Uniqueness and energy bounds for static AdS metrics

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We show that Wang's proof of uniqueness of anti–de Sitter spacetime can be adapted to provide uniqueness results for strictly static asymptotically locally hyperbolic vacuum metrics with toroidal infinity, and to prove negativity of the free energy $E - TS$ of static asymptotically AdS black holes with toroidal or higher-genus horizons.

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I. INTRODUCTION

In [1] Wang derived an identity which allowed him to prove, in all space-dimensions $n \geq 3$, uniqueness of anti–de Sitter space within the class of strictly static conformally compactifiable solutions of the vacuum Einstein equations with a negative cosmological constant and a spherical conformal infinity. (The identity already appears in [2].) The aim of this paper is to explore further consequences of this identity. Namely, we consider static solutions of the vacuum Einstein equations with a negative cosmological constant and prove:

(1) The cuspidal Birmingham-Kottler metrics are unique in the class of strictly static solutions containing asymptotically locally hyperbolic ends and other controlled asymptotic ends or suitable boundaries, cf. Theorems 6.2–6.5 and 6.7 below.
(2) We establish a new lower bound for entropy of horizons in terms of the genus of the horizon, cf. Eq. (7.9) below.
(3) We give a simple proof of an upper bound in the spirit of [[3], Eq. (24)] on the free energy of a connected static black hole in terms of the genus of the horizon, cf. Eq. (8.5) below.

When Λ is positive we review the argument of [2] (compare [4]), that the identity mentioned above provides a simple proof of an upper bound for entropy of horizons, Eq. (9.10) below. This upper bound has been rediscovered in [5,6], with different proofs.

This work can be thought-of as a continuation of [7]. It reviews, in a slightly different manner, some results from [7] and considers further applications. Similar ideas have been independently pursued in [8].

II. THE EQUATIONS

We consider the vacuum Einstein equations with a cosmological constant Λ for a static spacetime metric which we denote by $\bar{g}$:

$$\bar{g} = -V^2 dt^2 + \underbrace{g_{ij} dx^i dx^j}_{=: g}, \qquad \partial_t V = \partial_t g = 0. \qquad (2.1)$$

For definiteness we assume that the spacetime is diffeomorphic to $\mathbb{R} \times M$, where M is an n-dimensional manifold equipped with a Riemannian metric g. We always assume that $V \not\equiv 0$ and that (M, g) is complete, possibly with boundary. We shall say that (M, g, V) is strictly static if $V > 0$ on M.

It is convenient to rescale the metric by a constant to obtain

$$\Lambda = \varepsilon \frac{n(n-1)}{2}, \qquad (2.2)$$

with $\varepsilon \in \{-1, 0, 1\}$ according to the sign of Λ.[1] This leads to the following equations, where D is the covariant derivative of g, $\bar{R}_{\alpha\beta}$ is the Ricci tensor of $\bar{g}$ and R_{ij} the Ricci tensor of g:

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[1] Wang assumes $\Lambda < 0$ so he always has $\varepsilon = -1$.

$$\bar{R}_{\alpha\beta} = \varepsilon n \bar{g}_{\alpha\beta}, \tag{2.3}$$

$$D_i D_j V = V(R_{ij} - \varepsilon n g_{ij}), \tag{2.4}$$

$$\Delta V = -\varepsilon n V, \tag{2.5}$$

$$R \equiv g^{ij} R_{ij} = \varepsilon n(n-1). \tag{2.6}$$

By an abuse of terminology, triples (M, g, V) satisfying the above will also be called solutions of the static vacuum Einstein equations.

We note that there cannot be a static solution (M, g, V) with $\epsilon = -1$ on a compact manifold without boundary, as then (2.5) and the maximum principle imply $V \equiv 0$, contradicting the definition of staticity.

III. THE DIVERGENCE IDENTITY

As in [7], the key to our analysis is an identity which has been used by Shen [2] in dimension three in a context related to ours, and by Wang [1] in all dimensions $n \geq 3$ to prove uniqueness of anti–de Sitter spacetime. For the convenience of the reader we rederive this identity here.

We define the symmetric tensor field W on M by[2]

$$W_{ij} \equiv R_{ij} - \varepsilon(n-1) g_{ij} = V^{-1} D_i D_j V + \varepsilon g_{ij}, \tag{3.1}$$

where the last equality, which follows from (2.4), holds on the set where V does not vanish. Now, we have

$$\begin{aligned} V|W|_g^2 &= V W_{ij} g^{ik} g^{jl} W_{kl} \\ &= W_{ij} D^i D^j V + \varepsilon V W_{ij} g^{ij} \\ &= W_{ij} D^i D^j V + \varepsilon V(R - \varepsilon n(n-1)) \\ &= W_{ij} D^i D^j V, \end{aligned} \tag{3.2}$$

where in the last step we used (2.6) and where $|W|_g^2 = \langle W, W \rangle_g$ is the squared norm of W. Thus

$$V|W|_g^2 = W_{ij} D^i D^j V. \tag{3.3}$$

Strictly speaking, the calculation (3.2) is valid only on the region where V has no zeros. But in (3.3) both sides are smooth everywhere. Furthermore, it is well known that the set where V does not vanish is dense. This implies that (3.3) is true throughout M, regardless of zeros or sign of V.

Since (M, g) is Riemannian, $|W|_g^2$ is non-negative and equal to zero if and only if $W \equiv 0$.

Equation (3.3) implies

$$\begin{aligned} V|W|_g^2 &= W_{ij} D^i D^j V \\ &= D^i (W_{ij} D^j V) - (D^i W_{ij})(D^j V) \\ &= D^i (W_{ij} D^j V), \end{aligned} \tag{3.4}$$

since

$$\begin{aligned} D^i W_{ij} &= D^i R_{ij} - \varepsilon(n-1) D^i g_{ij} \\ &= D^i R_{ij} = \frac{1}{2} D_j R \\ &= 0, \end{aligned} \tag{3.5}$$

where we used again (2.6). We can integrate over M with the measure $\mathrm{d}\mu_g = \sqrt{\det(g)}\, \mathrm{d}^n x$ to obtain

$$\begin{aligned} \int_M V|W|_g^2 \mathrm{d}\mu_g &= \int_M D^i (W_{ij} D^j V) \mathrm{d}\mu_g \\ &= \int_{\partial M} W_{ij} D^i V N^j \mathrm{d}\sigma, \end{aligned} \tag{3.6}$$

where we applied Stokes' theorem with ∂M the boundary of M, $\mathrm{d}\sigma$ the measure on ∂M and N the unit outer directed normal vector field of ∂M.

Now, let us suppose that V is positive on the interior of M and that V vanishes on its boundary

$$H \equiv \{p \in M; V(p) = 0\},$$

with H not necessarily connected. We further assume that M has a conformal boundary at infinity ∂M_∞ (which we always assume to be compact, but not necessarily connected), and write

$$\partial M = \partial M_\infty \cup H. \tag{3.7}$$

We have

$$\begin{aligned} \int_M V|W|_g^2 \mathrm{d}\mu_g = \int_{\partial M_\infty} & W_{ij} D^i V N^j \mathrm{d}\sigma \\ &+ \int_H W_{ij} D^i V N^j \mathrm{d}\sigma_H, \end{aligned} \tag{3.8}$$

where we assumed that $M \cup \partial M_\infty$ is compact, and where we denote by $d\sigma_H$ the measure induced by g on H. By an abuse of notation, we continue to use the symbol $\int \cdot d\sigma$ for the integral on the boundary at infinity of M, which should of course be understood by a limiting process.

IV. THE INTEGRAL OVER THE HORIZON

The integral over the horizon H in (3.8) has been rewritten in a convenient form in [7]. We rederive the

[2] In his paper Wang denotes his tensor by T but we use W in order to avoid confusion with the stress-energy tensor.

formula for completeness. Recall that the surface gravity $\kappa = \sqrt{g(DV,DV)}|_H$ of H is a nonzero constant on each connected component H_p of $H = \cup_{p=1}^{P} H_p$, for some $P \in \mathbb{N}$. Thus there exists a locally constant function $\kappa: H \to \mathbb{R}^{+*} := \mathbb{R}^{+}\backslash\{0\}$ such that on H

$$|DV|_g = \kappa. \tag{4.1}$$

Then, as N is the outer normal to H, on each connected component H_p we have

$$N = -\frac{DV}{|DV|_g} = -\frac{DV}{\kappa_p}, \tag{4.2}$$

where $\kappa_p \in \mathbb{R}^{+*}$ is the value of κ on H_p and the minus sign comes from the fact that V decreases approaching H as $V \geq 0$ on M and $V = 0$ on H. Thus

$$\begin{aligned}\int_H & W_{ij} D^i V N^j \mathrm{d}\sigma_H \\ &= -\sum_{H_p} \frac{1}{\kappa_p} \int_{H_p} W_{ij} D^i V D^j V \mathrm{d}\sigma_H \\ &= -\sum_{H_p} \frac{1}{\kappa_p} \int_{H_p} (R_{ij} - \varepsilon(n-1) g_{ij}) D^i V D^j V \mathrm{d}\sigma_H \\ &= -\sum_{H_p} \frac{1}{\kappa_p} \int_{H_p} (R_{ij} D^i V D^j V - \varepsilon(n-1)\kappa_p^2) \mathrm{d}\sigma_H. \end{aligned} \tag{4.3}$$

We denote by g_H the metric induced by g on H. Letting R_H denote the Ricci scalar of the metric g_H, we will need the Gauss embedding equation

$$R_H = R - 2g(N,N) R_{ij} N^i N^j + g(N,N)((h^{ij} A_{ij})^2 - |A|_h^2), \tag{4.4}$$

where A is the extrinsic curvature tensor of H in M, defined for two vector fields X, Y tangent to H as

$$A(X,Y) = g(D_X N, Y). \tag{4.5}$$

It is well known that H is totally geodesic, i.e., $A \equiv 0$, which can be seen as follows:

$$\begin{aligned} A(X,Y) &= -\frac{1}{\kappa} g(D_X DV, Y) \\ &= -\frac{1}{\kappa} g_{ij} X^k D_k D^i V Y^j \\ &= -\frac{1}{\kappa} X^k Y^j D_k D_j V \\ &= -\frac{1}{\kappa} X^k Y^j V (R_{kj} - \varepsilon n g_{kj}) = 0, \end{aligned} \tag{4.6}$$

since V is zero on H. Thus $A = 0$ and (4.4) becomes, using $g(N,N) = 1$ and $N = -DV/\kappa$,

$$R_H = R - \frac{2}{\kappa^2} R_{ij} D^i V D^j V. \tag{4.7}$$

We can now rewrite (4.3) as

$$\begin{aligned}\int_H & W_{ij} D^i V N^j \mathrm{d}\sigma_H \\ &= -\sum_{H_p} \frac{1}{\kappa_p} \int_{H_p} \left(\frac{\kappa_p^2}{2} (R - R_H) - \varepsilon(n-1)\kappa_p^2 \right) \mathrm{d}\sigma_H \\ &= \sum_{H_p} \frac{\kappa_p}{2} \int_{H_p} (R_H - R + 2\varepsilon(n-1)) \mathrm{d}\sigma_H \\ &= \sum_{H_p} \frac{\kappa_p}{2} \int_{H_p} (R_H - \varepsilon n(n-1) + 2\varepsilon(n-1)) \mathrm{d}\sigma_H \\ &= \sum_{H_p} \frac{\kappa_p}{2} \int_{H_p} (R_H - \varepsilon(n-1)(n-2)) \mathrm{d}\sigma_H. \end{aligned} \tag{4.8}$$

Using this result we obtain the key identity

$$\begin{aligned}\int_M & V|W|_g^2 \mathrm{d}\mu_g \\ &= \int_{\partial M_\infty} W_{ij} D^i V N^j \mathrm{d}\sigma \\ &\quad + \sum_{H_p} \frac{\kappa_p}{2} \int_{H_p} (R_H - \varepsilon(n-1)(n-2)) \mathrm{d}\sigma_H. \end{aligned} \tag{4.9}$$

V. THE BOUNDARY TERM AT INFINITY

To avoid ambiguities, we emphasise that in this section $\varepsilon = -1$. A region $M_{\mathrm{ext}} = (0, x_0] \times Q \subset M$, where Q is a compact $(n-1)$-dimensional manifold, is called an *asymptotically locally hyperbolic* (ALH) end if the sectional curvatures of g approach a (negative) constant as x tends to zero, where x is the coordinate along the first factor of M_{ext}, and if the metric $x^2 g$ extends smoothly to a Riemannian metric on $[0, x_0] \times Q$. (Assuming the last property, the sectional curvatures condition is equivalent to the requirement that $|dx|_{x^2 g}$ (i.e., the norm of dx in the metric $x^2 g$) tends to one as the "conformal boundary at infinity" $\{x = 0\}$ is approached.)

A Riemannian manifold (M, g) will be called ALH if it is complete and contains a finite number of ALH ends. The boundary at infinity ∂M_∞ of M will thus be the union of a finite number of manifolds Q as above.

A Hamiltonian analysis of general relativity leads, after many integrations by parts, to the following formula for the mass of an ALH end [9][3] (compare [10])

[3]We note a misprint in [[9], Eq. (4.40)], where a prefactor 1/16 should be replaced by 1/8.

$$m = -\frac{1}{8(n-2)\pi}\lim_{x\to 0}\int_{x=\mathrm{const}}\left(R^i{}_j - \frac{R}{n}\delta^i_j\right)\nabla^j V N_i d\sigma, \quad (5.1)$$

where the multiplicative prefactor in front of the integral arises from the Hilbert Lagrangian $\bar{R}/(16\pi)$, as relevant for the physical spacetime dimension $n+1=4$. Here N^j is the unit normal to the level sets of x such that $N^x < 0$. It follows that the integral over the conformal boundary at infinity in (4.9) is related to the total mass m (i.e., the sum of the masses over all ALH ends) of the spacetime as

$$\int_{\partial M_\infty} W_{ij} D^i V N^j \mathrm{d}\sigma = -8(n-2)m\pi, \quad (5.2)$$

and thus we get

$$\int_M V|W|_g^2 \mathrm{d}\mu_g = -8(n-2)m\pi + \sum_{H_p}\frac{\kappa_p}{2}\int_{H_p}(R_H + (n-2)(n-1))\mathrm{d}\sigma_H. \quad (5.3)$$

Recall, an ALH static triple (M,g,V) is *strictly static* if V is positive on M. In the strictly static, conformally compact and boundaryless case we obtain

$$\int_M V|W|_g^2 \mathrm{d}\mu_g = -8(n-2)m\pi. \quad (5.4)$$

Since the left-hand side is non-negative, we recover a result of [7]:

Theorem 5.1: Consider a strictly static solution of the static Einstein equations (M,g,V) with negative cosmological constant on a conformally compact manifold without boundary. Then the total mass is negative or zero, vanishing if and only if (M,g) is the hyperbolic space.

Proof.—The only thing that remains to be justified is that the vanishing of the mass implies hyperbolic space, this proceeds as follows: When the mass vanishes, (5.4) shows that the metric is Einstein. Thus the Hessian of V is proportional to the metric, which is the well studied Obata's equation. It follows e.g., from [[11] Theorem 2] that all complete metrics for which V does not change sign and DV has no zeros are *not* compactifiable (compare [[12] Proposition 4.1], see also Remark 6.3 below). We conclude that, under the current assumptions, DV must have a zero, which leads to hyperbolic space again by [11]. ■

As emphasized in [1], Theorem 5.1 leads to uniqueness of the anti–de Sitter spacetime, which has spherical conformal infinity, and thus non-negative mass by [13]. (Wang refers to [14–16] for positivity results; these last papers contain restrictive hypotheses, which have been meanwhile removed through the work in [13,17,18]. See also [19] for the rigidity case of these positive mass theorems, where spherical conformal infinity is assumed.) Note that examples of metrics, as in the theorem, with negative mass are provided by the Horowitz-Myers metrics. The theorem shows that if any further such solutions exist, they would have to have nonspherical infinity and negative mass.

Related negativity results for the mass, in the spirit of Theorem 6.1, can be found in [7,20].

VI. UNIQUENESS THEOREMS FOR THE CUSPIDAL BIRMINGHAM-KOTTLER METRICS

In this section we continue to assume that $\varepsilon = -1$.

Both the technique of the proof and the argument generalize to cover somewhat more general geometries, which we describe now. The *cuspidal Birmingham-Kottler (BK) metrics* provide a guiding example. By definition, these are the metrics which can be written in the form

$$g = \frac{dr^2}{V^2} + r^2 h, \quad V = r, \quad (6.1)$$

where h is a Ricci-flat metric on a compact $(n-1)$-dimensional manifold Q. One checks that g is Einstein, and has zero mass in the *asymptotically locally hyperbolic (ALH) end*, defined as the region where r tends to infinity. The underlying manifold $(0,\infty)\times Q$ has two asymptotic regions, with the already mentioned ALH end, together with the region $(0,r_0]\times Q$ where r is allowed to approach zero, which is metrically complete, and which will be referred to as a *cuspidal end*.

The example suggests a natural generalization of Wang's argument to manifolds which contain two kinds of asymptotic regions: the usual asymptotically locally hyperbolic ones, as well as ends with mildly controlled asymptotic behavior, as captured by the following definition: We will say that a triple (M,g,V) is *asymptotically locally hyperbolic with mild ends* if (M,g) is a complete manifold which admits an exhaustion $M = \cup_{i\in\mathbb{N}} M_i$ by smooth compact manifolds $M_i \subset M_{i+1}$ with boundaries

$$\partial M_i = \partial_1 M_i \cup \partial_2 M_i$$

where the (not necessarily connected) boundaries $\partial_2 M_i$ are a union of smooth hypersurfaces which approach a (compact) conformal boundary at infinity of M, while the (not necessarily connected) boundaries $\partial_1 M_i$ are a union of smooth hypersurfaces on which

$$|dV|_g|_{\partial_1 M_i} \times |W|_g|_{\partial_1 M_i} \times A(\partial_1 M_i) \to_{i\to\infty} 0, \quad (6.2)$$

where $A(\partial_1 M_i)$ is the "area" of the submanifolds $\partial_1 M_i$. Here $|\cdot|_g$ denotes the norm with respect to the metric g, and we assume that the number of boundary components is bounded by a number independent of i. The "mild ends" are then the regions associated with the boundaries satisfying (6.2).

As formulated so far, the definition allows some ALH ends to be mild ends. This occurs for example for hyperbolic space, where $W \equiv 0$. To avoid this issue, which would lead to the need to add annoying trivial comments when formal statements are made, we add to the definition of a mild end the requirement that a mild end is *not* ALH.

The conditions above are clearly satisfied by the metric (6.1), where both $A(\partial_1 M_i)$ and $|dV|_g|_{\partial_1 M_i}$ tend to zero when $\partial_1 M_i$ is taken to be $\{r = 1/i\}$, $1 \leq i \in \mathbb{N}$, with in fact $|W|_g|_{\partial_1 M_i}$ identically zero. But note that the above definition allows for degenerate black holes, such as extreme Kottler black holes with higher-genus topology, which contain asymptotically cylindrical ends along which both V and $|dV|_g$ tend to zero when receding to infinity along the end, with both the area of the cross sections of the cylindrical end and $|W|_g|_{\partial_1 M_i}$ approaching finite nonzero limits.

We have the following extension of Theorem 5.1:

Theorem 6.1: Consider a strictly static asymptotically locally hyperbolic solution (M, g, V) of the static Einstein equations with at least one mild end. Then the total mass is negative or zero, vanishing if and only if (g, V) is given by (6.1).

Proof.—Applying the divergence identity on M_i and passing with i to infinity one obtains that the sum of the masses of the ALH ends is nonpositive. If the mass vanishes we obtain that g is Einstein, and the result follows from [[11] Theorem 2, case (II.A)], compare the discussion after Theorem 5.1 above. (The cuspidal BK metric corresponds to the pseudohyperbolic space of zero type in the nomenclature of [11]. Note that case (II.A) [[11] Theorem 2] also allows for the pseudohyperbolic space of negative type, but this case does not admit a mild end.) ■

Recall that one of the obstructions, when attempting to prove the positive mass theorem for asymptotically locally hyperbolic manifolds using spinorial methods *à la Witten*, is that of lack of existence of nontrivial spinor fields which asymptote to Killing spinors of the asymptotic background near the conformal boundary at infinity. Such spinor fields will be called *asymptotic Killing spinors*. We shall say that an asymptotically locally hyperbolic spin manifold (M, g) has a *compatible spin structure* if all components of the conformal boundary at infinity admit nontrivial asymptotic Killing spinors.

The BK cuspidal metrics with a flat h provide examples of manifolds with compatible spin structure. Examples which do not have a compatible spin structure are the Kottler black holes with higher genus topology, or the Horowitz-Myers metrics. (Indeed, if they admitted a compatible spin structure, all static AH metrics on these manifolds would have positive mass, but some of the metrics do not.)

Theorem 6.1 leads to the following uniqueness theorem for the BK cuspidal metrics, seemingly unnoticed in the literature so far. To avoid ambiguities: we assume here and below that M has no boundary.

Theorem 6.2: Let (M, g, V) be a strictly static solution of the vacuum Einstein equations which is the union of a finite number of mild ends (at least one), a finite number of ALH ends (at least one), and of a compact set. If (M, g) carries a compatible spin structure, then (M, g, V) is the cuspidal BK metric (6.1).

Proof.—Choose any of the ALH ends of (M, g). One can run the generalization of Witten's proof of the positive energy theorem as in [14,16], using spinor fields which asymptote to a nontrivial Killing spinor in the chosen end and to zero on all other ends (if any), to conclude that the mass of each ALH end is positive or vanishes. The result follows by Theorem 6.1. ■

Remark 6.3: An example, not covered by the analysis so far, of a static but *not strictly static* ALH metric with zero mass is the "hyperbolic Einstein-Rosen bridge,"

$$g = dr^2 + \cosh^2(r) h, \qquad V = \sinh(r), \qquad (6.3)$$

with $r \in \mathbb{R}$, where h is a negatively curved Einstein metric on a compact manifold. In this case (M, g, V) is conformally compactifiable, with two ALH ends, and note that V changes sign. Changing the coordinate $r > 0$ to $R = \cosh(r)$ one recognizes the space-part of the zero-mass Kottler black hole with a negatively-curved h,

$$g = \frac{dR^2}{R^2 - 1} + R^2 h, \qquad V = \sqrt{R^2 - 1}. \qquad (6.4)$$

We are not aware of a positive-energy theorem which would hold for this topology, and which could lead directly to a uniqueness theorem for this metric. See, however, Theorem 7.1 below, and [21] for a uniqueness result for the metric (6.3) within the class of Einstein metrics. ■

A completely different uniqueness theorem for the cuspidal BK metrics, without spin assumptions, has been recently proved by the second author and H.C. Jang [12]. The results there are motivated by the fact that the level sets of the coordinate r of (6.1) have mean curvature $H = n - 1$, so that one can cut the manifold along any such set to obtain a conformally compactifiable manifold with boundary satisfying $H = n - 1$. To obtain a result like Theorem 6.2, but without spin assumption, we will make use of the following slight refinement of part 3 (the toroidal case) of Theorem 1.1 in [22].

Theorem 6.4: Let (M, g) be an n-dimensional, $4 \leq n \leq 7$, asymptotically locally hyperbolic manifold with flat toroidal conformal infinity $(Q, \mathring{h})$, such that M is diffeomorphic to $[r_0, \infty) \times Q$. Suppose that:

(1) The boundary $Q_0 = \{r_0\} \times Q$ has mean curvature $H \le n-1$, where H is the divergence $D_i N^i$ of the unit normal N^i pointing into M.
(2) The scalar curvature R of M satisfies $R \ge -n(n-1)$.

Then (M, g) has non-negative mass, $m \ge 0$.

Comment on the proof. The only difference in this version is that the condition $H < n-1$ in [[22] Theorem 1.1, part 3] has been replaced by the condition $H \le n-1$. To explain this weakening, we indicate briefly how the proof goes. Suppose by contradiction the mass is negative. Then, as in the proof of [[22] Theorem 1.1, part 3] there exists a compact hypersurface Q_1 out near infinity, homologous to Q_0, with mean curvature $H_1 > n-1$. Then, with respect to the initial data set $(M, g, K = -g)$, Q_0 has null expansion $\theta_0 \le 0$ (and < 0 if $H < n-1$), and Q_1 has null expansion $\theta_1 > 0$, both with respect to the null normal fields pointing toward the ALH end. In the case $\theta_0 < 0$, the basic existence result for marginally outer trapped surfaces (MOTS) (see e.g., [[23] Theorem 3.3]) guarantees the existence of an outermost MOTS in the region between Q_0 and Q_1. However, as was carefully shown in Theorem 5.1 in [24], the assumption $\theta_1 < 0$ can be weakened to $\theta_1 \le 0$. The only difference is that the outermost MOTS Σ, whose existence is guaranteed by this theorem, may have some component that agrees with Q_0. Now, as discussed in the proof of [[22] Theorem 1.1, part 3], Σ (or some component of Σ) cannot carry a metric of positive scalar curvature. But then Theorem 3.1 in [25] implies that Σ cannot be outermost. Hence the mass must be non-negative.

We further remark, as was similarly noted in [22], the condition that (Q, h) is a flat torus can be replaced by the somewhat more general condition that $(Q, \mathring{h})$ is a compact flat manifold, provided the product assumption in Theorem 6.4 extends to the conformal boundary. This follows from a covering space argument, using the fact that any compact flat manifold is finitely covered by a flat torus.

Using Theorem 6.4, one can now argue in a manner similar to the proof of Theorem 6.2 to obtain the following:

Theorem 6.5: Let (M, g, V) be a strictly static solution of the vacuum Einstein equations diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$, $4 \le n \le 7$, where $\mathbb{T}^{n-1}$ is a torus, with one ALH end with flat toroidal conformal infinity. If there exists $r_0 \in \mathbb{R}$ such that $\mathbb{T}^{n-1} \times \{r_0\}$ has mean curvature satisfying $H \le n-1$ with respect to the normal pointing toward the ALH end, then (M, g, V) is the BK cuspidal solution $((0,\infty) \times \mathbb{T}^{n-1}, r^{-2}dr^2 + r^2 h)$.

The general BK rigidity result considered in recent work of L.-H. Huang and H.C. Jang [8] (see Remark 6.8 below) involves ALH manifolds with compact boundary, and without "internal" cuspidal ends. In a similar vein, we consider below a uniqueness result for the BK cuspidal spaces in the context of static vacuum ALH manifolds with boundary. This result makes use of certain properties of constant mean curvature (CMC) hypersurfaces in Riemannian manifolds, which we now describe; cf., e.g., [26].

Let Σ be a two-sided compact hypersurface in a Riemannian manifold (M, g) of dimension n. Hence, Σ admits a smooth unit normal field N. Consider a normal variation $t \to \Sigma_t$ of $\Sigma = \Sigma_0$, i.e., a variation with variation vector field $V = \frac{\partial}{\partial t}|_{t=0} = \phi N$. $\phi \in C^\infty(\Sigma)$. Let $\mathcal{B}(t) = \mathcal{A}(t) - (n-1)\mathcal{V}(t)$, where $\mathcal{A}(t)$ is the area of Σ_t and $\mathcal{V}(t)$ is the (signed) volume of the region bounded by Σ_t and Σ. Then a computation shows that Σ has mean curvature $H = n-1$ if and only if $\mathcal{B}'(0) = 0$ for all normal variations $t \to \Sigma_t$. We say that Σ is a *stable* CMC hypersurface, with mean curvature $H = n-1$, provided $\mathcal{B}''(0) \ge 0$ for all normal variations $t \to \Sigma_t$. Consider the operator $L: C^\infty(\Sigma) \to C^\infty(\Sigma)$, defined by

$$L(\phi) = -\Delta\phi + \frac{1}{2}(R_\Sigma - R - |A|^2 - H^2)\phi \qquad (6.5)$$

where, as before, A is the second fundamental form of Σ. It is a well known fact that Σ is stable if and only if the principal eigenvalue of L is non-negative, $\lambda_1(L) \ge 0$. We will take this analytic characterization as our definition of stability. Using this characterization, the following was proved in [15].

Lemma 6.6: Let (M, g) be an n-dimensional Riemannian manifold with scalar curvature R satisfying, $R \ge -n(n-1)$. Let Σ be a compact 2-sided stable CMC hypersurface in M, with mean curvature $H = n-1$. Suppose Σ does not carry a metric of positive scalar curvature. Then the following holds.
(i) Σ is umbilic, in fact $A = h$, where h is the induced metric on Σ.
(ii) Σ is Ricci flat and $R = -n(n-1)$ along Σ.

We now consider the following uniqueness result for the cuspidal BK space.

Theorem 6.7: Let (M, g) be a strictly static ALH manifold with compact boundary Σ, and with static potential V, such that (M, g, V) satisfies the static vacuum Einstein equations. Suppose that:
(1) Σ is a stable CMC hypersurface with mean curvature $H = n-1$ (with respect to the inward pointing unit normal N).
(2) Σ does not carry a metric of positive scalar curvature.
(3) Conditions hold which imply that (M, g) has non-negative mass, $m \ge 0$ (compare the positivity results in [16] for manifolds with compatible spin structure, or Theorem 6.4 above.)

Then $m = 0$ and (M, g) is Einstein, $R_{ij} = -(n-1)g_{ij}$. Furthermore, if Σ is a regular level set of the static potential, $\Sigma = \{V = V_0\}$, then (M, g) is isometric to the (truncated) cuspidal BK space $([r_0, \infty) \times \Sigma, r^{-2}dr^2 + r^2 h)$.

Proof.—Using the Gauss equation (4.4), together with Lemma 6.6, one easily computes,

$$R_{ij}N^i N^j = -(n-1). \qquad (6.6)$$

Furthermore, by the Codazzi equation and part (i) of Lemma 6.6, one has

$$R_{ij}X^iN^j = 0 \quad \text{for all } X \text{ tangent to } \Sigma. \tag{6.7}$$

Now, applying (3.8), we obtain

$$\begin{aligned}\int_M V|W|^2 d\mu &= \int_{\partial M_\infty} W_{ij}D^iVN^j d\sigma + \int_\Sigma W_{ij}D^iVN^j d\sigma_\Sigma \\ &= -8(n-2)m\pi + \int_\Sigma W_{ij}D^iVN^j d\sigma_\Sigma;\end{aligned} \tag{6.8}$$

recall that $W_{ij} = R_{ij} + (n-1)g_{ij}$.

Along Σ we can write DV as,

$$DV = \lambda N + X \tag{6.9}$$

where X is tangent to Σ. Hence, along Σ, we have

$$\begin{aligned}W_{ij}D^iVN^j &= \lambda W_{ij}N^iN^j + W_{ij}X^iN^j = \lambda W_{ij}N^iN^j \\ &= \lambda(R_{ij}N^iN^j + (n-1)g_{ij}N^iN^j) \\ &= \lambda(-(n-1) + (n-1)) = 0.\end{aligned}$$

Thus,

$$\int_M V|W|^2 d\mu = -8(n-2)m\pi. \tag{6.10}$$

and hence $m \le 0$. Assumption 3 in Theorem 6.7 then implies $m = 0$, and hence by (6.10), $R_{ij} = -(n-1)g_{ij}$.

We now assume Σ is the level set, $\Sigma = \{V = V_0\}$, and show that (M, g) is isometric to the cuspidal BK space, as in the statement of the theorem. We have,

$$D_iD_jV = V(R_{ij} + ng_{ij}) = Vg_{ij}. \tag{6.11}$$

Hence, in view of Proposition 4.2 in [12], it suffices to show that, along Σ, $|DV| = V_0$.

Along Σ, we have $DV = \lambda N$, where $\lambda = \pm|DV|$. Let X be any unit tangent vector to Σ. From (6.11), $D_iD_jVX^iX^j = V_0$. On the other hand, using the definition of the Hessian,

$$\begin{aligned}D_iD_jVX^iX^j &= g_{ij}X^kD_kD^iVX^j = g_{ij}X^kD_k(\lambda N^i)X^j \\ &= \lambda g_{ij}X^kD_kN^iX^j + (X^kD_k\lambda)g_{ij}N^iX^j \\ &= \lambda A_{ij}X^iX^j = \lambda h_{ij}X^iX^j = \lambda,\end{aligned}$$

where in the last line we have used part (i) of Lemma 6.6. Thus $\lambda = V_0 > 0$, and hence $|DV| = V_0$ along Σ. ■

Remark 6.8: Theorem 4.1 in [12] shows that under a strengthening of the stability assumption, the assumption in Theorem 6.7 that the boundary is a level set of the static potential, used to conclude that (M, g) is isometric to the cuspidal BK space, can be removed. This strengthened assumption ("locally weakly outermost") is used in forthcoming work of Lan-Hsuan Huang and Hyun Chul Jang [8], in which they establish the uniqueness of the cuspidal BK space in a more general (not necessarily static) ALH setting with boundary, assuming the mass vanishes. We thank them for communications regarding their work.

VII. LOWER BOUND FOR ENTROPY (AREA) WHEN $m \geq 0$

We continue to assume that $\Lambda < 0$ but now we consider solutions with a horizon H. If $m \geq 0$ (which holds if the conformal boundary at infinity is a sphere by the Riemannian asymptotically hyperbolic positive mass theorem [13]) we have

$$\begin{aligned}&\sum_{H_p}\kappa_p \int_{H_p}(R_H + (n-2)(n-1))\mathrm{d}\sigma_H \\ &= 2\int_M V|W|_g^2 \mathrm{d}\mu_g + 16(n-2)m\pi \geq 0,\end{aligned} \tag{7.1}$$

and the inequality is saturated if and only if $m = 0$ and $W = 0$. This last condition is equivalent to, since $\varepsilon = -1$,

$$R_{ij} = -(n-1)g_{ij}, \tag{7.2}$$

so (M, g) is an Einstein manifold (the Ricci tensor is proportional to the metric).

The identity (7.1) leads to the following uniqueness result for the metric (6.3) on $\{r \geq 0\}$, which has $m = 0$ and $R_H \equiv -(n-2)(n-1)$:

Theorem 7.1: Consider a static solution (M, g, V) of the vacuum Einstein equations on a manifold with boundary, with V positive on the interior of M and vanishing at ∂M. Suppose that M is the union of a finite number of ALH ends (at least one), a finite number of mild ends, and a compact set. If

$$m \geq 0 \quad \text{and} \quad R_H \leq -(n-2)(n-1) \tag{7.3}$$

then M is diffeomorphic to $[0, \infty) \times \partial M$ with the metric (6.3).

Proof.—Under the current hypotheses the left-hand side of (7.1) is smaller than or equal to zero, while the right-hand side is larger than or equal to zero. As already pointed-out the metric must be Einstein, and the result follows from case (II.A) of [[11] Theorem 2]. ■

If the manifold M is 3-dimensional, $n = 3$, the identity (7.1) implies

$$\sum_{H_p}\kappa_p \int_{H_p}(R_H + 2)\mathrm{d}\sigma_H \geq 0, \tag{7.4}$$

that is to say

$$2\sum_{H_p}\kappa_p A_p \geq -\sum_{H_p}\kappa_p \int_{H_p} R_H \mathrm{d}\sigma_H, \tag{7.5}$$

where A_p is the area of the connected component H_p of the boundary H.

Assuming that the H_p's are all closed and orientable, the Gauss-Bonnet theorem,

$$\int_{H_p} R_H \mathrm{d}\sigma_H = 8\pi(1-g_p), \tag{7.6}$$

where g_p is the genus of H_p, gives the inequality

$$\sum_{H_p}\kappa_p A_p \geq 4\pi\sum_{H_p}\kappa_p(g_p-1). \tag{7.7}$$

with equality holding if and only if $m=0$, $A_p = 4\pi(g_p-1)$, with M diffeomorphic to $[0,\infty)\times\partial M$ and g given by (6.3).

In order to express (7.7) in terms of the cosmological constant Λ we have to use the fact that according to (2.2) for $n=3$ we set $\Lambda=-3$, so in this case we have

$$\sum_{H_p}\kappa_p A_p \geq \frac{12\pi}{|\Lambda|}\sum_{H_p}\kappa_p(g_p-1). \tag{7.8}$$

Note that the cosmological constant Λ has the dimension of inverse length squared so this is the only way to reintroduce it while preserving the homogeneity of the dimensions.

In the case where H is connected this reads

$$A_H \geq \frac{12\pi(g_H-1)}{|\Lambda|}, \tag{7.9}$$

with A_H the area of H and g_H its genus. This can be compared to a weaker inequality of Gibbons [[27] Eq. (45)], where time symmetry but no staticity is assumed (see also Woolgar [28]):

$$A_H \geq \frac{4\pi(g_H-1)}{|\Lambda|}. \tag{7.10}$$

VIII. AN UPPER BOUND FOR THE FREE ENERGY

Let $k\in\mathbb{R}$. In [3] one defines

$$F_k = E - kTS, \tag{8.1}$$

where E is the total mass, T is the Hawking temperature of a Killing horizon, and S its entropy [29]

$$E = m, \qquad T = \frac{\kappa}{2\pi}, \qquad S = \frac{A}{4}. \tag{8.2}$$

The functional F_k equals the total mass when $k=0$ and the "free energy" when $k=1$.

From (7.1) we have

$$\sum_{H_p}\kappa_p \int_{H_p}(R_H + (n-2)(n-1))\mathrm{d}\sigma_H = 2\int_M V|W|_g^2 \mathrm{d}\mu_g + 16(n-2)m\pi, \tag{8.3}$$

which in dimension $n=3$ and for a single Killing horizon H reads

$$2\pi T\left(\int_H R_H \mathrm{d}\sigma_H + 2A\right) - 16\pi m \geq 0. \tag{8.4}$$

If the Killing horizon is a torus or an orientable manifold of higher genus g_H (not to be confused with the metric g_H on the horizon…) we obtain

$$E - TS \leq \frac{1}{2}\kappa(1-g_H) \leq 0. \tag{8.5}$$

Using the notation of (8.1), we conclude that the free energy $F\equiv F_1$ of static solutions containing a connected horizon of higher genus is negative:

$$F_1 \leq 0. \tag{8.6}$$

This gives a simple proof of the inequality [[3] Eq. (24)] for a connected horizon with $g_H \geq 1$.

In the current context it is instructive to consider the Birmingham-Kottler metrics (A1) of the Appendix below. For these metrics we find, in space-dimension n and using the normalization $\ell = 1$,

$$\begin{aligned}\kappa &= |DV|_g|_{V=0} = \frac{1}{2}(V^2)'|_{V=0} = \left(r + \frac{(n-2)\mu}{r^{n-1}}\right)\Big|_{r^2=\frac{2\mu}{r^{n-2}}-k} \\ &= \left(\frac{n\mu}{r^{n-1}} - \frac{k}{r}\right)\Big|_{V=0}, \\ 8\pi TS &= \kappa A_H = \kappa r^{n-1}|_{V=0}A_\infty = (n\mu - kr^{n-2})|_{V=0}A_\infty, \\ 8\pi E &= 8\pi m = (n-1)A_\infty\mu.\end{aligned}$$

The toroidal case $k=0$ leads to

$$\frac{E}{TS} = \frac{n-1}{n} \Leftrightarrow E - \frac{n-1}{n}TS = 0 \Leftrightarrow F_{\frac{n-1}{n}} = 0, \tag{8.7}$$

which suggests that in this case and in dimension $n=3$ the sharp inequality is

$$F_{\frac{2}{3}} \leq 0.$$

In the case $k=-1$, for solutions with $\mu > 0$ we get

$$\frac{E}{TS} = \frac{(n-1)\mu}{(n\mu + r^{n-2})|_{V=0}} \leq \frac{n-1}{n}, \tag{8.8}$$

(as $n \geq 2$ and $r > 0$) which is equivalent to

$$E - \frac{n-1}{n} TS \leq 0 \Leftrightarrow F_{\frac{n-1}{n}} \leq 0. \quad (8.9)$$

When $k = -1$, $n = 3$, and $\mu < 0$ belongs to the black-hole range $\mu > -(3\sqrt{3})^{-1}$, the ratio E/TS varies from $-\infty$ to zero which does not seem to lead to any new information, but note that (8.9) remains true trivially since $E \leq 0$ in this case.

IX. POSITIVE COSMOLOGICAL CONSTANT

In this section we consider a positive cosmological constant so $\Lambda > 0$ and $\varepsilon = 1$ in (2.2). In this case there is no conformal boundary at infinity, and the models (M, g, V) of interest are compact manifolds with a boundary H on which V vanishes. Thus (4.9) becomes

$$\int_M V|W|_g^2 \mathrm{d}\mu_g = \sum_{H_p} \frac{\kappa_p}{2} \int_{H_p} (R_H - (n-1)(n-2)) \mathrm{d}\sigma_H \geq 0, \quad (9.1)$$

and the inequality is saturated with a nontrivial V if and only if $W = 0$ on M, that is to say if and only if

$$R_{ij} = (n-1) g_{ij}. \quad (9.2)$$

If M is 3-dimensional, $n = 3$, from (9.1) we obtain

$$\sum_{H_p} \kappa_p A_p \leq \frac{1}{2} \sum_{H_p} \kappa_p \int_{H_p} R_H \mathrm{d}\sigma_H. \quad (9.3)$$

Assuming the H_p's are all closed and orientable and applying the Gauss-Bonnet theorem we are led to

$$\sum_{H_p} \kappa_p A_p \leq 4\pi \sum_{H_p} \kappa_p (1 - g_p). \quad (9.4)$$

Reintroducing the cosmological constant ($\Lambda = +3$ so far) yields

$$\sum_{H_p} \kappa_p A_p \leq \frac{12\pi}{\Lambda} \sum_{H_p} \kappa_p (1 - g_p). \quad (9.5)$$

If H is connected then this inequality becomes

$$A_H \leq \frac{12\pi}{\Lambda} (1 - g_H), \quad (9.6)$$

and since $A_H > 0$ and $g_H \in \mathbb{N}$ we must have

$$g_H = 0, \quad (9.7)$$

so the genus of H is zero, which means that H is homeomorphic to a two-sphere, and we get

$$A_H \leq \frac{12\pi}{\Lambda}. \quad (9.8)$$

If one sets $\Lambda = 3$ through a constant conformal rescaling then the result becomes

$$A_H \leq 4\pi. \quad (9.9)$$

The above[4] gives a simple proof of a result already obtained by Boucher-Gibbons-Horowitz [[5] Eq. (3.1)] (compare Ambrozio [[4] Theorem 2] and Borghini-Mazzieri [[6] Corollary 2.6]):

Theorem 9.1: Let (M, g, V) be a 3-dimensional compact static solution to the vacuum Einstein equation with positive cosmological constant Λ. If the boundary H of M is connected, closed, and orientable then H is homeomorphic to a two-sphere and its area A_H satisfies

$$A_H \leq \frac{12\pi}{\Lambda}, \quad (9.10)$$

with equality holding only in de Sitter space.

We note that the equality case is handled as before by an analysis of Obata's equation, and that the condition of orientability can be removed by passing to a finite covering of M, in which the areas of each horizon will be larger than or equal to the original ones.

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APPENDIX: THE MASS OF THE BIRMINGHAM-KOTTLER METRICS

Consider a Birmingham-Kottler (BK) metric [30,31] in space-time dimension $d \equiv n + 1$:

$$g = -V^2 dt^2 + \frac{dr^2}{V^2} + r^2 \underbrace{h_{AB} dx^A dx^B}_{=: h_k}, \qquad V^2 = \frac{r^2}{\ell^2} + k - \frac{2\mu}{r^{n-2}}, \quad (A1)$$

with $\mu \in \mathbb{R}$, where ℓ is related to the cosmological constant Λ as

$$\Lambda = -\frac{n(n-1)}{2\ell^2},$$

and where h_k is an Einstein metric with scalar curvature $(n-1)(n-2)k$, with $k \in \{\pm 1, 0\}$. When $k \neq 0$ the

[4]Once this paper was written we realized that Theorem 9.1, as well as the proof presented here, can already be found in [2].

coordinates above are uniquely defined except when h_k is the unit round metric on a sphere S^{n-1}, in which case the coordinates are defined up to a conformal transformation of the sphere. On the other hand the case $k=0$ allows for the rescaling

$$(r,t)\mapsto(\bar r=\lambda r,\bar t=\lambda^{-1}t), \tag{A2}$$

where λ is a positive constant, in which case (A1) becomes

$$g=-\bar V^2 d\bar t^2+\frac{d\bar r^2}{\bar V^2}+\bar r^2\underbrace{\bar h_{AB}dx^A dx^B}_{=:\bar h_k}, \tag{A3}$$

$$\bar V^2=\frac{\bar r^2}{\ell^2}-\frac{2\bar\mu}{\bar r^{n-2}}, \tag{A4}$$

with

$$\bar h_k=\lambda^{-2}h_k,\qquad \bar\mu=\lambda^n\mu,\qquad \text{and note that}\quad \bar V=\lambda V. \tag{A5}$$

If we take the point of view that only the conformal class of h_k matters as far as the asymptotic data are concerned, we conclude that, when $k=0$, the only information carried by the number μ is its vanishing or its sign.

Now, we see from (A5) that

$$\bar V\times\bar\mu\times\sqrt{\det\bar h_k}=V\times\mu\times\sqrt{\det h_k} \tag{A6}$$

so that the product

$$V\times\mu\times\sqrt{\det h_k}, \tag{A7}$$

which has the same scaling as the integrand of the formula (5.1) defining the Hamiltonian energy, is invariant under (A2).

In conclusion, (1) μ by itself is *not* the Hamiltonian mass, and (2) the inclusion of V and of the area factor associated with the metric h_k in the integrand are essential for an invariant definition of mass.

1. The space Ricci tensor for Birmingham-Kottler metrics

We wish to calculate the boundary integral at infinity which arises when integrating the divergence identity (3.4) for a BK metric. For this we need the space-part of its Ricci tensor. The simplest way to calculate this tensor is to use the static KID equation (2.4) written backwards

$$R_{ij}=V^{-1}D_iD_jV+\frac{2\Lambda}{n-1}g_{ij}. \tag{A8}$$

One readily finds, in the scaling where $2\Lambda=-n(n-1)$, in an ON frame with $\theta^{\hat 1}=V^{-1}dr$, and where $\theta^{\hat A}$ is orthonormal frame for the metric $r^2h_k\equiv r^2h_{AB}dx^Adx^B$,

$$\begin{aligned}R_{\hat 1\hat 1}&=V^{-1}D_{\hat 1}D_{\hat 1}V-ng_{\hat 1\hat 1}\\&=1-n-\frac{(n-2)(n-1)m}{r^n},\end{aligned} \tag{A9}$$

$$\begin{aligned}R_{\hat A\hat B}&=V^{-1}D_{\hat A}D_{\hat B}V-ng_{\hat A\hat B}\\&=\left(1-n+\frac{(n-2)\mu}{r^n}\right)\delta_{\hat A\hat B},\end{aligned} \tag{A10}$$

with the remaining components of the Ricci tensor being zero by symmetry considerations. These formulas readily lead to (compare [10])

$$-\lim_{R\to\infty}\int_{r=R}\nabla^jV\left(R^i{}_j-\frac{R}{n}\delta^i_j\right)N_i d\sigma=(n-1)(n-2)A_\infty\mu, \tag{A11}$$

where A_∞ is the area of the boundary at infinity in the metric h_k.

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