is a closed coordinate ball of radius R, thus the manifold N is an $(n-1)$-dimensional sphere. In dimension $n \geq 3$ we take $\mathring{g} = -\mathrm{d}t^2 \oplus \mathring{\gamma}$, where $\mathring{\gamma} = \delta \oplus \mathring{k}$, and where δ is the Euclidean metric on $\mathbb{R}^n$. If $n = 2$, in (4.1) we take $\mathring{\lambda} = \mathring{\nu} = 0$ and $\mathring{m}_r = r^2 d\varphi^2$, where φ is a coordinate on S^1 which does not *necessarily* range over $[0, 2\pi]$. Thus, for all $n \geq 2$ we have $\mathring{\lambda} = \mathring{\nu} = 0$ and $\mathring{m}_r = r^2\mathrm{d}\Omega^2$, where $\mathrm{d}\Omega^2$ is the round metric on S^{n-1}; strictly speaking, $\mathbb{R}_r$ is then (R, ∞), a set diffeomorphic to $\mathbb{R}$. Metrics g which asymptote to this $\mathring{g}$ as r tends to infinity will be said to have an *asymptotically flat base manifold*. Equation (4.2) gives

$$\pm\,\mathring{\theta}_\pm = \frac{n-1}{r} \tag{4.3}$$

which is positive, as required.

We shall say that a Riemannian metric γ on $\mathscr{S}_{\rm ext}$ is *Kaluza–Klein asymptotically flat*, or *KK-asymptotically flat* for short, if there exists $\alpha > 0$ and $k \geq 1$ such that for $0 \leq \ell \leq k$ we

$$\mathring{D}_{i_1} \dots \mathring{D}_{i_\ell}(\gamma - \mathring{\gamma}) = O(r^{-\alpha-\ell}), \tag{4.4}$$

where $\mathring{D}$ denotes the Levi-Civita connection of $\mathring{\gamma}$, and r is the radius function in $\mathbb{R}^n$, $r := \sqrt{(x^1)^2 + \dots (x^n)^2}$, with the x^i's being any Euclidean coordinates of $(\mathbb{R}^n, \delta)$. We shall say that a general relativistic initial data set $(\mathscr{S}_{\rm ext}, \gamma, K)$ is *Kaluza–Klein asymptotically flat*, or *KK-asymptotically flat*, if $(\mathscr{S}_{\rm ext}, \gamma)$ is KK-asymptotically flat and if for $0 \leq \ell \leq k-1$ we have

$$\mathring{D}_{i_1} \dots \mathring{D}_{i_\ell} K = O(r^{-\alpha-1-\ell}). \tag{4.5}$$

A space-time $(\mathscr{M}, g)$ will be said to contain a *Kaluza–Klein asymptotically flat region* if there exists a subset of $\mathscr{M}$, denoted by $\mathscr{M}_{\rm ext}$, and a time function t on $\mathscr{M}_{\rm ext}$, such that the initial data (g, K) induced by g on the level sets of t are KK-asymptotically flat.

All this reduces to the usual notion of asymptotic flatness when Q is the manifold consisting of a single point; a similar comment applies to the next section.

Let $\mathscr{T}_R = \{r = R\}$ be a level set of r in $\mathscr{M}_{\rm ext}$. Then the unit outwards pointing conormal $n^\flat$ to $\mathscr{T}_R$ is $n^\flat = (1 + O(r^{-\alpha}))dr$. This implies that the future directed null vector fields normal to the foliation of $\mathscr{T}_R$ by the level sets of t take the form $n^\pm = \partial_t \pm \frac{x^i}{r}\partial_i + O(r^{-\alpha})$, leading to [compare (4.3)]

$$\pm\, \theta^\pm = \frac{n-1}{r} + O(r^{-\alpha-1}) > 0 \quad \text{for } r \text{ large enough.} \tag{4.6}$$

4.2. Asymptotically Anti-de Sitter Base Manifolds

We consider, now, manifolds with asymptotically anti-de Sitter base metrics. The base reference metric is taken of the form

$$-\, e^{2\mathring{\lambda}(r)}\mathrm{d}t^2 + e^{-2\mathring{\nu}(r)}\mathrm{d}r^2 + \mathring{m}_r, \quad \mathring{m}_r = r^2\mathring{m}, \tag{4.7}$$

which can be thought of as being a generalised Kottler metric, where $\mathring{m}$ is an Einstein metric on the compact $(n{-}1)$-dimensional manifold N, $n \geq 2$. Furthermore,

$$e^{2\mathring{\lambda}(r)} = e^{2\mathring{\nu}(r)} = \mathring{\alpha}r^2 + \mathring{\beta},$$

for some suitable constants $\mathring{\alpha} > 0$ and $\mathring{\beta} \in \mathbb{R}$, which can be chosen so that (4.7) is an Einstein metric: Indeed, if Q has dimension k, then $\mathring{g}$ will satisfy the vacuum Einstein equations with cosmological constant Λ if $\mathring{k}$ is an Einstein metric with scalar curvature $2k\Lambda/(n+k-1)$, while $\mathring{\alpha} = -2\Lambda/n(n+k-1)$, and $\mathring{\beta} = R(\mathring{m})/(n-1)(n-2)$ for $n > 2$, while $\mathring{\beta}$ is arbitrary if $n = 2$, where $R(\mathring{m})$ is the scalar curvature of the metric $\mathring{m}$ (compare [1,3]).

In a manner somewhat analogous to the previous section, with decay requirements adapted to the problem at hand, we shall say that a Riemannian metric

γ on $\mathscr{S}_{\rm ext}$ is *KK-asymptotically adS* if there exist a real number $\alpha > 1$ and an integer $k \geq 1$ such that for $0 \leq \ell \leq k$ we have

$$|\mathring{D}_{i_1} \dots \mathring{D}_{i_\ell}(\gamma - \mathring{\gamma})|_{\mathring{\gamma}} = O(r^{-\alpha}), \tag{4.8}$$

where $|\cdot|_{\mathring{\gamma}}$ is the norm of a tensor with respect to $\mathring{\gamma}$, and r is a "radial coordinate" as in (4.7). We shall say that a general relativistic initial data set $(\mathscr{S}_{\rm ext}, \gamma, K)$ is *KK-asymptotically adS*, if $(\mathscr{S}_{\rm ext}, \gamma)$ is KK-asymptotically adS and if for $0 \leq \ell \leq k-1$ we have

$$|\mathring{D}_{i_1} \dots \mathring{D}_{i_\ell} K|_{\mathring{\gamma}} = O(r^{-\alpha}). \tag{4.9}$$

Finally, a space-time $(\mathscr{M}, g)$ will be said to contain a *Kaluza–Klein asymptotically adS region* if there exists a subset of $\mathscr{M}$, denoted by $\mathscr{M}_{\rm ext}$, and a time function t on $\mathscr{M}_{\rm ext}$, such that the initial data (g, K) induced by g on the level sets of t are KK-asymptotically adS.

The fact that KK-asymptotically adS metrics have the right null convexity properties is easiest to see using the conformal compactifiability properties of the base metric. Suppose, to start with, that Q consists only of one point, so that $\mathring{k} = 0$. Suppose further that g has a conformal compactification in the usual Penrose sense, so that the unphysical metric $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ extends smoothly to a conformal boundary at which Ω vanishes; this is certainly the case for the reference metrics $\mathring{g}$ of (4.7), and will also hold for a large class of asymptotically adS metrics as defined above. We define the outwards directed $\tilde{g}$-unit normal to the level sets of Ω to be

$$\tilde{n}^\mu = -\frac{\tilde{g}^{\mu\nu}\partial_\nu \Omega}{\sqrt{\tilde{g}^{\alpha\beta}\partial_\alpha\Omega\partial_\beta\Omega}}$$

(the minus sign being justified by the fact that Ω decreases as the conformal boundary $\{\Omega = 0\}$ is approached). Let, finally, t be a time function on the conformally completed manifold $\widetilde{\mathscr{M}}$ such that the $\tilde{g}$-unit timelike vector field $\tilde{T}$ normal to the level sets of t is tangent to the conformal boundary; thus $\tilde{T}(\Omega) = \Omega\psi$ for some function ψ which is smooth on $\widetilde{\mathscr{M}}$. Then $n^\mu = \Omega\tilde{n}^\mu$ and $T^\mu = \Omega\tilde{T}^\mu$ are unit and normal to $\{t = \mathrm{const}, \Omega = \mathrm{const}'\}$. So, in space-time dimension $n+1$,

$$\begin{aligned}
\pm\,\theta_\pm &= \nabla_\mu(\pm T^\mu + n^\mu)\\
&= \frac{1}{\sqrt{|\det g|}}\partial_\mu\left(\sqrt{|\det g|}(\pm T^\mu + n^\mu)\right)\\
&= \frac{\Omega^{n+1}}{\sqrt{|\det \tilde{g}|}}\partial_\mu\left(\Omega^{-n-1}\sqrt{|\det \tilde{g}|}\,\Omega(\pm\tilde{T}^\mu + \tilde{n}^\mu)\right)\\
&= n|d\Omega|_{\tilde{g}} + O(\Omega),
\end{aligned} \tag{4.10}$$

which is positive for Ω small enough (in the last equation n is the space-dimension, not to be confused with the unit normal to the level sets of Ω). It is now a simple exercise to check that, for KK-asymptotically adS metrics, the correction terms arising from $\mathring{k}$, and from the error terms in (4.8)–(4.9) will not affect positivity of $\pm\theta_\pm$ whenever $\alpha > 1$, as required above.

4.3. Uniform KK-asymptotic Ends

We shall say that a KK-asymptotically flat region, or a KK-asymptotically adS region, is *uniform of order* k if there exists a time function t such that the estimates (4.4)–(4.5), or (4.8)–(4.9), hold with constants independent of t.

5. Topological Censorship for Uniform KK-asymptotic Ends

In this section we shall consider manifolds with KK-asymptotically flat or KK-asymptotically adS regions. Now, our approach to topological censorship in this work requires uniformity in time of the mean null extrinsic curvatures of the spheres $\{t = \text{const}, \ r = \text{const}'\}$. This might conceivably hold for a wide class of dynamical metrics, but how large is the corresponding class of metrics remains to be seen. Now, the applications we have in mind for our results [6] concern stationary metrics, in which case the uniformity is easy to guarantee by an obvious choice of time functions. Hence the uniformity hypothesis is quite reasonable from this perspective.

Consider, first, a space-time with a Killing vector field X, with complete orbits, containing a KK-asymptotic end $\mathscr{S}_{\text{ext}}$. Then X will be called *stationary* if X is timelike on $\mathscr{S}_{\text{ext}}$ and approaches, as r goes to infinity, ∂_t in the coordinate system of (4.1);[2]

$(\mathscr{M}, g)$ will then be called stationary. Similarly to the standard asymptotically flat case, we set

$$\mathscr{M}_{\text{ext}} := \cup_{t\in\mathbb{R}}\phi_t[X](\mathscr{S}_{\text{ext}}),$$

where $\phi_t[X]$ denotes the flow of X. Assuming stationarity, the *domain of outer communications* is defined as in [9,7]:

$$\langle\langle \mathscr{M}_{\text{ext}}\rangle\rangle := I^-(\mathscr{M}_{\text{ext}}) \cap I^+(\mathscr{M}_{\text{ext}}). \tag{5.1}$$

More generally, let $(\mathscr{M}, g)$ admit a time function t ranging over an open interval I (not necessarily equal to $\mathbb{R}$), and a radius function r as in (4.1), with KK-asymptotic level sets which are *uniform of order zero*. We then set

$$\mathscr{M}_{\text{ext}} := \{p \in \mathscr{M} : \ r(p) \geq R_0\}$$

for some R_0 chosen large enough so that for any $R \geq R_0$ we have

$$J^\pm(\mathscr{M}_{\text{ext}}) = J^\pm(\{r = R\}). \tag{5.2}$$

[2] For metrics which are asymptotically flat in the usual (rather than KK) sense, the existence of such coordinates can be established for Killing vectors which are timelike on $\mathscr{S}_{\text{ext}}$, whenever the initial data set satisfies the conditions of the positive energy theorem. It is likely that a similar result holds for KK-asymptotically flat or adS metrics, but we have not investigated this issue any further.

To see that such an R_0 exists, note that the inclusion $J^\pm(\mathscr{M}_{\rm ext}) \supset J^\pm(\{r = R\})$ is obvious whenever $\{r = R\} \subset \mathscr{M}_{\rm ext}$. To justify the opposite inclusion let, say, $x \in J^-(\mathscr{M}_{\rm ext})$, so there exists a future directed causal curve from x to some point $(t, p) \in \mathscr{M}_{\rm ext}$, thus $p \in \mathscr{S}_{\rm ext}$. We need to show that there exists a future directed causal curve from (t, p) to a point $(t', q) \in \{r = R\}$. This follows from the somewhat more general fact, that for any t and for any two points $p, q \in \mathscr{S}_{\rm ext}$ such that $r(p) \geq R_0$ and $r(q) \geq R_0$, there exists a causal curve $\gamma(s) = (t + \alpha s, \sigma(s))$ such that $\sigma(0) = p$ and $\sigma(0) = q$, with α and σ independent of t. Now, existence of σ follows from connectedness of $\mathscr{S}_{\rm ext}$. Next, the existence of a t-independent (large) constant α so that γ is causal for $\mathring{g}$ follows immediately from the form of the metric $\mathring{g}$. Finally, it should be clear from uniformity in time of the error terms that, increasing α if necessary, γ will also be causal for g, independently of t, provided R_0 is chosen large enough.

The domain of outer communications is again defined by (5.1).

If the asymptotic estimates are moreover uniform to order one, we choose R_0 large enough so that all level sets of $\{r = R\}$, with R sufficiently large are *future and past inner trapped.*

Remark 5.1. As shown in Appendix A, there exist vacuum space-times which are uniformly asymptotically flat to order zero, and for which the null convexity conditions needed for our arguments are satisfied even though the asymptotic flatness estimates (4.4)–(4.5) are *not* uniform to order one. For simplicity, in this section we shall only formulate our theorems assuming uniformity to order one, but it should be clear to the reader that the results hold e.g. for metrics with the asymptotic behavior as in Appendix A.

Let us consider a space-time $(\mathscr{M}, g)$ with several KK-asymptotic regions $\mathscr{M}^\lambda_{\rm ext}$, $\lambda \in \Lambda$, each generating its own domain of outer communications. We assume that all regions are uniform to order one with respect to a Cauchy time function t. Let $\mathscr{T}_\lambda \subset \mathscr{M}^\lambda_{\rm ext}$ be defined as $\{r = \hat{R}_\lambda\}$ for an appropriately large R_λ. Consider the manifold obtained by removing from the original space-time the asymptotic regions $\{r > \hat{R}_\lambda\} \subset \mathscr{M}^\lambda_{\rm ext}$; this is a manifold with boundary $\mathscr{T} = \cup_\lambda \mathscr{T}_\lambda$, each connected component $\mathscr{T}_\lambda$ being both future and past inwards trapped. From (5.2) we have $J^\pm(\mathscr{M}^\lambda_{\rm ext}) = J^\pm(\mathscr{T}_\lambda)$. Then the following result is a straightforward consequence of Theorem 3.1:

Theorem 5.2. *If $(\mathscr{M}, g)$ is a globally hyperbolic with KK-asymptotic ends, uniform to order one, satisfying the null energy condition* (3.1)*, then*

$$J^+(\mathscr{M}^{\lambda_1}_{\rm ext}) \cap J^-(\mathscr{M}^{\lambda_2}_{\rm ext}) = \emptyset \quad \textit{whenever} \quad \mathscr{M}^{\lambda_1}_{\rm ext} \cap \mathscr{M}^{\lambda_2}_{\rm ext} = \emptyset. \tag{5.3}$$

Next, Theorems 3.5 and 3.6 yield the following result on topological censorship for stationary KK-asymptotically flat space-times:

Theorem 5.3. *Let $(\mathscr{M}, g)$ be a space-time satisfying the null energy condition, and containing a KK-asymptotic end $\mathscr{M}_{\rm ext}$, uniform to order one. Suppose further* *that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic. Then every causal curve in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ with end-points in $\mathscr{M}_{\rm ext}$ is homotopic to a curve in $\mathscr{M}_{\rm ext}$. Moreover the map $j_*\colon \pi_1(\mathscr{M}_{\rm ext}) \to \pi_1(\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle)$ is surjective.*

Proof. It suffices to prove the second statement. Let $\hat{R} > R$ and $\mathscr{T} = \{r = \hat{R}\}$ be defined as in the previous result. Let α be a loop in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ based at p_0, and let c be the radial curve from p_0 to $p \in \mathscr{T}$. Then since $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle = \langle\langle \mathscr{T}\rangle\rangle$, by Theorem 3.6 the loop $c * \alpha * c^-$, where c^- denotes c followed backwards, is homotopic to a loop β in $\mathscr{T}$ based at p. Thus α is in turn homotopic to $c^- * \beta * c$, which is a loop that lies entirely in $\mathscr{M}_{\rm ext}$ hence establishing the result. □

For future reference, we point out the following special case of Proposition 3.3, which follows immediately from the fact that large level sets of r are inner trapped:

Proposition 5.4. *Let $(\mathscr{M}, g)$ be a stationary, asymptotically flat, or KK-asymptotically flat globally hyperbolic space-time satisfying the null energy condition. Let $S \subset \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ be future inwards marginally trapped. There exists a large constant R_1 such that for all $R_2 \geq R_1$ there are no future directed null geodesics starting inwardly at S, ending inwardly at $\{r = R_2\} \subset \mathscr{M}_{\rm ext}$, and locally minimising the time of flight.*

Now we proceed to prove the main theorem for quotients of KK-asymptotically flat space-times.

Theorem 5.5. *Let $(\mathscr{M}, g)$ be a space-time satisfying the null energy condition, and containing a KK-asymptotically flat region, or a KK-asymptotically adS region, with the asymptotic estimates uniform to order one. Suppose that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and that there exists an action of a group G_s on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ by isometries which, on $\mathscr{M}_{\rm ext} \approx \mathbb{R} \times \mathscr{S}_{\rm ext}$, takes the form*

$$g \cdot (t, p) = (t, g \cdot p).$$

If $\mathscr{S}_{\rm ext}/G_s$ simply connected, then so is $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/G_s$.

Remark 5.6. A variation on the proof below, using an exhaustion argument, shows that the result remains valid if the asymptotic decay estimates are uniform in t to order zero, and uniform over compact sets in t to order one. In this case the hypersurfaces $\{r = R\}$ are not necessarily trapped, but there exists a sequence R_k such that the hypersurfaces $\{r = R_k, |t| < k\}$ are.

Proof. If the action of G_s is such that the projection $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \to \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/G_s$ has the homotopy lifting property (see, e.g., [20]), then the following argument applies: Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{M}_{\rm ext} & \xrightarrow{i} & \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \\ q\downarrow & & \downarrow p \\ \mathscr{M}_{\rm ext}/G_s & \xrightarrow{j} & \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/G_s \end{array}$$

where p and q are the standard projections, i the standard inclusion and j the map induced by i. Thus we have the corresponding commutative diagram

$$\begin{array}{ccc} \pi_1(\mathscr{M}_{\rm ext}) & \xrightarrow{i_*} & \pi_1(\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle) \\ q_* \downarrow & & \downarrow p_* \\ \pi_1(\mathscr{M}_{\rm ext}/G_s) & \xrightarrow{j_*} & \pi_1(\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G_s) \end{array}$$

of fundamental groups. By Theorem 5.2, i_* is onto. Finally notice that p_* and q_* are onto since p and q have the homotopy lifting property. Hence j_* is onto and as a consequence $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G_s$ is simply connected if $\mathscr{M}_{\rm ext}/G_s = \mathbb{R}\times(\mathscr{S}_{\rm ext}/G_s)$ is.

The homotopy lifting property of the action is known to hold in many significant cases (e.g., when the action is free), but it is not clear whether it holds in sufficient generality. However, one can proceed as follows: Let π denote the projection map

$$\pi : \mathscr{M} \to \mathscr{M}/G_s .$$

We start by constructing a covering space, $\widehat{\mathscr{M}}$, of $\mathscr{M}$: Choose $p\in\mathscr{M}$ and let Ω be the set of continuous paths in $\mathscr{M}$ starting at p. We shall say that the paths $\gamma_a\in\Omega$, $a=1,2$, are equivalent, writing $\gamma_1\sim\gamma_2$, if they share their end point, and if the projection $\pi(\gamma_1 * \gamma_2^-)$ of the path $\gamma_1 * \gamma_2^-$, obtained by concatenating γ_1 with γ_2 followed backwards, is homotopically trivial in $\mathscr{M}/G_s$. We set

$$\widehat{\mathscr{M}} := \Omega/\sim .$$

By the usual arguments (see, e.g., the proof of [22, Theorem 12.8]) $\widehat{\mathscr{M}}$ is a topological covering of $\mathscr{M}$, while [23, Proposition 2.12] shows that $\widehat{\mathscr{M}}$ is a smooth manifold. (In fact, $\widehat{\mathscr{M}}$ is the covering space of $\mathscr{M}$ associated with the subgroup $\operatorname{Ker}\pi_*\subset\pi_1(M)$.) The covering is trivial if and only if $\mathscr{M}/G_s$ is simply connected.

Since $\mathscr{S}_{\rm ext}/G_s$ is simply connected, the quotient $\mathscr{M}_{\rm ext}/G_s=\mathbb{R}\times(\mathscr{S}_{\rm ext}/G_s)$ also is, which implies that $\pi^{-1}(\mathscr{M}_{\rm ext})\subset\widehat{\mathscr{M}}$ is the union of pairwise disjoint diffeomorphic copies $\mathscr{M}^\lambda_{\rm ext}$, $\lambda\in\Lambda$, of $\mathscr{M}_{\rm ext}$, for some index set Λ. Each $\mathscr{M}^\lambda_{\rm ext}$ comes with an associated open domain of dependence $\langle\langle\mathscr{M}^\lambda_{\rm ext}\rangle\rangle\subset\widehat{\mathscr{M}}$. As in the proof of Theorem 3.6, the $\langle\langle\mathscr{M}^\lambda_{\rm ext}\rangle\rangle$'s form an open cover of $\widehat{\mathscr{M}}$. Moreover, by Theorem 5.2 they are pairwise disjoint. Connectedness of $\widehat{\mathscr{M}}$ implies that Λ is a singleton $\{\lambda_*\}$, with $\widehat{\mathscr{M}}=\langle\langle\mathscr{M}^{\lambda_*}_{\rm ext}\rangle\rangle$, hence $\widehat{\mathscr{M}}=\mathscr{M}$, and the result follows. □

5.1. Existence of Twist Potentials

We turn now our attention to the question of existence of twist potentials. The problem is the following: suppose that ω is a closed one form on a domain of outer communications $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$. For $i=1,\dots r$ let X_i be the basis of a Lie algebra of Killing vectors generating a connected group G of isometries and suppose that

$$\forall i \quad \mathscr{L}_{X_i}\omega = 0 = \omega(X_i). \tag{5.4}$$

If $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ is simply connected, then there exists a G-invariant function v such that $\omega = dv$. More generally, if $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G$ is a simply connected *manifold*, then

ω descends to a closed one-form on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G$, and again existence of the potential v follows. Let us show that the hypothesis that $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G$ is a manifold can be replaced by the weaker condition, that the projection map $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle \to \langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G$ has the *path homotopy lifting property*, namely: every homotopy of paths in $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G$ can be lifted to a continuous family of paths in $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$:

Proposition 5.7. *If $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle/G$ is simply connected, and if the path homotopy lifting property holds, then there exists a G-invariant function v on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ so that $\omega = {\rm d}v$.*

Proof. To simplify notations, let $\mathscr{M} = \langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ with the induced metric. Choose a point $p \in \mathscr{M}$, let $\gamma : [0,1] \to \mathscr{M}$ be any path with $\gamma(0) = p$, set

$$v_\gamma = \int_\gamma \omega,$$

we need to show that $v_\gamma = 0$ whenever $\gamma(0) = \gamma(1)$. Let $\mathring{\gamma}^\flat$ be the projection to $\mathscr{M}/G$ of a loop $\mathring{\gamma}$ through p, since $\mathscr{M}/G$ is simply connected there exists a continuous one-parameter family of paths $\gamma_t^\flat$, $t \in [0,1]$, so that $\gamma_0^\flat = \mathring{\gamma}^\flat$, $\gamma_t^\flat(0) = \gamma_t^\flat(1) = \mathring{\gamma}^\flat(0)$, $\gamma_1^\flat(s) = \mathring{\gamma}^\flat(0)$. Let γ_t be any continuous lift of $\gamma_t^\flat$ to $\mathscr{M}$ which is also continuous in t, such that $\gamma_t(1) = p$. Then $\gamma_t(0) = g_t p$ for some continuous $g_t \in G$. We can thus obtain a closed path through p, denoted by $\hat\gamma_t$, by following γ_t from p to $\gamma_t(0)$, and then following the path

$$[0,t] \ni s \mapsto g_{t-s}p.$$

Since γ_1 is trivial, so is $\hat\gamma_1 = \gamma_1$, so that $v_{\hat\gamma_1} = 0$. The family $\hat\gamma_t$ provides thus a homotopy of $\hat\gamma_0$ with $\hat\gamma_1$, and by homotopy invariance

$$0 = v_{\gamma_1} = v_{\hat\gamma_1} = v_{\hat\gamma_0} = v_{\gamma_0}.$$

Next, using the fact that both γ_0 and $\mathring{\gamma}$ project to $\mathring{\gamma}^\flat$, we will show that

$$v_{\gamma_0} = v_{\mathring{\gamma}}, \tag{5.5}$$

which will establish the result.

Let $s \in [0,1]$, set $r := \mathring{\gamma}(s)$, let $\mathscr{O}_r \subset \mathscr{O}$ denote any sufficiently small simply connected neighborhood of r, and let v_r denote the solution on $\mathscr{O}_r$ of

$$dv_r = \omega, \quad v_r(r) = 0. \tag{5.6}$$

Let $\mathscr{U}_r = G\mathscr{O}_r$ be the orbit of G through $\mathscr{O}_r$, for $p' \in \mathscr{U}_p$ there exists $\hat p \in \mathscr{O}_r$ and $g \in G$ such that $p' = g\hat p$. Set $v_r(p') := v_r(\hat p)$, this is well defined as the right-hand-side is independent of the choice of g and q by (5.4). Then v_r is a solution of (5.6) on $\mathscr{U}_r$, and for all s such that $\mathring{\gamma}(s) \in \mathscr{U}_r$ we have

$$v_r(\mathring{\gamma}(s)) = v_r(\gamma_0(s)).$$

It follows that for any interval $[s_1, s_2]$ such that $\mathring{\gamma}([s_1, s_2]) \subset \mathscr{O}_r$ we have

$$\int_{\mathring{\gamma}([s_1,s_2])} \omega = v_r(\mathring{\gamma}(s_2)) - v_r(\mathring{\gamma}(s_1)) = v_r(\gamma_0(s_2)) - v_r(\gamma_0(s_1)) = \int_{\gamma_0([s_1,s_2])} \omega.$$

A covering argument finishes the proof. □

6. Weakly Future Trapped Surfaces are Invisible

Yet another application of the ideas above is the following result, which is part of folklore knowledge in general relativity, without a satisfactory proof available elsewhere in the literature:

Theorem 6.1. *Let $(\mathscr{M}, g)$ be an asymptotically flat space-time, in the sense of admitting a regular future conformal completion $\widetilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}^+$, where $\mathscr{I}^+$ is a connected null hypersurface, such that,*

1. *$\widetilde{\mathscr{D}} = \mathscr{D} \cup \mathscr{I}^+$ is globally hyperbolic, where $\mathscr{D} = I^-(\mathscr{I}^+, \widetilde{\mathscr{M}})$, and*
2. *for any compact set $K \subset \mathscr{D}$, $J^+(K, \widetilde{\mathscr{D}})$ does not contain all of $\mathscr{I}^+$ ("i^0-avoidance").*

If the NEC holds on $\mathscr{D}$, then there are no compact future weakly trapped submanifolds within $\mathscr{D}$.

Remarks 6.2. 1. Note that if $\mathscr{M} \cup \mathscr{I}^+$ is globally hyperbolic, then $\widetilde{\mathscr{D}}$ also is.
2. Compare [8, Appendix B] for a discussion of issues that arise in a related context.

Proof. We begin by noting that the global hyperbolicity of $\widetilde{\mathscr{D}}$ implies that $\widetilde{\mathscr{D}}$ is causally simple, i.e., that sets of the form $J^+(K, \widetilde{\mathscr{D}})$ are closed in $\widetilde{\mathscr{D}}$ for all compact sets K. Suppose S is a compact future weakly trapped submanifold in $\mathscr{D}$. Let q be a point on $\partial(J^+(S, \widetilde{\mathscr{D}}) \cap \mathscr{I}^+) = \dot{J}^+(S, \widetilde{\mathscr{D}}) \cap \mathscr{I}^+$, which is nonempty by i^0-avoidance. Since $\dot{J}^+(S, \widetilde{\mathscr{D}}) = J^+(S, \widetilde{\mathscr{D}}) \setminus I^+(S, \widetilde{\mathscr{D}})$, there exists an achronal null geodesic $\gamma : [a, b] \to \widetilde{\mathscr{D}}$, with $\gamma(a) \in S$ and $\gamma(b) = q$, emanating orthogonally from S, without S-conjugate points on $[a, b)$. In particular, $\dot{J}^+(S)$ is a smooth null hypersurface near $\gamma([a, b))$. Below we show that for a suitably chosen point $q \in \dot{J}^+(S, \widetilde{\mathscr{D}}) \cap \mathscr{I}^+$, there exists a spacelike hypersurface S_+ in $\mathscr{I}^+$ that passes through q and does not meet $I^+(S, \widetilde{\mathscr{D}})$. Given this, the proof may now be completed along the lines of the proof of Proposition 3.3. Since S_+ does not meet $I^+(S, \widetilde{\mathscr{D}})$, one easily argues that $\dot{J}^-(S_+)$ is a smooth null hypersurface near S_+ that contains a segment $\gamma([b-\varepsilon, b])$ and lies to the causal past of $\dot{J}^+(S)$.

Let $\tilde{K}$ be a future directed outward pointing null vector at q orthogonal to S_+ in the unphysical metric $\tilde{g} = \Omega^2 g$. Since Ω decreases to the future along γ near q, we can choose $\tilde{K}$ so that $\tilde{K}(\Omega) = \tilde{g}(\tilde{K}, \tilde{\nabla}\Omega) = -1$. Now extend $\tilde{K}$ to a null vector field tangent to $\dot{J}^-(S_+)$ near q, and let $K = \Omega\tilde{K}$. A computation, using basic properties of conformal transformations, shows that the divergence θ of $\dot{J}^-(S_+)$

with respect to K in the physical metric g is related to the divergence $\tilde\theta$ of $\dot J^-(S_+)$ with respect to $\tilde K$ in the unphysical metric $\tilde g$ by, in space-time dimension $n+1$,

$$\theta = -(n-1)\tilde K(\Omega) + \Omega\,\tilde\theta.$$

It follows that $\dot J^-(S_+)$ will have positive null divergence at points of γ close to q. On the other hand, as in the proof of Proposition 3.3, $\dot J^+(S)$, has nonpositive null divergence along γ, and we are again led to a contradiction of the maximum principle for null hypersurfaces.

We conclude the proof by explaining how to choose q and S_+. For this purpose we introduce a Riemannian metric on $\mathscr{I}^+$, with respect to which the following constructions are carried out. Fix $q_0 \in \dot J^+(S,\widetilde{\mathscr{D}})\cap\mathscr{I}^+$, and let $U\subset\mathscr{I}^+$ be a convex normal neighborhood of q_0. By choosing a point $p\in U$, $p\notin J^+(S,\widetilde{\mathscr{D}})$, sufficiently close to q_0, we obtain a point $q\in\dot J^+(S,\widetilde{\mathscr{D}})\cap\mathscr{I}^+$, such that the geodesic segment $\overline{pq}$ in U realizes the distance from p to $\dot J^+(S,\widetilde{\mathscr{D}})\cap\mathscr{I}^+$. Now let S_+ be the distance sphere in U centered at p and passing through q. S_+ is a smooth hypersurface in $\mathscr{I}^+$ that does not meet $I^+(S,\widetilde{\mathscr{D}})$. It follows that S_+ intersects the generator of $\mathscr{I}^+$ through q transversely, and hence is spacelike near q. To see this, let γ be the null geodesic from S to q as in the preceding paragraph. For $x\in\gamma$ sufficiently close to q, $S'=\dot J^+(x,\widetilde{\mathscr{D}})\cap\mathscr{I}^+$ will be, in the vicinity of q, a smooth hypersurface in $\mathscr{I}^+$ transverse to the null generator of $\mathscr{I}^+$ through q. But, since $S'\subset J^+(S,\widetilde{\mathscr{D}})$, S_+ must meet S' tangentially at q. Hence q is the desired point in $\dot J^+(S,\widetilde{\mathscr{D}})\cap\mathscr{I}^+$ and S_+, suitably restricted, is the desired spacelike hypersurface in $\mathscr{I}^+$. $\square$

Remark 6.3. An entirely analogous result holds for asymptotically anti-de Sitter space-times, in the sense of admitting a regular conformal completion, with timelike conformal infinity $\mathscr{I}$, and can be proved in a similar fashion.

We further note that Proposition 3.3 may be used to obtain a version of Theorem 6.1 for space-times $(\mathscr{M},g)$ with KK-asymptotic ends, as follows.

Theorem 6.4. *Let $(\mathscr{M},g)$ be a KK-asymptotically flat or KK-asymptotically anti-de Sitter space-time with the asymptotic estimates uniform to order one. If $(\mathscr{M},g)$ contains a globally hyperbolic domain of outer communications $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ on which the NEC holds, then, there are no compact future weakly trapped submanifolds within $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$.*

Remark 6.5. Theorems 6.1 and 6.4 may be adapted to rule out the visibility from infinity of submanifolds S bounding compact acausal hypersurfaces $\mathscr{S}$ with weakly *outer* future trapped boundary. In fact $\mathscr{S}$ is allowed to have non-weakly outer trapped components of the boundary as long as those lie in a black hole region. Here the outer direction at S is defined as pointing away from $\mathscr{S}$.

Appendix A. Uniform Boundaries in Lindblad–Rodnianski–Loizelet Metrics

In this appendix we wish to point out that sufficiently small data vacuum space-times constructed using the Lindblad–Rodnianski method [24], as generalised by Loizelet to higher dimensions [25,26] (compare [5]), contain past inwards trapped, *closed to the future* (in a sense which should be made clear by what is said below), timelike hypersurfaces. This is irrelevant as far as the topological implications of our analysis are concerned, as in this case the space-time manifold is $\mathbb{R}^{n+1}$ anyway, but it illustrates the fact that such hypersurfaces can arise in vacuum space-times which are not necessarily stationary. Note that the resulting space-times are uniformly asymptotically flat to order zero, but *not* to order one in general, as the retarded-time derivatives of a radiating metric will not fall-off faster than $1/r$ when approaching future null infinity.

In order to proceed, we recall some facts about the space-times constructed in [24,25]. In Minkowski space-time $\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, \eta)$ let

$$q = r - t,$$

and let $H^{\mu\nu} := g^{\mu\nu} - \eta^{\mu\nu}$, where $g_{\mu\nu}$ is a small data vacuum metric on $\mathbb{R}^{n+1}$ as constructed in [24,25]. By [24, Corollary 9.3] for $n = 3$, and by [25, Corollary 5.1] for $n \geq 3$,[3] there exist constants C, $0 < \delta < \delta' < 1$ such that

$$|\partial H| \leq \begin{cases} C\varepsilon(1+t+|q|)^{\frac{1-n}{2}+\delta}(1+|q|)^{-1-\delta'}, & q \geq 0, \\ C\varepsilon(1+t+|q|)^{\frac{1-n}{2}+\delta}(1+|q|)^{-1/2}, & q < 0, \end{cases} \tag{A.1}$$

$$|H| \leq \begin{cases} C\varepsilon(1+t+|q|)^{\frac{1-n}{2}+\delta}(1+|q|)^{-\delta'}, & q \geq 0, \\ C\varepsilon(1+t+|q|)^{\frac{1-n}{2}+\delta}(1+|q|)^{1/2}, & q < 0, \end{cases} \tag{A.2}$$

$$|\bar{\partial} H| \leq \begin{cases} C\varepsilon(1+t+|q|)^{\frac{-1-n}{2}+\delta}(1+|q|)^{-\delta'}, & q \geq 0, \\ C\varepsilon(1+t+|q|)^{\frac{-1-n}{2}+\delta}(1+|q|)^{1/2}, & q < 0. \end{cases} \tag{A.3}$$

Here ϵ and δ are small constants determined by the initial data, and δ can be chosen as small as desired by choosing the data close enough to the Minkowskian ones. Next, $\bar{\partial}$ denotes partial coordinate derivatives ∂_μ to which a projection operator in directions tangent to the outgoing coordinate cones $\{t - r = \text{const}\}$ has been applied, e.g., in spherical coordinates, $\bar{\partial} \in \text{Span}\{L := \partial_t + \partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\varphi\}$.

Examining separately the cases $0 \leq t \leq r/2$, $r/2 \leq t \leq r$, and $r \leq t$, it is easily seen that there exists a constant $C > 0$ such that, for all $n \geq 3$,[4]

$$|H| \leq \frac{C}{(1+r)^{1/2-\delta}}, \quad |\partial H| \leq \frac{C}{(1+r)^{1-\delta}}, \quad |\bar{\partial} H| \leq \frac{C}{(1+r)^{3/2-\delta}}. \tag{A.4}$$

[3] At the end of the bootstrap argument one concludes that the inequalities there are satisfied by the solution.

[4] The estimates are actually better in higher dimensions, which is irrelevant for our purposes here.

The first inequality implies that $(\mathscr{M}, g)$ is uniformly asymptotically flat to order zero. On the other hand, $(\mathscr{M}, g)$ is *not* uniformly asymptotically flat to order one. However, the third inequality shows that one can choose R_0 large enough so that for all $R \geq R_0$ the hypersurfaces $\{r = R, t \geq 0\}$ are *inward past null convex*, in the sense that the level sets of t within $\{r = R\}$ have negative definite past inwards null second fundamental form (compare [14]). Indeed, from the first inequality one can choose null normals to $\{r = R\}$ of the form $\pm\partial_0 \pm \partial_r + O(r^{1/2-\delta})$, with a uniform error term. It now follows from the second and third estimate that the null second fundamental forms differ from their Minkowskian counterparts by terms which are uniformly $O(r^{-3/2+2\delta})$ and $O(r^{-3/2+\delta})$. For sufficiently small initial data one can choose $\delta < 1/4$, and the result follows.

A corresponding result holds for $t < 0$ by invariance of the Einstein equations under the map $t \mapsto -t$.

In particular the traces of the null second fundamental forms have the right signs for the results in our work to apply.

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