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Photon surfaces with equipotential time slices

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ABSTRACT

Photon surfaces are timelike totally umbilic hypersurfaces of Lorentzian spacetimes. In the first part of this paper, we locally characterize all possible photon surfaces in a class of static spherically symmetric spacetimes that includes (exterior) Schwarzschild, Reissner–Nordström, and Schwarzschild–anti de Sitter in $n+1$ dimensions. In the second part, we prove that any static, vacuum, and "asymptotically isotropic" $n+1$-dimensional spacetime that possesses what we call an "equipotential" and "outward directed" photon surface is isometric to the Schwarzschild spacetime of the same (necessarily positive) mass using a uniqueness result obtained by the first author.

I. INTRODUCTION

One of the cornerstone results in the theory of black holes (in 3 + 1 dimensions) is the static black hole uniqueness theorem first proposed by Israel[1] for a single horizon and later by Bunting and Masood-ul-Alam[2] for multiple horizons, which establishes the uniqueness of the Schwarzschild spacetime among all static asymptotically flat black hole solutions to the vacuum Einstein equations. Refer to the book by Heusler[3] and the review article[4] by Robinson for a (then) complete list of references on further contributions, Simon's spinor proof recently described in the article by Raulot (Ref. 5, Appendix A), and the recent article by Agostiniani and Mazzieri[6] for newer approaches in the case of a single horizon.

A well-known and intriguing feature of (positive mass) Schwarzschild spacetime is the existence of a *photon sphere*, namely, the timelike cylinder P over the $\{r = (nm)^{\frac{1}{n-2}}, t = 0\}$ $n-1$-sphere. P has the property of being *null totally geodesic* in the sense that any null geodesic tangent to P remains in P, i.e., P traps all light rays tangent to it.

In Ref. 7, the first author introduced and studied the notion of a photon sphere for general static spacetimes (see also Ref. 8 in the spherically symmetric case). Based on this study, in Ref. 9 she adapted Israel's argument (which requires the static lapse function to have nonzero gradient) to obtain a *photon sphere uniqueness result*, thereby establishing the uniqueness of the Schwarzschild spacetime among all static asymptotically flat solutions to the vacuum Einstein equations, which admit a single photon sphere. Subsequent to that work, by adapting the argument of Bunting and Masood-ul-Alam,[2] the authors[10] were able to improve this result by particularly avoiding the gradient condition and allowing *a priori* multiple photon spheres. For further results on photon spheres, in particular uniqueness results in the electro-vacuum case and the case of other matter fields, see Refs. 11–20.

In this paper, we will be concerned with the notion of *photon surfaces* in spacetimes (see Refs. 8 and 21 for slightly more general versions of this notion). A photon surface in an $n+1$-dimensional spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is a timelike hypersurface P^n, which is null totally geodesic, as described above. As shown in Refs. 8 and 21, a timelike hypersurface P^n is a photon surface if and only if it is totally umbilic. By definition, a *photon sphere* in a *static spacetime* is a photon surface P^n along which the static lapse function N is constant (see Sec. II for details). While the Schwarzschild spacetime [in dimension $n+1$, $n \geq 3$, and with $r > (2m)^{\frac{1}{n-2}}$, $m > 0$] admits a single photon sphere, it admits infinitely many photon surfaces of various types, as briefly described in Sec. III.

In Sec. III, we derive the relevant ordinary differential equation (ODE) describing spherically symmetric photon surfaces for a class of static spherically symmetric spacetimes, which includes (exterior) Schwarzschild, Reissner–Nordström, and Schwarzschild–AdS (anti–de Sitter). In the generic case, a photon surface in this setting is given by a formula for the derivative of the radius-time-profile curve $r = r(t)$ (see Theorem 3.5). This formula is used in Ref. 22 to give a detailed qualitative description of all spherically symmetric photon surfaces in many (exterior) black hole spacetimes within the class $\mathcal{S}$ (defined in Sec. III), including the (positive mass) Schwarzschild spacetime. In addition, in Sec. III, we present a result that shows, for generic static, isotropic spacetimes, which include positive mass Schwarzschild and sub-extremal Reissner–Nordström, and that, apart from some (partial) timelike hyperplanes, all photon surfaces are isotropic (see Theorem 3.8 and Corollary 3.9). As a consequence, one obtains a complete characterization of all photon surfaces in Reissner–Nordström and Schwarzschild.

Finally, in Sec. IV, we obtain a new rigidity result pertaining to photon surfaces, rather than just to photon spheres. We prove that any static, vacuum, and asymptotically isotropic spacetime possessing an (possibly disconnected) "outward directed" photon surface inner boundary with the property that the static lapse function N is constant on each component of each time slice $\Sigma^{n-1}(t) := P^n \cap \{t = \text{const.}\}$ must necessarily be a Schwarzschild spacetime of positive mass, with the photon surface being one of the spherically symmetric photon surfaces in Schwarzschild classified in Sec. III. We call such photon surfaces *equipotential*. This generalizes static vacuum photon sphere uniqueness to certain photon surfaces and to higher dimensions.

The proof makes use of a new higher dimensional uniqueness result for the Schwarzschild spacetime due to the first author[23] (see Sec. IV for a statement). This result generalizes in various directions the higher dimensional Schwarzschild uniqueness result of Gibbons *et al.*[24] In particular, it does not *a priori* require the spacetime to be vacuum or static. A different proof of the result we use from Ref. 23 has since been given by Raulot,[5] assuming that the manifolds under consideration are spin. These results rely on the rigidity case of a (low regularity version) of the Riemannian positive mass theorem.[25–30]

II. PRELIMINARIES

The static spherically symmetric $(n+1)$-dimensional *Schwarzschild spacetime of mass* $m \in \mathbb{R}$, with $n \geq 3$, is given by $(\overline{\mathfrak{L}}^{n+1} := \mathbb{R} \times (\mathbb{R}^n \backslash \overline{B_{r_m}(0)}), \overline{\mathfrak{g}})$, where the Lorentzian metric $\overline{\mathfrak{g}}$ is given by

$$\overline{\mathfrak{g}} = -\overline{N}^2 dt^2 + \overline{N}^{-2} dr^2 + r^2 \Omega, \quad \overline{N} = \left(1 - \frac{2m}{r^{n-2}}\right)^{1/2}, \tag{2.1}$$

with Ω denoting the standard metric on $\mathbb{S}^{n-1}$, and $r_m := (2m)^{\frac{1}{n-2}}$ for $m > 0$ and $r_m := 0$ for $m \leq 0$ (see also the work of Tangherlini).[31] For $m > 0$, the timelike, cylindrical hypersurface $\overline{P}^n := \{r = (nm)^{\frac{1}{n-2}}\}$ is called the *photon sphere* of the Schwarzschild spacetime because any null geodesic (or "photon") $\gamma : \mathbb{R} \to \overline{\mathfrak{L}}^{n+1}$ that is tangent to $\overline{P}^n$ for some parameter $\tau_0 \in \mathbb{R}$ is necessarily tangent to it for all parameters $\tau \in \mathbb{R}$. In particular, the Schwarzschild photon sphere is a timelike hypersurface ruled by null geodesics spiraling around the central black hole of mass $m > 0$ "at a fixed distance."

The Schwarzschild photon sphere can be seen as a special case of what is called a "photon surface"[8,21] in a general *spacetime* (or smooth Lorentzian manifold).

Definition 2.1 (Photon surface). *A timelike embedded hypersurface $P^n \hookrightarrow \mathfrak{L}^{n+1}$ in a spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is called a* photon surface *if any null geodesic initially tangent to P^n remains tangent to P^n as long as it exists or, in other words, if P^n is* null totally geodesic.

The one-sheeted hyperboloids in the Minkowski spacetime (Schwarzschild spacetime with $m = 0$) are also examples of photon surfaces (see Sec. III). It will be useful to know that, by an algebraic observation, being a null totally geodesic timelike hypersurface is equivalent to being an umbilic timelike hypersurface.

Proposition 2.2 (See Theorem II.1 of Ref. 8 and Proposition 1 of Ref. 21). *Let $(\mathfrak{L}^{n+1}, \mathfrak{g})$ be a spacetime and $P^n \hookrightarrow \mathfrak{L}^{n+1}$* be *an embedded timelike hypersurface. Then, P^n is a photon surface if and only if it is* totally umbilic, *that is, if and only if its second fundamental form is pure trace.*

As stated above, the Schwarzschild spacetime is "static" by the following definition.

Definition 2.3 (Static spacetime). *A spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is called* (standard) *static if it is a warped product of the form*

$$\mathfrak{L}^{n+1} = \mathbb{R} \times M^n, \quad \mathfrak{g} = -N^2 dt^2 + g, \tag{2.2}$$

where (M^n, g) is a smooth Riemannian manifold and $N : M^n \to \mathbb{R}^+$ is a smooth function called the (static) lapse function *of the spacetime.*

Remark 2.4 [Static spacetime cont. and (canonical) time slices]. *We will slightly abuse the standard terminology and also call a spacetime static if it is a subset (with boundary) of a warped product static spacetime $(\mathbb{R} \times M^n, \mathfrak{g} = -N^2 dt^2 + g)$, $\mathfrak{L}^{n+1} \subseteq \mathbb{R} \times M^n$, to allow for*

inner boundary $\partial\mathfrak{L}$ not arising as a warped product. We will denote the (canonical) time slices $\{t = const.\}$ *of a static spacetime* $(\mathfrak{L}^{n+1}, \mathfrak{g})$, $\mathfrak{L}^{n+1} \subseteq \mathbb{R} \times M^n$, *by* $M^n(t)$ *and continue to denote the induced metric and (restricted) lapse function on* $M^n(t)$ *by g and N, respectively.*

In the context of static spacetimes, we will use the following definition of "photon spheres," extending that of Ref. 7–9. Consistently, the Schwarzschild photon sphere clearly is a photon surface in the Schwarzschild spacetime in this sense.

Definition 2.5 (Photon sphere). *Let* $(\mathfrak{L}^{n+1}, \mathfrak{g})$ *be a static spacetime and* $P^n \hookrightarrow \mathfrak{L}^{n+1}$ *be a photon surface. Then,* P^n *is called a* photon sphere *if it is of the form* $P^n = \mathbb{R} \times \Sigma^{n-1}$ *for some smooth hypersurface* $\Sigma^{n-1} \subset \mathfrak{L}^{n+1}$ *and if the lapse function N of the spacetime is constant along each connected component of* P^n.

For our discussions in Secs. III, IV, we will make use of the following definitions.

Definition 2.6 (Equipotential photon surface). *Let* $(\mathfrak{L}^{n+1}, \mathfrak{g})$ *be a static spacetime and* $P^n \hookrightarrow \mathfrak{L}^{n+1}$ be *a photon surface. Then,* P^n *is called* equipotential *if the lapse function N of the spacetime is constant along each connected component of each time slice* $\Sigma^{n-1}(t) := P^n \cap M^n(t)$ *of the photon surface.*

Definition 2.7 (Outward directed photon surface). *Let* $(\mathfrak{L}^{n+1}, \mathfrak{g})$ *be a static spacetime and* $P^n \hookrightarrow \mathfrak{L}^{n+1}$ be *a photon surface arising as the inner boundary of* $\mathfrak{L}^{n+1}$, $P^n = \partial\mathfrak{L}$, *and let* η *be the "outward" unit normal to* P^n *(i.e., the normal pointing into* $\mathfrak{L}^{n+1}$*). Then,* P^n *is called outward directed if the* η*-derivative of the lapse function N of the spacetime is positive,* $\eta(N) > 0$, *along* P^n.

As usual, a spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is said to be *vacuum* or to satisfy the *Einstein vacuum equation* if

$$\mathfrak{Ric} = 0 \tag{2.3}$$

on $\mathfrak{L}^{n+1}$, where $\mathfrak{Ric}$ denotes the Ricci curvature tensor of $(\mathfrak{L}^{n+1}, \mathfrak{g})$. For a static spacetime, the Einstein vacuum Eq. (2.3) is equivalent to the *static vacuum equations*,

$$N\,\mathrm{Ric} = {}^{g}\nabla^2 N, \tag{2.4}$$

$$\mathrm{R} = 0, \tag{2.5}$$

on M^n, where Ric, R, and ${}^{g}\nabla^2$ denote the Ricci and scalar curvature and the covariant Hessian of (M^n, g), respectively. Combining the trace of (2.4) with (2.5), one obtains the covariant Laplace equation on M^n,

$$\triangle_g N = 0. \tag{2.6}$$

It is clear that, provided (2.4) holds, (2.5) and (2.6) can be interchanged without losing information. Of course, the Schwarzschild spacetime $(\mathbb{R} \times \overline{M}^n, \overline{\mathfrak{g}})$ is vacuum, and thus, (2.4) and (2.6) hold for the Schwarzschild spatial metric $\overline{g} = \overline{N}^{-2}dr^2 + r^2\Omega$ and lapse $\overline{N}$ on its canonical time slice $\overline{M}^n = \mathbb{R}^n \backslash \overline{B_{r_m}(0)}$.

Curvature quantities of a spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$ such as the Riemann curvature endomorphism $\mathfrak{Rm}$, the Ricci curvature tensor $\mathfrak{Ric}$, and the scalar curvature $\mathfrak{R}$ will be denoted in gothic print, and the corresponding covariant derivative will be denoted by ${}^{\mathfrak{g}}\nabla$. The Lorentzian metric induced on a timelike embedded hypersurface $P^n \hookrightarrow \mathfrak{L}^{n+1}$ will be denoted by p, the (outward, see Definition 2.7) unit normal by η, and the corresponding second fundamental form and mean curvature by $\mathfrak{h}$ and $\mathfrak{H} = \mathrm{tr}_p\mathfrak{h}$, respectively. With this notation, Proposition 2.2 can be restated to state that a photon surface is characterized by

$$\mathfrak{h} = \frac{\mathfrak{H}}{n}\, p. \tag{2.7}$$

Set sign conventions $\mathfrak{h}(X, Y) = \mathfrak{g}({}^{\mathfrak{g}}\nabla_X \eta, Y)$ for vectors X, Y tangent to P.

If the spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is static, its time slices $M^n(t)$ have vanishing second fundamental form $K = 0$ by the warped product structure, or in other words, the time slices are totally geodesic. The *time slices of a photon surface* $P^n \hookrightarrow \mathfrak{L}^{n+1}$ will be denoted by $\Sigma^{n-1}(t) := P^n \cap M^n(t)$, with induced metric $\sigma = \sigma(t)$, second fundamental form $h = h(t)$, and mean curvature $H = H(t) = \mathrm{tr}_{\sigma(t)}h(t)$ with respect to the outward pointing unit normal $\nu = \nu(t)$. As an intersection of a totally geodesic time slice and a totally umbilic photon surfaces, $\Sigma^{n-1}(t)$ is necessarily totally umbilic, and we have

$$h(t) = \frac{H(t)}{n-1}\,\sigma(t). \tag{2.8}$$

Our choice of sign of the mean curvature is such that the mean curvature of $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ is positive with respect to the outward unit normal in Euclidean space.

The following proposition will be useful to characterize photon surfaces in vacuum spacetimes.

Proposition 2.8 (Ref. 9, Proposition 3.3). *Let $n \geq 2$ and let $(\mathfrak{L}^{n+1}, \mathfrak{g})$ be a smooth semi-Riemannian manifold possessing a totally umbilic embedded semi-Riemannian hypersurface $P^n \hookrightarrow \mathfrak{L}^{n+1}$. If the semi-Riemannian manifold $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is Einstein, or, in other words, if $\mathfrak{Ric} = \Lambda \mathfrak{g}$ for some constant $\Lambda \in \mathbb{R}$, then each connected component of P^n has constant mean curvature $\mathfrak{H}$ and constant scalar curvature,*

$$\mathrm{R}_p = (n + 1 - 2\tau)\Lambda + \tau \frac{n-1}{n} \mathfrak{H}^2, \tag{2.9}$$

where $\tau := \mathfrak{g}(\eta, \eta)$ denotes the causal character of the unit normal η to P^n.

In particular, connected components of photon surfaces in vacuum spacetimes ($\Lambda = 0$) have constant mean curvature and constant scalar curvature, related via

$$\mathrm{R}_p = \frac{n-1}{n} \mathfrak{H}^2. \tag{2.10}$$

We will now proceed to define and discuss the assumption of asymptotic flatness and asymptotic isotropy of static spacetimes.

Definition 2.9 (Asymptotic flatness). *A smooth Riemannian manifold (M^n, g) with $n \geq 3$ is called asymptotically flat if the manifold M^n is diffeomorphic to the union of a (possibly empty) compact set and an open end E^n, which is diffeomorphic to $\mathbb{R}^n \backslash \overline{B}$, $\Phi = (x^i) : E^n \to \mathbb{R}^n \backslash \overline{B}$, where B is some centered open ball in $\mathbb{R}^n$, and*

$$(\Phi_* g)_{ij} - \delta_{ij} = \mathcal{O}_k\left(r^{1-\frac{n}{2}-\varepsilon}\right), \tag{2.11}$$

$$\Phi_* R = \mathcal{O}_0\left(r^{-n-\varepsilon}\right) \tag{2.12}$$

for $i, j = 1, \ldots, n$ on $\mathbb{R}^n \backslash \overline{B}$ as $r := \sqrt{(x^1)^2 + \cdots + (x^n)^2} \to \infty$ for some $k \in \mathbb{Z}$, $k \geq 2$, and $\varepsilon > 0$. Here, δ denotes the flat Euclidean metric, and δ_{ij} denotes its components in the Cartesian coordinates (x^i).

A static spacetime $(\mathfrak{L}^{n+1} = \mathbb{R} \times M^n, \mathfrak{g} = -N^2 dt^2 + g)$ is called asymptotically flat if its Riemannian base (M^n, g) is asymptotically flat as a Riemannian manifold and, in addition, its lapse function satisfies

$$\Phi_* N - 1 = \mathcal{O}_{k+1}\left(r^{1-\frac{n}{2}-\varepsilon}\right) \tag{2.13}$$

on $\mathbb{R}^n \backslash \overline{B}$ as $r \to \infty$ with respect to the same coordinate chart Φ and numbers $k \in \mathbb{Z}$, $k \geq 2$, and *$\varepsilon > 0$. We will abuse language and call $\mathfrak{L}^{n+1} \subseteq \mathbb{R} \times M^n$ asymptotically flat as long as $\mathfrak{L}^{n+1}$ has a timelike inner boundary $\partial \mathfrak{L}$.*

Here and in the following, we say that a smooth function $f : \mathbb{R}^n \backslash \overline{B} \to \mathbb{R}$ satisfies $f = \mathcal{O}_k(r^p)$ for $k \in \mathbb{N}_0$ and $p \in \mathbb{R}$ as $r \to \infty$ if there exists a constant $C > 0$ and a centered open ball centered open ball $B_1 \supseteq B$ such that $|D^\alpha f| \leq C r^{p-|\alpha|}$ in $\mathbb{R}^n \backslash \overline{B_1}$ for every multi-index α satisfying $|\alpha| \leq k$.

One can expect a well-known result by Kennefick and Murchadha[32] to generalize to higher dimensions, which would assert that static vacuum asymptotically flat spacetimes are automatically "asymptotically isotropic" under suitable asymptotic coordinates. Here, we will resort to assuming asymptotic isotropy, leaving the higher dimensional generalization of this result to be dealt elsewhere.

Definition 2.10 (Asymptotic isotropy[23]). *A smooth Riemannian manifold (M^n, g) of dimension $n \geq 3$ is called asymptotically isotropic (of mass m) if the manifold M^n is diffeomorphic to the union of a (possibly empty) compact set and an open end E^n, which is diffeomorphic to $\mathbb{R}^n \backslash \overline{B}$, $\Psi = (y^i) : E^n \to \mathbb{R}^n \backslash \overline{B}$, where B is some centered open ball in $\mathbb{R}^n$, and if there exists a constant $m \in \mathbb{R}$ such that*

$$(\Psi_* g)_{ij} - (\widetilde{g}_m)_{ij} = \mathcal{O}_2(s^{1-n}), \tag{2.14}$$

for $i, j = 1, \ldots, n$ on $\mathbb{R}^n \backslash \overline{B}$ as $s := \sqrt{(y^1)^2 + \cdots + (y^n)^2} \to \infty$, where

$$\widetilde{g}_m := \varphi_m^{\frac{4}{n-2}}(s)\, \delta, \tag{2.15}$$

$$\varphi_m(s) := 1 + \frac{m}{2s^{n-2}} \tag{2.16}$$

denotes the spatial Schwarzschild metric in isotropic coordinates.

A static spacetime $(\mathfrak{L}^{n+1} = \mathbb{R} \times M^n, \mathfrak{g} = -N^2 dt^2 + g)$ is called asymptotically isotropic (of mass m) *if its Riemannian base (M^n, g) is asymptotically isotropic of mass $m \in \mathbb{R}$ as a Riemannian manifold and, in addition, its lapse function N satisfies*

$$N - \widetilde{N}_m = \mathcal{O}_2(s^{1-n}) \tag{2.17}$$

on $\mathbb{R}^n \backslash \overline{B}$ *as* $s \to \infty$ *with respect to the same coordinate chart* Ψ *and mass* m. *Here,* $\widetilde{N}_m$ *denotes the Schwarzschild lapse function in isotropic coordinates, given by*

$$\widetilde{N}_m(s) := \frac{1 - \frac{m}{2s^{n-2}}}{1 + \frac{m}{2s^{n-2}}}. \tag{2.18}$$

As before, we will abuse language and call $\mathfrak{L}^{n+1} \subseteq \mathbb{R} \times M^n$ asymptotically isotropic *as long as it has a timelike inner boundary.*

Here, we have rewritten the Schwarzschild spacetime, spatial metric, and lapse function in isotropic coordinates via the radial coordinate transformation,

$$r =: s\, \varphi_m^{\frac{2}{n-2}}(s). \tag{2.19}$$

For $m > 0$, this transformation bijectively maps $r \in (r_m, \infty) \mapsto s \in (s_m, \infty)$, with $s_m := \left(\frac{m}{2}\right)^{\frac{1}{n-2}}$. For $m = 0$, this transformation is the identity on $\mathbb{R}^+$, while for $m < 0$, it only provides a coordinate transformation for r suitably large, corresponding to $s > \left(\frac{|m|}{2}\right)^{\frac{1}{n-2}}$.

Remark 2.11. A simple computation shows that the parameter m *in Definition 2.10 equals the ADM-mass of the Riemannian manifold* (M^n, g) *defined in Refs. 33 and 34.*

Remark 2.12. One can analogously define asymptotically flat and asymptotically isotropic Riemannian manifolds and static spacetimes with multiple ends E_l^n *and associated masses* m_l.

With these definitions at hand, let us point out that photon spheres are always outward directed in static, vacuum, asymptotically isotropic spacetimes, a fact that is a straightforward generalization to higher dimensions of Lemma 2.6 and Eq. (2.13) of Ref. 10.

Lemma 2.13. Let $P^n \hookrightarrow \mathfrak{L}^{n+1}$ *be a photon sphere in a static vacuum asymptotically flat spacetime* $(\mathfrak{L}^{n+1}, \mathfrak{g})$. *Then,* P^n *is outward directed.*

III. PHOTON SURFACES IN A CLASS OF STATIC SPHERICALLY SYMMETRIC SPACETIMES

In this section, we will give a local characterization of photon surfaces in a certain class $\mathcal{S}$ of static spherically symmetric spacetimes $(\mathbb{R} \times M^n, \mathfrak{g})$, which includes the $n+1$-dimensional (exterior) Schwarzschild spacetime. We will first locally characterize the spherically symmetric photon surfaces in $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ in Theorem 3.5 and then show in Theorem 3.8 and particularly in Corollary 3.9 that there are essentially no other photon surfaces in spacetimes, $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$. As mentioned above, these results have been used in Ref. 22 to give a detailed description of all photon surfaces in many spacetimes in class $\mathcal{S}$, including the (positive mass) Schwarzschild spacetime.

The class $\mathcal{S}$ is defined as follows: Let $(\mathbb{R} \times M^n, \mathfrak{g})$ be a smooth Lorentzian spacetime such that

$$M^n = \mathcal{I} \times \mathbb{S}^{n-1} \ni (r, \xi) \tag{3.1}$$

for an open interval $\mathcal{I} \subseteq (0, \infty)$, finite or infinite, so that there exists a smooth positive function $f : \mathcal{I} \to \mathbb{R}$ for which we can express the spacetime metric $\mathfrak{g}$ as

$$\mathfrak{g} = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\Omega \tag{3.2}$$

in the global coordinates $t \in \mathbb{R}$, $(r, \xi) \in \mathcal{I} \times \mathbb{S}^{n-1}$, where Ω denotes the canonical metric on $\mathbb{S}^{n-1}$ of area ω_{n-1}. A Lorentzian spacetime $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ is clearly spherically symmetric and moreover naturally (standard) static via the hypersurface orthogonal timelike Killing vector field ∂_t.

Remark 3.1. Note that we do not assume that spacetimes $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ *satisfy any kind of Einstein equations or have any special type of asymptotic behavior toward the boundary of the radial interval* $\mathcal{I}$, *such as being asymptotically flat or asymptotically hyperbolic as* $r \nearrow \sup \mathcal{I}$, *or such as forming a regular minimal surface as* $r \searrow \inf \mathcal{I}$.

Remark 3.2. As ∂_t *is a Killing vector field, the time translation of any photon surface in a spacetime* $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ *will also be a photon surface in* $(\mathbb{R} \times M^n, \mathfrak{g})$. *As all spacetimes* $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ *are also time-reflection symmetric (i.e.,* $t \to -t$ *is an isometry), the time reflection of any photon surface in* $(\mathbb{R} \times M^n, \mathfrak{g})$ *will also be a photon surface in* $(\mathbb{R} \times M^n, \mathfrak{g})$.

While the form of the metric (3.2) is certainly non-generic even among static spherically symmetric spacetimes, the class $\mathcal{S}$ contains many important examples of spacetimes, such as Minkowski spacetime and the regions exterior to the black holes and white holes in Schwarzschild spacetime, Reissner–Nordström spacetime, and Schwarzschild–anti de Sitter spacetime (in $n+1$ dimensions), each for a specific choice of f.

Before we proceed with characterizing photon surfaces in spacetimes in this class $\mathcal{S}$, let us first make the following natural definition.

Definition 3.3. Let $(\mathbb{R}\times M^n, \mathfrak{g}) \in \mathcal{S}$. A connected timelike hypersurface $P^n \hookrightarrow (\mathbb{R}\times M^n, \mathfrak{g})$ will be called spherically symmetric if, for each $t_0 \in \mathbb{R}$ for which the intersection $M^n(t_0) := P^n \cap \{t = t_0\} \neq \emptyset$, there exists a radius $r_0 \in \mathcal{I}$ (where $M^n = \mathcal{I}\times\mathbb{S}^{n-1}$) such that

$$M^n(t_0) = \{t_0\}\times\{r_0\}\times\mathbb{S}^{n-1} \subset \{t_0\}\times M^n. \tag{3.3}$$

A future timelike curve $\gamma : I \to P^n$, parametrized by the arclength on some open interval $I \subset \mathbb{R}$, is called a radial profile *of P^n if $\gamma' \in span\{\partial_t, \partial_r\} \subset T_{\gamma'}(\mathbb{R}\times M^n)$ on I and if the orbit of γ under the rotation generates P^n.*

With this definition at hand, we will now prove the following lemma, which will be used in the Proof of Theorem 3.5.

Lemma 3.4. Let $(\mathbb{R}\times M^n, \mathfrak{g}) \in \mathcal{S}$ and let $P^n \hookrightarrow (\mathbb{R}\times M^n, \mathfrak{g})$ be a spherically symmetric timelike hypersurface. Assume that $P^n \hookrightarrow (\mathbb{R}\times M^n, \mathfrak{g})$ has a radial profile $\gamma : I \to P^n$, which may be written as $\gamma(s) = (t(s), r(s), \xi_*) \in \mathbb{R}\times\mathcal{I}\times\mathbb{S}^{n-1}$ for some fixed $\xi_* \in \mathbb{S}^{n-1}$. If $P^n \hookrightarrow (\mathbb{R}\times M^n, \mathfrak{g})$ is a photon surface, i.e., is totally umbilic with the umbilicity factor λ, then the following first order ODEs hold on I:

$$\dot{t} = \frac{\lambda r}{f(r)}, \tag{3.4}$$

$$(\dot{r})^2 = \lambda^2 r^2 - f(r), \tag{3.5}$$

where λ is constant (also where $\dot{} = \frac{d}{ds}$). Conversely, provided $\dot{r} \neq 0$, if the ODEs (3.4) and (3.5) hold, with λ being constant, then P is a photon surface with the umbilicity factor λ.

Proof. To simplify the notation, we write P for $P^n \hookrightarrow (\mathbb{R}\times M^n, \mathfrak{g})$ and f for $f(r)$. As in Sec. II, let p and $\mathfrak{h}$ denote the induced metric and second fundamental form of P, respectively.

Set $e_0 = \dot{\gamma}$, and extend it to all of P by making it invariant under the rotational symmetries. Thus, e_0 is the future directed unit tangent vector field to P orthogonal to each time slice $\{t(s) = \text{const.}\}$, $s \in I$. In terms of coordinates, we have

$$e_0 = \dot{t}\partial_t + \dot{r}\partial_r. \tag{3.6}$$

Let η be the outward pointing unit normal field to P. From (3.2) and (3.6), we obtain

$$\eta = \frac{\dot{r}}{f}\partial_t + \dot{t}f\partial_r. \tag{3.7}$$

Claim. P is a photon suface, with umbilicity factor $\lambda = \frac{f}{r}\dot{t}$, if and only if e_0 satisfies

$$^{\mathfrak{g}}\nabla_{e_0} e_0 = \lambda\eta. \tag{3.8}$$

□

Proof of the claim. Extend e_0 to an orthonormal basis $\{e_0, e_1, \ldots, e_{n-1}\}$ in a neighborhood of an arbitrary point in P. Thus, each e_I, $I = 1, \ldots, n-1$, where defined, is tangent to the time slices. A simple computation then gives

$$^{\mathfrak{g}}\nabla_{e_I}\eta = \frac{\dot{r}}{f}\,^{\mathfrak{g}}\nabla_{e_I}\partial_t + \dot{t}f\,^{\mathfrak{g}}\nabla_{e_I}\partial_r = \frac{\dot{t}f}{r}e_I, \tag{3.9}$$

from which it follows that

$$\mathfrak{h}(e_I, e_J) = \mathfrak{g}({}^{\mathfrak{g}}\nabla_{e_I}\eta, e_J) = \lambda\delta_{IJ}, \quad I, J = 1, \ldots, n-1, \tag{3.10}$$

where δ_{IJ} is the Kronecker delta and

$$\lambda = \frac{f}{r}\dot{t}. \tag{3.11}$$

Similarly,

$$\mathfrak{h}(e_0, e_I) = \mathfrak{h}(e_I, e_0) = \mathfrak{g}({}^{\mathfrak{g}}\nabla_{e_I}\eta, e_0)) = 0. \tag{3.12}$$

Hence,

$$[\mathfrak{h}(e_I, e_J)]_{I,J=0,\ldots,n-1} \tag{3.13}$$

is a diagonal matrix with $\mathfrak{h}(e_I, e_I) = \lambda$ for $I = 1, \ldots, n-1$. It remains to consider $\mathfrak{h}(e_0, e_0)$.

The profile curve γ and its rotational translates are "longitudes" in the "surface of revolution" P. As such, each is a unit speed geodesic in P, from which it follows that

$${}^{\mathfrak{g}}\nabla_{e_0} e_0 = \ell\eta \tag{3.14}$$

for some scalar ℓ. This implies that

$$\mathfrak{h}(e_0, e_0) = \mathfrak{g}({}^{\mathfrak{g}}\nabla_{e_0}\eta, e_0) = -\mathfrak{g}({}^{\mathfrak{g}}\nabla_{e_0} e_0, \eta) = -\ell. \tag{3.15}$$

From this and (3.13), we conclude that P is a photon surface if and only if $\ell = \lambda = \frac{f}{r}\dot{t}$, which establishes the claim.

Using the coordinate expressions for e_0, η, and λ, a straightforward computation shows that (3.8), with $\lambda = \frac{f}{r}\dot{t}$, is equivalent to the following system of second order ODEs in the coordinate functions $t = t(s)$ and $r = r(s)$:

$$\ddot{t} + \frac{f'}{f}\dot{r}\dot{t} = \frac{\dot{r}}{r}\dot{t}, \tag{3.16}$$

$$\ddot{r} + \frac{ff'}{2}\dot{t}^2 - \frac{f'}{2f}\dot{r}^2 = \frac{(f\dot{t})^2}{r}. \tag{3.17}$$

Now, assume that P is a photon sphere with the umbilicity factor λ so that, particularly, (3.16) and (3.11) hold. Treating (3.16) as a first order linear equation in $\dot{t}$, we have

$$\ddot{t} + \left(\frac{f'}{f}\dot{r} - \frac{\dot{r}}{r}\right)\dot{t} = 0,$$

which, multiplying through by the integrating factor $\frac{f}{r}$, gives $\frac{d}{ds}\left(\frac{f}{r}\dot{t}\right) = 0$ so that (3.4) holds, with $\lambda = \frac{f}{r}\dot{t} > 0$ being a constant on P. The assumption that γ is parameterized with respect to arc length gives

$$f(\dot{t})^2 - \frac{1}{f}(\dot{r})^2 = 1. \tag{3.18}$$

Together with (3.4), we see that (3.5) also holds.

Conversely, now assume that (3.4) and (3.5) hold, with $\lambda = \frac{f}{r}\dot{t} =$ being constant, and, in addition, that $\dot{r} \neq 0$. Differentiating (3.4) with respect to s and then using (3.11) easily implies (3.16). Differentiating (3.5) with respect to s, then using (3.11), and dividing out by $\dot{r}$ gives

$$\ddot{r} + \frac{f'}{2} = \frac{(f\dot{t})^2}{r}. \tag{3.19}$$

Together with (3.18) [which follows from (3.4) and (3.5)], this implies (3.17). We have shown that (3.16) and (3.17) hold, from which it follows that (3.8) holds with $\lambda = \frac{f}{r}\dot{t}$. Invoking the claim then completes the Proof of Lemma 3.4. □

From Lemma 3.4, we obtain the following.

Theorem 3.5. *Let $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ and let $P^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$ be a spherically symmetric timelike hypersurface. Assume that $P^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$ is a photon surface, with the umbilicity factor λ, i.e.,*

$$\mathfrak{h} = \lambda p,$$

where p and $\mathfrak{h}$ are the induced metric and second fundamental form induced on P^n by $P^n \hookrightarrow (\mathbb{R}\times M^n, \mathfrak{g})$, respectively.

Let $\gamma : I \to P^n$ be a radial profile for P^n and write $\gamma(s) = (t(s), r(s), \xi_*) \in \mathbb{R}\times\mathcal{I}\times\mathbb{S}^{n-1}$ for some $\xi_* \in \mathbb{S}^{n-1}$. Then, λ is a positive constant and either *$r \equiv r_*$ along γ for some $r_* \in \mathcal{I}$ at which the* photon sphere condition

$$f'(r_*)r_* = 2f(r_*) \tag{3.20}$$

holds, $\lambda = \frac{\sqrt{f(r_*)}}{r_*}$, and $(P^n, p) = (\mathbb{R}\times\mathbb{S}^{n-1}, -f(r_*)dt^2 + r_*^2\,\Omega)$ is a cylinder and thus a photon sphere or *$r = r(t)$ can globally be written as a smooth non-constant function of t in the range of γ and $r = r(t)$ satisfying the* photon surface ODE,

$$\left(\frac{dr}{dt}\right)^2 = \frac{f(r)^2\left(\lambda^2 r^2 - f(r)\right)}{\lambda^2 r^2}. \tag{3.21}$$

Conversely, whenever the photon sphere condition $f'(r_*)r_* = 2f(r_*)$ holds for some $r_* \in \mathcal{I}$, then the cylinder $(P^n, p) = (\mathbb{R}\times\mathbb{S}^{n-1}, -f(r_*)dt^2 + r_*^2\,\Omega)$ is a photon sphere in $(\mathbb{R}\times M^n, \mathfrak{g})$ with the umbilicity factor $\lambda = \frac{\sqrt{f(r_*)}}{r_*}$. In addition, any smooth non-constant solution $r = r(t)$ of (3.21) for some constant $\lambda > 0$ gives rise to a photon surface in $(\mathbb{R}\times M^n, \mathfrak{g})$ with the umbilicity factor λ.

Proof. From Lemma 3.4, we know that λ is a positive constant. Moreover, we know that $t = t(s)$ and $r = r(s)$ satisfy Eqs. (3.4) and (3.5). In the case when $r \equiv r_*$ for some constant r_*, these equations immediately imply

$$\dot{t} = \frac{1}{\sqrt{f(r_*)}}, \quad \lambda = \frac{\sqrt{f(r_*)}}{r_*}. \tag{3.22}$$

Furthermore, (3.17) implies

$$f'(r_*)r_* = 2f(r_*). \tag{3.23}$$

In the general case, Eqs. (3.4) and (3.5) clearly imply (3.21). The converse statements are easily obtained from (3.21) and the unit speed condition (3.18). □

Remark 3.6. In view of Remark 3.2, note that in the "either" case, the photon sphere is time-translation and time-reflection invariant in itself. In the "or" case, note that the photon surface ODE (3.21) is time-translation and time-reflection invariant and will thus allow for time-translated and time-reflected solutions corresponding to the same $\lambda > 0$.

Example 3.7. Choosing $(\mathbb{R}\times M^n, \mathfrak{g}) = (\mathbb{R}^{1,n}, \mathfrak{m})$, where $\mathfrak{m}$ is the Minkowski metric and $f : (0,\infty) \to \mathbb{R} : r \mapsto 1$, the photon sphere condition cannot be satisfied for any $r_* \in (0,\infty)$ so that every spherically symmetric photon surface in the Minkowski spacetime must satisfy ODE (3.21), which reduces to

$$\left(\frac{dr}{dt}\right)^2 = \frac{\lambda^2 r^2 - 1}{\lambda^2 r^2} \quad \Leftrightarrow \quad r(t) = \sqrt{\lambda^{-2} + (t - t_0)^2} \quad \text{for some} \quad t_0 \in \mathbb{R}$$

and describes the rotational one-sheeted hyperboloids of radii λ^{-1} for any $0 < \lambda < \infty$.

This is, of course, consistent with the well-known fact that the only timelike totally umbilic hypersurfaces in the Minkowski spacetime are, apart from (parts of) timelike hyperplanes, precisely (parts of) these hyperboloids and their spatial translates, the formula for which explicitly displays the time-translation and time-reflection invariance of the photon surface characterization problem.

Note that the photon sphere condition is satisfied precisely at the well-known photon sphere radius $r_* = (nm)^{\frac{1}{n-2}}$ in the $n+1$-dimensional Schwarzschild spacetime, where $f(r) = 1 - \frac{2m}{r^{n-2}}$ for $m > 0$ and $r > r_H = (2m)^{\frac{1}{n-2}}$ and there is no photon sphere radius for $m \le 0$ and $r > 0$. While there are no other photon spheres, there are many non-cylindrical photon surfaces in the Schwarzschild spacetime. The analysis in Ref. 22, based on Theorem 3.5, shows that, up to time translation and time reflection (cf. Remarks 3.2 and 3.6), there are five classes of non-cylindrical spherically symmetric photon surfaces in the (exterior) positive mass Schwarzschild spacetime [as well as in many other (exterior) black hole spacetimes in class $\mathcal{S}$]; the profile curves for representatives from each class are depicted in Fig. 1.

In each case, they approach asymptotically the event horizon $r = r_H$ and/or the photon sphere $r = r_*$ and/or become asymptotically null at infinity in (t, r)-coordinates. An analysis of the behavior of the asymptotics of the non-cylindrical spherically symmetric photon is performed for both Schwarzschild and many other (exterior) black hole spacetimes in class $\mathcal{S}$ in (generalized) Kruskal–Szekeres coordinates

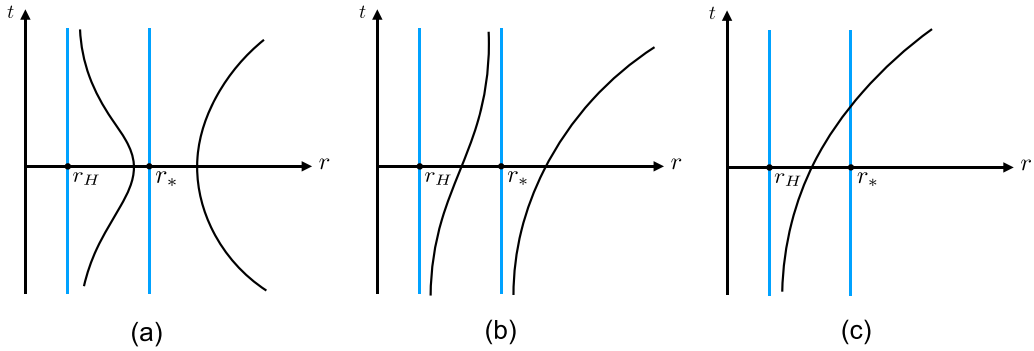

FIG. 1. Profile curves for all types of spherically symmetric photon surfaces in Schwarzschild spacetime grouped according to the umbilicity factor (see Ref. 22 for details).

in Ref. 35. There, it is found that the photon surfaces appearing to approach $r = r_H$ in (t, r)-coordinates, in fact, cross the event horizon, while those approaching the photon sphere $r = r_*$ or asymptotically becoming null in (t, r)-coordinates do so in (generalized) Kruskal–Szekeres coordinates too.

Using quite different methods, in Ref. 36, the same types of photon surfaces are found in a 2 + 1-dimensional spacetime obtained by dropping an angle coordinate from 3 + 1-dimensional Schwarzschild of positive mass.

The following question naturally arises: What about non-spherically symmetric photon surfaces? This is addressed in the following theorem (see Corollary 3.9).

Theorem 3.8. *Let $n \geq 3$, $I \subseteq \mathbb{R}^+$* be *an open interval, and $D^n := \{y \in \mathbb{R}^n \,|\, |y| = s \in I\}$, and let $\widetilde{N}, \psi : I \to \mathbb{R}^+$ be smooth positive functions. Consider the* static isotropic *spacetime*

$$\left(\mathbb{R} \times D^n, \mathfrak{g} = -\widetilde{N}^2 dt^2 + \psi^2\, \delta\right) \tag{3.24}$$

of lapse $\widetilde{N} = \widetilde{N}(s)$ and conformal factor $\psi = \psi(s)$, with $s := |y|$ for $y \in D^n$ here and in the following. We write $\widetilde{g} := \psi^2\, \delta$. A timelike hypersurface P^n in $(\mathbb{R} \times D^n, \mathfrak{g})$ is called isotropic *if $P^n \cap \{t = const.\} = \mathbb{S}^{n-1}_{s(t)}(0) \subset D^n$ for some radius $s(t) \in I$ for every t for which $P^n \cap \{t = const.\} \neq \emptyset$. A* (partial) centered vertical hyperplane *in $(\mathbb{R} \times D^n, \mathfrak{g})$ is the restriction of a timelike hyperplane in the Minkowski spacetime containing the t-axis to $\mathbb{R} \times D^n$, i.e., a set of the form*

$$\{(t, y) \in \mathbb{R} \times D^n \,|\, y \cdot u = 0\} \tag{3.25}$$

for some fixed Euclidean unit vector $u \in \mathbb{R}^n$, where $\cdot$ denotes the Euclidean inner product. Centered vertical hyperplanes are totally geodesic in $(\mathbb{R} \times D^n, \mathfrak{g})$.

Furthermore, assume that the functions $\widetilde{N}$ and ψ satisfy

$$\frac{\widetilde{N}'(s)}{\widetilde{N}(s)} \neq \frac{\psi'(s)}{\psi(s)} \tag{3.26}$$

for all $s \in S$ for some dense subset $S \subseteq I$. Then, any photon surface in $(\mathbb{R} \times D^n, \mathfrak{g})$ is either (part of) an isotropic photon surface or (part of) a centered vertical hyperplane.

Corollary 3.9. Let $n \geq 3$ and $m > 0$, and consider the $n + 1$-dimensional Schwarzschild spacetime of mass m. Then, any connected photon surface is either (part of) a centered vertical hyperplane, as described above, or (part of) a spherically symmetric photon surface, as described in Theorem 3.5.

Proof of Corollary 3.9. Recall the isotropic form of the Schwarzschild spacetime (2.15), (2.16), and (2.18), with $I = (s_m, \infty)$, and note that (3.26) corresponds to $s^{n-2} \neq \frac{m}{2(n-1)}$ (which can be quickly seen when exploiting $\widetilde{N} = \frac{2-\varphi}{\varphi}$, $\psi = \varphi^{\frac{2}{n-2}}$). This, however, is automatic as $\left(\frac{m}{2(n-1)}\right)^{\frac{1}{n-2}} < s_m = \left(\frac{m}{2}\right)^{\frac{1}{n-2}} < s$ holds for all $s \in I = (s_m, \infty)$. □

Remark 3.10. Condition (3.26) has the following geometric interpretation: If

$$\frac{\widetilde{N}'(s)}{\widetilde{N}(s)} = \frac{\psi'(s)}{\psi(s)}$$

holds for all $s \in O$, with $O \subseteq I$ being an open subset, then, on each connected component $J \subseteq O$, there exists a positive constant $A > 0$ such that $\widetilde{N}(s) = A\psi(s)$ for all $s \in J$, where we used that $\widetilde{N}, \psi > 0$. This shows that $\mathfrak{g} = -\widetilde{N}^2 dt^2 + \psi^2 \delta = \psi^2(-A^2 dt^2 + \delta)$ or, in other words, the static isotropic spacetime $(\mathbb{R} \times (O \times \mathbb{S}^{n-1}), \mathfrak{g})$ is locally conformally flat and hence possesses additional photon surfaces corresponding to the totally geodesic timelike hyperplanes that do not contain the t-axis and to the spatially translated totally umbilic rotational one-sheeted hyperboloids of the (time-rescaled) Minkowski spacetime (see Example 3.7).

Hence, Theorem (3.8) gives a full characterization of photon surfaces in nowhere locally conformally flat static isotropic spacetimes.

Remark 3.11. A static isotropic spacetime $(\mathbb{R} \times D^n, -\widetilde{N}^2 dt^2 + \psi^2 \delta)$ can be globally rewritten as a spacetime of class $\mathcal{S}$ if and only if $\widetilde{N}^2(s) = (1 + \frac{s\psi'(s)}{\psi(s)})^2 > 0$ for all $s \in I$ by setting $r(s) := s\psi(s)$ and $f(r) := \widetilde{N}(s(r))$, where $s = s(r)$ denotes the inverse function of $r = r(s)$. In this case, the photon sphere and photon surface conditions on the isotropic radius profile $s = S_*$ and $s = S(t)$ [3.57 and (3.60)] reduce to the much simpler photon sphere and photon surface conditions for the area radius profile $r = r_*$ and $r = r(t)$ [3.20 and (3.21)], respectively.

Conversely, a spacetime of class $\mathcal{S}$ can always be locally rewritten in isotropic form by picking a suitable $r_0 \in \mathcal{I}$ (or $r_0 \in \overline{\mathcal{I}}$) and setting

$$s = s(r) := \exp\left(\int_{r_0}^{r} \left(\rho\sqrt{(f(\rho)}\right)^{-1} d\rho\right),$$

$\psi(s) := \frac{r(s)}{s}$, and $\widetilde{N}(s) := \sqrt{f(r(s))}$, where $r = r(s)$ denotes the inverse of $s = s(r)$.

The main reason for switching into the isotropic picture lies in the spatial conformal flatness allowing us to easily describe centered vertical hyperplanes and to exclude photon surfaces that are neither centered vertical hyperplanes nor isotropic.

Proof of Theorem 3.8. Let P^n be a connected photon surface in a static isotropic spacetime $(\mathbb{R} \times D^n, \mathfrak{g} = -\widetilde{N}^2 dt^2 + \psi^2 \delta)$. As before, set $M^n(t) := \{t = \text{const.}\}$. Let $T := \{t \in \mathbb{R} \,|\, P^n \cap M^n(t) \neq \emptyset\}$ and note that T is an open, possibly infinite, interval. Set $\Sigma^{n-1}(t) := P^n \cap M^n(t)$ for $t \in T$. As timelike and spacelike submanifolds are always transversal, $\Sigma^{n-1}(t)$ is a smooth surface. Furthermore, $\Sigma^{n-1}(t)$ is umbilic in $M^n(t)$ by the time symmetry of $M^n(t)$, or, in other words, because the second fundamental form of $M^n(t)$ in a static spacetime vanishes. As $(M^n(t), \widetilde{g})$ is conformally flat, exploiting the conformal invariance of umbilicity, the only umbilic hypersurfaces in $(M^n(t), \widetilde{g})$ are the conformal images of pieces of Euclidean round spheres and pieces of Euclidean hyperplanes. Slightly abusing the notation and denoting points in the spacetime by their isotropic coordinates, by continuity, and by connectedness of P^n, $\left(\Sigma^{n-1}(t)\right)_{t\in T}$ is thus either a family of pieces of spheres,

$$\Sigma^{n-1}(t) \subseteq \{y \in \mathbb{R}^n \,|\, |y - c(t)| = S(t)\}, \tag{3.27}$$

with centers $c(t) \in \mathbb{R}^n$ and radii $S(t) > 0$ for all $t \in T$, or a family of pieces of hyperplanes,

$$\Sigma^{n-1}(t) \subseteq \{y \in \mathbb{R}^n \,|\, y \cdot u(t) = a(t)\} \tag{3.28}$$

for some δ-unit normal vectors $u(t) \in \mathbb{R}^n$ and altitudes $a(t) \in \mathbb{R}$ for all $t \in T$, where $|\cdot|$ and $\cdot$ denote the Euclidean norm and inner product, respectively. The (outward, where appropriate) unit normal η to P^n can be written as $\eta = \alpha\nu + \beta\partial_t$, with $\alpha > 0$, recalling that ν denotes the (outward, where appropriate) unit normal to $\Sigma^{n-1}(t)$ in $M^n(t)$. By time symmetry of $M^n(t)$, the second fundamental form $\mathfrak{h}$ of P^n in the spacetime restricted to $\Sigma^{n-1}(t)$ can be expressed in terms of the second fundamental form h of $\Sigma^{n-1}(t)$ in $M^n(t)$ via $\mathfrak{h}|_{T\Sigma^{n-1}(t)\times T\Sigma^{n-1}(t)} = \alpha h$. By umbilicity, $\mathfrak{h} = \lambda p$, with p denoting the induced metric on P^n, and this implies that

$$h = \frac{\lambda}{\alpha}\sigma, \tag{3.29}$$

where σ is the induced metric on $\Sigma^{n-1}(t)$. We will treat the planar and the spherical cases separately. We will denote t-derivatives by $\dot{}$ and s-derivatives by $'$.

A. Planar case

Let $\Sigma^{n-1}(t)$ be as in (3.28) for all $t \in T$. We will show that $a(t) = 0$ and $\dot{u}(t) = 0$ for all $t \in T$ and, moreover, that $\lambda = 0$ along P^n. This then implies that P^n is contained in a centered vertical hyperplane with unit normal $\eta = \nu = \psi^{-1}(s)u^i\partial_{y^i}$ and, moreover, that all centered vertical hyperplanes are totally geodesic as they can be written in the form (3.28) with $\dot{u}(t) = 0$ and $a(t) = 0$ for all $t \in T$.

For each $t \in T$, extend $u(t)$ to a δ-orthonormal basis $\{e_1(t) = u(t), e_2(t), \ldots, e_n(t)\}$ of $\mathbb{R}^n$ so that $e_I(t)$ is smooth in t, and constant for each t, for all $I = 2, \ldots, n$. Then, clearly, $X_I(t, y) := e_I^k(t)\partial_{y^k}$ is tangent to P^n for all $I = 2, \ldots, n$ and $\{X_I(t, \cdot)\}_{I=2}^n$ is an orthogonal frame for $\Sigma^{n-1}(t)$ with respect to $\widetilde{g}$ by conformal flatness. To find the missing (spacetime-) orthogonal tangent vector to P^n, consider a curve

$\mu(t) = (t, y(t))$ in P^n with tangent vector $\dot{\mu}(t) = \partial_t + \dot{y}^i(t)\partial_{y^i}$. Let the capital Latin indices run from $2, \ldots, n$. Now decompose $\dot{y}(t) = \rho(t)u(t) + \xi^I(t)e_I(t) \in \mathbb{R}^n$. By (3.28), we find that $\rho(t) = \dot{y}(t) \cdot u(t) = \dot{a}(t) - y(t) \cdot \dot{u}(t)$. Hence, a future pointing tangent vector to P^n orthogonal in the spacetime to all X_I is given by

$$X_1(t, y) := \partial_t + (\dot{a}(t) - y \cdot \dot{u}(t))u^i(t)\partial_{y^i} \tag{3.30}$$

so that we have constructed a smooth orthogonal tangent frame $\{X_i\}_{i=1}^n$ for P^n. Hence, we can compute the (spacetime) unit normal to P^n to be

$$\eta(t, y) = \frac{\frac{\psi(s)}{\widetilde{N}(s)}(\dot{a}(t) - y \cdot \dot{u}(t))\partial_t + \frac{\widetilde{N}(s)}{\psi(s)} u^i(t)\partial_{y^i}}{\sqrt{\widetilde{N}^2(s) - \psi^2(s)(\dot{a}(t) - y \cdot \dot{u}(t))^2}}. \tag{3.31}$$

In other words, using that $v(t, y) = \psi^{-1}(s)\, u^i(t)\partial_{y^i}$, we have

$$\alpha(t, y) = \frac{\widetilde{N}(s)}{\sqrt{\widetilde{N}^2(s) - \psi^2(s)(\dot{a}(t) - y \cdot \dot{u}(t))^2}}, \tag{3.32}$$

$$\beta(t, y) = \frac{\psi(s)(\dot{a}(t) - y \cdot \dot{u}(t))}{\widetilde{N}(s)\sqrt{\widetilde{N}^2(s) - \psi^2(s)(\dot{a}(t) - y \cdot \dot{u}(t))^2}}. \tag{3.33}$$

We are now in a position to compute the second fundamental forms explicitly and take advantage of the umbilicity of P^n. Using $u(t) \cdot \dot{e}_J(t) = -\dot{u}(t) \cdot e_J(t)$ for all $t \in T$, the condition $\mathfrak{h}(X_1, X_J) = 0$ gives

$$-\dot{u}(t) \cdot e_J(t) + \frac{1}{s}\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\}(\dot{a}(t) - y \cdot \dot{u}(t))y \cdot e_J(t) = 0 \tag{3.34}$$

for $J = 2, \ldots, n$. As $\{u(t), e_J(t)\}_{J=2}^n$ is a δ-orthonormal frame, this is equivalent to

$$\frac{1}{s}\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\}(\dot{a}(t) - y \cdot \dot{u}(t))(y - a(t)\, u(t)) = \dot{u}(t). \tag{3.35}$$

As $\Sigma^{n-1}(t)$ has the dimension $n - 1$, (3.35) tells us that $\dot{u}(t) = 0$ for all $t \in T$ by linear dependence considerations (otherwise, if the term in braces $\{\ldots\}$ vanishes). Hence, (3.35) simplifies to

$$\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\}\dot{a}(t)\,(y - a(t)\,u) = 0 \tag{3.36}$$

so that, for a given $t \in T$, again using that $\Sigma^{n-1}(t)$ has dimension $n - 1$ and linear dependence considerations, we find $\dot{a}(t) = 0$ if the term in braces $\{\ldots\}$ does not vanish along P^n, i.e., when assuming (3.26).

Let us now compute the umbilicity factor λ, exploiting that u and a are constant. Note that e_I is also constant, (3.32) and (3.33) reduce to $\alpha = 1$ and $\beta = 0$, and using (3.31) and (3.30), we get $\eta = v$ and $X_1 = \partial_t$. As e_I is independent of y, we find

$$h(X_I, X_J) = \frac{\psi'(s)}{s}\, a\, \delta_{IJ} \tag{3.37}$$

so that by (3.29), the photon surface umbilicity factor λ satisfies

$$\lambda(t, y) = \lambda(y) = \frac{\psi'(s)}{s\psi^2(s)}\, a \tag{3.38}$$

and is particularly independent of t. From $\mathfrak{h}(X_1, X_1) = \lambda p(X_1, X_1)$, we find

$$\lambda(y) = \frac{\widetilde{N}'(s)}{s\widetilde{N}(s)\psi(s)}\, a. \tag{3.39}$$

Thus, (3.38) and (3.39) combine to

$$\lambda(y) = \frac{\psi'(s)}{s\psi^2(s)}\, a = \frac{\widetilde{N}'(s)}{s\widetilde{N}(s)\psi(s)}\, a, \tag{3.40}$$

which implies that $a = 0$ and, indeed, $\lambda(y) = \lambda = 0$ is also independent of y when assuming that (3.26) holds along P^n. This shows that centered vertical hyperplanes are totally geodesic and that any photon surface P^n as in (3.28) along which (3.26) holds is (part of) a centered vertical hyperplane.

B. Spherical case

Let $\Sigma^{n-1}(t)$ be as in (3.27) for all $t \in T$. We will show that $c(t) = 0$ for all $t \in T$. This then implies that P^n is contained in an isotropic photon surface as desired, namely, in a photon sphere with isotropic radius $s = S_*$ satisfying (3.57) or with isotropic radius profile $s = S(t)$ as in (3.60). We will use the abbreviation

$$u(t,y) := \frac{y - c(t)}{S(t)} \tag{3.41}$$

to reduce the notational complexity.

The outward unit normal ν to $\Sigma^{n-1}(t)$ in $M^n(t)$ is given by

$$\nu = \psi^{-1}(s)\, u^i(t,y)\, \partial_{y^i}. \tag{3.42}$$

Now, choose a smooth δ-orthonormal system of vectors $e_I(t,y)$ locally along P^n such that $e_I(t,y) \cdot u(t,y) = 0$ for all $(t,y) \in P^n$ and set $X_I(t,y) := e_I^k(t,y)\partial_{y^k}$ for all $(t,y) \in P^n$ and all $I = 2, \dots, n$ so that $\{X_I(t,\cdot)\}_{I=2}^n$ is an orthogonal frame for $\Sigma^{n-1}(t)$ with respect to $\widetilde{g}$ by conformal flatness. To find the missing (spacetime-) orthogonal tangent vector to P^n, consider a curve $\mu(t) = (t, y(t))$ in P^n with the tangent vector $\dot{\mu}(t) = \partial_t + \dot{y}^i(t)\partial_{y^i}$. Let capital Latin indices again run from $2, \dots, n$. Now, decompose

$$\dot{y}(t) = \rho(t)u(t,y(t)) + \xi^I(t)e_I(t,y(t)) \in \mathbb{R}^n. \tag{3.43}$$

By (3.27) and the fact that $\frac{d}{dt}|u(t,y(t))|_\delta^2 = 0$ for all $t \in T$, we find

$$\rho(t) = \dot{y}(t) \cdot u(t,y(t)) = \dot{c}(t) \cdot u(t,y(t)) + \dot{S}(t). \tag{3.44}$$

Hence, a future pointing tangent vector to P^n orthogonal in the spacetime to all X_I is given by

$$X_1(t,y) := \partial_t + \big(\dot{c}(t) \cdot u(t,y) + \dot{S}(t)\big)u^i(t,y)\, \partial_{y^i} \tag{3.45}$$

so that we have constructed a smooth orthogonal tangent frame $\{X_i\}_{i=1}^n$ for P^n. Hence, we can compute the outward (spacetime) unit normal to P^n to be

$$\eta(t,y) = \frac{\frac{\psi(s)}{\widetilde{N}(s)}\big(\dot{c}(t) \cdot u(t,y) + \dot{S}(t)\big)\partial_t + \frac{\widetilde{N}(s)}{\psi(s)}u^i(t,y)\partial_{y^i}}{\sqrt{\widetilde{N}^2(s) - \psi^2(s)\big(\dot{c}(t) \cdot u(t,y) + \dot{S}(t)\big)^2}}. \tag{3.46}$$

In other words, using that $\nu(t,y) = \psi^{-1}(s)u^i(t,y)\partial_{y^i}$, we have

$$\alpha(t,y) = \frac{\widetilde{N}(s)}{\sqrt{\widetilde{N}^2(s) - \psi^2(s)\big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)^2}}, \tag{3.47}$$

$$\beta(t,y) = \frac{\psi(s)\big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)}{\widetilde{N}(s)\sqrt{\widetilde{N}^2(s) - \psi^2(s)\big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)^2}}. \tag{3.48}$$

Extending the fields e_J trivially in the radial direction, let us first collect the following explicit formulas arising from differentiating e_J in direction u:

$$e_I^i(t,y)\big(e_{J,y^i}(t,y)\big)\cdot u(t,y) = -\frac{e_I^i(t,y)\, e_J(t,y)\cdot y_{,y^i}}{S(t)} = -\frac{\delta_{IJ}}{S(t)}, \tag{3.49}$$

$$u^i(t,y)\big(e_{J,y^i}(t,y)\big)\cdot u(t,y) = -\frac{u^i(t,y)\, e_J(t,y)\cdot y_{,y^i}}{S(t)} = 0. \tag{3.50}$$

Now, let us compute the second fundamental forms explicitly. Using (3.50), we find that the umbilicity condition $\mathfrak{h}(X_1, X_J) = 0$ gives

$$\begin{aligned} 0 &= u(t,y)\cdot \dot{e}_J(t,y) + \big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)u_i(t,y)u^k(t,y) \\ &\quad \times \left[e^i_{J,y^k}(t,y) + \frac{\psi'(s)}{s\psi(s)}\big(y_k\, e_J^i(t,y) + y\cdot e_J(t,y)\,\delta^i_k - y^i(e_J)_k(t,y)\big)\right] \\ &\quad - \big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)\frac{\widetilde{N}'(s)}{s\widetilde{N}(s)}\, y\cdot e_J(t,y) \\ &= \frac{\dot{c}(t)}{S(t)}\cdot e_J(t,y) + \big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)\frac{1}{s}\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\} y\cdot e_J(t,y) \end{aligned}$$

for all $J = 2,\dots,n$. As $\{u(t,y), e_J(t,y)\}_{J=2}^n$ is a δ-orthonormal frame, this turns out to be equivalent to

$$\begin{aligned} -\frac{\dot{c}(t)}{S(t)} &= \big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)\frac{1}{s}\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\} y \\ &\quad - \left(\big(\dot{c}(t)\cdot u(t,y) + \dot{S}(t)\big)\frac{1}{s}\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\} y\cdot u(t,y) + \frac{\dot{c}(t)\cdot u(t,y)}{S(t)}\right) u(t,y). \end{aligned} \tag{3.51}$$

As $\Sigma^{n-1}(t)$ has the dimension $n-1$ for all $t\in T$, (3.51) tells us by linear dependence considerations that $\dot{c}(t) = 0$ for all $t\in T$. Consequently, (3.51) simplifies to

$$0 = \dot{S}(t)\left\{\frac{\psi'(s)}{\psi(s)} - \frac{\widetilde{N}'(s)}{\widetilde{N}(s)}\right\}(y - (y\cdot u(t,y))u(t,y)). \tag{3.52}$$

Assuming (3.26), the term in braces $\{\dots\}$ does not vanish and (3.52) implies that, for a fixed $t\in T$, $\dot{S}(t) = 0$ or $y = (y\cdot u(t,y))u(t,y)$ for all $y\in\Sigma^{n-1}(t)$. However, $y = (y\cdot u(t,y))u(t,y)$ for all $y\in\Sigma^{n-1}(t)$ is equivalent to $c = 0$ and $S(t) = s$ for all $y\in\Sigma^{n-1}(t)$, again by linear dependence considerations and as $\Sigma^{n-1}(t)$ has dimension $n-1$. In other words, assuming (3.26), we now know that $c = 0$ unless S is constant along P^n, the case in which a constant center $c\neq 0$ is potentially possible.

Let us now continue with our computation of the conformal factor λ using the simplification $\dot{c}(t) = 0$ for all $t\in T$. We first treat the **case $\dot{S}(t) = 0$ for all $t\in T$**: We know that $S(t) = S = s$. Moreover, (3.47) and (3.48) give $\alpha = 1$ and $\beta = 0$, and by (3.46) and (3.45), $\eta = \nu$ and $X_1 = \partial_t$. Moreover, $u(t,y) = u(y)$ and $e_J(t,y)$ and, hence, X_J are independent of t. By (3.49) and $y\cdot u(y) = c\cdot u(y) + S$ via (3.41), we find

$$h(X_I, X_J) = \frac{\psi(S)}{S}\left(1 + \frac{\psi'(S)}{\psi(S)}(c\cdot u(y) + S)\right)\delta_{IJ}, \tag{3.53}$$

so that, by (3.29), the photon surface umbilicity factor λ satisfies

$$\lambda(t,y) = \frac{1}{S\psi(S)}\left(1 + \frac{\psi'(S)}{\psi(S)}(c\cdot u(y) + S)\right), \tag{3.54}$$

and thus, particularly, $\lambda(t,y) = \lambda(y)$ is independent of t. Similarly, from $\mathfrak{h}(X_1,X_1) = \lambda p(X_1,X_1)$, we find

$$\lambda(y) = \frac{\widetilde{N}'(S)}{S\widetilde{N}(S)\psi(S)}(c\cdot u(y) + S), \tag{3.55}$$

and hence,

$$1 + \left\{\frac{\psi'(S)}{\psi(S)} - \frac{\widetilde{N}'(S)}{\widetilde{N}(S)}\right\}(c\cdot u(y) + S) = 0 \tag{3.56}$$

for all $y \in \Sigma^{n-1}(t)$ and all $t \in T$. As $\Sigma^{n-1}(t)$ has the dimension $n-1$ and S is constant, we conclude that $c\cdot u(y)$ is constant and, hence, by (3.41) that $c\cdot y$ must be constant along P^n. Using again that $\Sigma^{n-1}(t)$ has the dimension $n-1$, this leads to $c = 0$ as desired. Hence, the isotropic radii $s = S_*$ for which this photon sphere can occur are the solutions of the implicit photon sphere equation,

$$1 + \left\{\frac{\psi'(S_*)}{\psi(S_*)} - \frac{\widetilde{N}'(S_*)}{\widetilde{N}(S_*)}\right\}S_* = 0, \tag{3.57}$$

provided that such solutions exist.

Let us now treat the other **case $c = 0$**: We find $X_1 = \partial_t + \dot{S}(t)u^i(t,y)\partial_{y^i}$ by (3.45), and indeed, $e_J(t,y) = e_J(y)$ is independent of t as $u(t,y) = \frac{y}{S(t)}$. Moreover, $s = S(t)$ holds for all $(t,y) \in P^n$. Thus, using (3.49), we can compute

$$h(X_I,X_J) = \frac{\psi(S(t))}{S(t)}\left(1 + \frac{\psi'(S(t))}{\psi(S(t))}S(t)\right)\delta_{IJ}, \tag{3.58}$$

so that, by (3.29), the photon surface umbilicity factor λ satisfies

$$\lambda(t,y) = \frac{\widetilde{N}(S(t))\left(1 + \frac{\psi'(S(t))}{\psi(S(t))}S(t)\right)}{S(t)\psi(S(t))\sqrt{\widetilde{N}^2(S(t)) - \psi^2(S(t))\dot{S}^2(t)}} \tag{3.59}$$

from which we see that $\lambda(t,y) = \lambda(t)$ only depends on t. From the remaining umbilicity condition $\mathfrak{h}(X_1,X_1) = \lambda p(X_1,X_1)$, we obtain

$$\begin{aligned}\lambda(t) = {} & \frac{\widetilde{N}(S(t))\psi(S(t))}{\sqrt{\widetilde{N}^2(S(t)) - \psi^2(S(t))\dot{S}^2(t)}^3} \\ & \times\left(\frac{\widetilde{N}'(S(t))\widetilde{N}(S(t))}{\psi^2(S(t))} + \ddot{S}(t) + \left\{\frac{\psi'(S(t))}{\psi(S(t))} - \frac{2\widetilde{N}'(S(t))}{\widetilde{N}(S(t))}\right\}\dot{S}^2(t)\right)\end{aligned}$$

and can conclude that the implicit equation,

$$\begin{aligned}&\left(1 + \frac{\psi'(S(t))}{\psi(S(t))}S(t)\right)\left(\widetilde{N}^2(S(t)) - \psi^2(S(t))\dot{S}^2(t)\right) \\ &\quad = S(t)\widetilde{N}'(S(t))\widetilde{N}(S(t)) \\ &\qquad + S(t)\psi^2(S(t))\left(\ddot{S}(t) + \left\{\frac{\psi'(S(t))}{\psi(S(t))} - \frac{2\widetilde{N}'(S(t))}{\widetilde{N}(S(t))}\right\}\dot{S}^2(t)\right),\end{aligned} \tag{3.60}$$

holds for the isotropic radius profile $s = S(t)$. □

Now that we know that all photon surfaces except some vertical partial hyperplanes in "most" spacetimes of class $\mathcal{S}$ are spherically symmetric, let us gain a complementary perspective on these by relating spherically symmetric photon surfaces to null geodesics. The following notion of maximality will be useful for this endeavor.

Definition 3.12 (Maximal photon surface). *Let $(\mathbb{R}\times M^n,\mathfrak{g})\in\mathcal{S}$ and let $P^n\hookrightarrow(\mathbb{R}\times M^n,\mathfrak{g})$ be a connected spherically symmetric photon surface. We say that P^n is maximal if P^n does not lie inside a strictly larger connected spherically symmetric photon surface.*

If $\zeta: J\to\mathbb{R}\times M^n$ is a null geodesic in $(\mathbb{R}\times M^n,\mathfrak{g})\in\mathcal{S}$ defined on some interval $J\subseteq\mathbb{R}$, then the *energy* $E:=-\mathfrak{g}_\zeta(\partial_t|_\zeta,\dot\zeta)$ is constant along ζ. Exploiting the spherical symmetry of the spacetime, one can find a set $\{X_i\}_{i=1}^{\frac{(n-1)(n-2)}{2}}$ of Killing vector fields tangential to the orbits of the spherical symmetry, which span (over $\mathbb{R}$) the Killing subalgebra of the spacetime corresponding to spherical symmetry. Now, locally select a linearly independent system $\{X_{i_I}\}_{I=1}^{n-1}$ among these and observe that this gives us

$$[X_{i_I},X_{i_J}]=:\alpha_{IJ}^K X_{i_K} \tag{3.61}$$

for some smooth coefficients α_{IJ}^K satisfying $\alpha_{IJ}^K=-\alpha_{JI}^K$, where $I,J,K=1,\dots,n-1$, and where we have used that the spheres of symmetry are $n-1$-dimensional. Next, we set $\ell_I:=\mathfrak{g}(X_{i_I},\dot\zeta)$, and note that ℓ_I is constant along ζ. We then define the *(total) angular momentum ℓ of ζ* as $\ell:=(G^{IJ}\ell_I\ell_J)^{\frac12}\geq 0$, where (G^{IJ}) is the inverse of $(G_{IJ}):=(\mathfrak{g}_\zeta(X_{i_I}|_\zeta,X_{i_J}|_\zeta))$. Then, ℓ is constant along ζ because, locally, we have

$$\begin{aligned}
\frac{d}{ds}\ell^2&=-G^{IA}G^{JB}\Big(\frac{d}{ds}G_{AB}\Big)\ell_I\ell_J\\
&=-G^{IA}G^{JB}\Big(\mathfrak{g}_\zeta({}^{\mathfrak{g}}\nabla_{\dot\zeta}X_{i_A},X_{i_B}|_\zeta)+\mathfrak{g}_\zeta({}^{\mathfrak{g}}\nabla_{\dot\zeta}X_{i_B},X_{i_A}|_\zeta)\Big)\ell_I\ell_J\\
&=G^{IA}G^{JB}\Big(\mathfrak{g}_\zeta({}^{\mathfrak{g}}\nabla_{X_{i_B}}X_{i_A}\big|_\zeta,\dot\zeta)+\mathfrak{g}_\zeta({}^{\mathfrak{g}}\nabla_{X_{i_A}}X_{i_B}\big|_\zeta,\dot\zeta)\Big)\ell_I\ell_J\\
&=G^{IA}G^{JB}\alpha_{BA}^K\,\mathfrak{g}_\zeta(X_{i_K}|_\zeta,\dot\zeta)\ell_I\ell_J\\
&=G^{IA}G^{JB}\alpha_{BA}^K\ell_I\ell_J\ell_K,
\end{aligned}$$

where s denotes the parameter along ζ and where we have used the Killing property of X_{i_A} and X_{i_B} to get from the second to the third line, (3.61) to get from the third to the fourth, and the definition of ℓ_K to get from the fourth to the last one. As α is antisymmetric in its lower indices, ℓ must be constant along ζ. Moreover, ℓ is independent of the local choice of the linearly independent system $\{X_{i_I}\}_{I=1}^{n-1}$, which can be seen as follows: Suppose that $\{Y_I\}_{I=1}^{n-1}$ is another local linearly independent system of spherical Killing vector fields sharing an open domain with $\{X_{i_I}\}_{I=1}^{n-1}$, set $\widehat{G}_{IJ}:=(\mathfrak{g}_\zeta(Y_I|_\zeta,Y_J|_\zeta))$, let $(\widehat{G}^{IJ})$ be the inverse of $(\widehat{G}_{IJ})$, and set $\widehat{\ell}_I:=\mathfrak{g}(Y_I,\dot\zeta)$ for $I,J=1,\dots,n-1$. Observe that, by linear independence, $Y_I=:T_I^KX_{i_K}$ for a smooth family of invertible matrices (T_I^K) with inverses $((T^{-1})_K^I)$. Then,

$$\begin{aligned}
\widehat{G}_{IJ}&=G_{KL}T_I^KT_J^L,\\
\widehat{\ell}_I&=T_I^K\ell_K,\ \text{and hence,}\\
\widehat{G}^{IJ}\widehat{\ell}_I\widehat{\ell}_J&=G^{AB}(T^{-1})_A^I(T^{-1})_B^JT_I^K\ell_KT_J^L\ell_L=G^{AB}\ell_A\ell_B=\ell^2.
\end{aligned}$$

We will use the following definition of null geodesics generating a hypersurface.

Proposition 3.13. *Let $(\mathbb{R}\times M^n,\mathfrak{g})\in\mathcal{S}$ and let $\zeta:J\to\mathbb{R}\times M^n$ and $\zeta(s)=(t(s),r(s),\xi(s))$ be a (not necessarily maximal) null geodesic defined on some open interval $J\subseteq\mathbb{R}$ with angular momentum ℓ, where $\xi(s)\in\mathbb{S}^{n-1}$. Then, ζ generates the hypersurface H_ζ^n defined as*

$$H_\zeta^n:=\{(t,p)\in\mathbb{R}\times M^n\mid\exists s_*\in J,\xi_*\in\mathbb{S}^{n-1}:t=t(s_*),p=(r(s_*),\xi_*)\},$$

which is a smooth, connected, and spherically symmetric hypersurface in $\mathbb{R}\times M^n$, which is timelike if $\ell>0$ and null if $\ell=0$.

Proof. The claim that H_ζ^n is a smooth spherically symmetric hypersurface is verified by recalling that $r>0$ for all $(t,p)\in\mathbb{R}\times M^n$ so that $\zeta_*\in\mathbb{S}^{n-1}$ is unique and by realizing that $t=t(s)$ is injective along the null geodesic ζ, which shows that s_* is unique. It is connected because the geodesic ζ is defined on an interval.

To show the claims about its causal character, set $e_0=\dot\zeta$ and extend it to all of H_ζ^n by making it invariant under the spherical symmetries. Thus, e_0 is a null tangent vector field to H_ζ^n, orthogonal to each time slice $\{t(s)=\text{const.}\}$ of H_ζ^n. Now, if $\ell=0$, it is easy to see that $\dot\xi=0$ along

ζ and, hence, $e_0 \perp_{\mathfrak{g}} \mathbb{S}^{n-1}$ or, in other words, $\mathfrak{g}(e_0, X) = 0$ for any vector field X tangent to H^n_ζ whence H^n_ζ is null. Otherwise, if $\ell > 0$, we will have $\dot{\xi} \neq 0$ everywhere along ζ, and hence, e_0 will not be $\mathfrak{g}$-orthogonal to $\mathbb{S}^{n-1}$ and, hence, $\mathfrak{g}$ will induce a Lorentzian metric on H^n_ζ. □

With these concepts at hand, we can write down a characterization of spherically symmetric photon surfaces via generating null geodesics.

Proposition 3.14. Let $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$ and let $H^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$ be a connected spherically symmetric timelike hypersurface. Then, H^n is generated by a null geodesic $\zeta : J \to \mathbb{R} \times M^n$ if and only if H^n is a photon surface. Moreover, H^n is a maximal photon surface if and only if any null geodesic $\zeta : J \to \mathbb{R} \times M^n$ generating H^n is maximal.

The umbilicity factor λ of a photon surface P^n is related to the energy E and angular momentum ℓ of its generating null geodesics by $\lambda = \frac{E}{\ell}$.

Proof. First, assume that H^n is generated by a null geodesic $\zeta : J \to \mathbb{R} \times M^n$ so that $H^n = H^n_\zeta$ and observe that H^n must then actually be ruled by the null geodesics arising by rotating ζ around $\mathbb{S}^{n-1}$. As H^n is timelike by assumption, we know from Definition and Proposition 3.13 that the angular momentum ℓ of ζ satisfies $\ell > 0$. Proceeding as above, let $\{e_I\}_{I=1}^{n-1}$ be a local orthonormal system tangent to $\mathbb{S}^{n-1}$ along ζ, and set $e_0 := \dot{\zeta}$ and extend it to all of H^n_ζ by making it invariant under the spherical symmetries so that $\{e_0, e_I\}_{I=1}^{n-1}$ is a local frame along ζ. Writing $\zeta(s) = (t(s), r(s), \xi(s))$ for $s \in J$ as before, we can write

$$\dot{\zeta}(s) = \dot{t}(s)\partial_t + \dot{r}(s)\partial_r + \dot{\xi}(s) \tag{3.62}$$

for $s \in J$ so that ζ being a null curve is equivalent to

$$-f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2|\dot{\xi}|^2_\Omega = 0 \tag{3.63}$$

along ζ. From this and the definition of energy E and angular momentum ℓ, we obtain

$$\dot{t} = \frac{E}{f(r)}, \tag{3.64}$$

$$\dot{r}^2 = E^2 - \frac{\ell^2 f(r)}{r^2}, \tag{3.65}$$

$$|\dot{\xi}|^2_\Omega = \frac{\ell^2}{r^4} \tag{3.66}$$

along ζ. To compute the second fundamental form $\mathfrak{h}$ of $H^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$, let us first compute the outward (growing r) spacelike unit normal η to H^n. By spherical symmetry, η must be a linear combination of ∂_t and ∂_r with no angular contribution. It must also be orthogonal to $e_0 = \dot{\zeta}$. From this, we find

$$\eta = \frac{1}{\ell}\left(\frac{r\dot{r}}{f(r)}\partial_t + Er\partial_r\right), \tag{3.67}$$

where we have used (3.64) and (3.65). For $I = 1, \ldots, n-1$, we find by a direct computation (exploiting spherical symmetry) that

$$^{\mathfrak{g}}\nabla_{e_I}\eta = \frac{E}{\ell}\, e_I, \tag{3.68}$$

and hence,

$$\mathfrak{h}(e_I, e_\beta) = \mathfrak{g}(^{\mathfrak{g}}\nabla_{e_I}\eta, e_\beta) = \frac{E}{\ell}\,\mathfrak{g}(e_I, e_\beta) \tag{3.69}$$

for $I = 1, \ldots, n-1$ and $\beta = 0, \ldots, n-1$. On the other hand, smoothly extending e_0 to a neighborhood of H^n, one finds that

$$\mathfrak{h}(e_0, e_0) = \mathfrak{g}(^{\mathfrak{g}}\nabla_{e_0}\eta, e_0) = -\mathfrak{g}(^{\mathfrak{g}}\nabla_{e_0}e_0, \eta) = 0 \tag{3.70}$$

holds as $e_0 = \dot{\zeta}$ and ζ is a geodesic. Hence, by symmetry of $\mathfrak{h}$ and $\mathfrak{g}$,

$$\mathfrak{h}(e_\alpha, e_\beta) = \frac{E}{\ell}\mathfrak{g}(e_\alpha, e_\beta) \tag{3.71}$$

for $\alpha, \beta = 0, \dots, n-1$ so that H^n is totally umbilic with umbilicity factor $\lambda = \frac{E}{\ell}$.

Conversely, assume that H^n is a spherically symmetric photon surface. Let $q \in H^n$ and let $X \in T_q H^n$ be any null tangent vector. Let $\zeta : J_{\max} \ni 0 \to \mathbb{R} \times M^n$ be the maximal null geodesic with $\zeta(0) = q$, $\dot{\zeta}(0) = X$. As H^n is totally umbilic, ζ must remain tangent to H^n on some maximal open interval $0 \in J \subseteq J_{\max}$ by Proposition 2.2. By spherical symmetry, H^n is, in fact, generated by $\zeta|_J$.

It remains to discuss the claim about maximality, which is a straightforward consequence of the above argument about the choice of the interval $J \subseteq J_{\max}$. □

Remark 3.15. In view of effective one-body dynamics (see Ref. 37), it may be of interest to point out, as a computation shows, that a spherically symmetric photon surface P^n in a static spherically symmetric spacetime with metric of the form

$$\mathfrak{g} = -G dt^2 + \frac{1}{f} dr^2 + r^2\, \Omega,$$

where $G = G(r)$ and $f = f(r)$ are smooth, positive functions on an open interval $I \subseteq (0, \infty)$, will have a constant umbilicity factor λ if and only if $G = \kappa f$ along P^n for some $\kappa > 0$, with $\lambda = \frac{E f(r)}{\ell G(r)}$ characterizing the umbilicity factor along any generating null geodesic with energy E and angular momentum ℓ.

Remark 3.16. Proposition 3.14 allows us to conclude the existence of (maximal) spherically symmetric photon surfaces in spacetimes of class $\mathcal{S}$ from the existence of (maximal) null geodesics.

A different view on spherically symmetric photon surfaces in spacetimes of class $\mathcal{S}$ can be gained by lifting them to the phase space (i.e., to the cotangent bundle). We will end this section by proving a partitioning property of the null section of phase space by maximal photon surfaces and by the so-called maximal principal null hypersurfaces.

Definition 3.17 [(Maximal) Principal null hypersurfaces]. Let $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$, $\zeta : J \to \mathbb{R} \times M^n$ be a null geodesic, and ℓ be its angular momentum. Then, ζ is called a principal null geodesic if $\ell = 0$. A hypersurface H^n_ζ generated by a (maximal) principal null geodesic ζ will be called a (maximal) principal null hypersurface.

Recall that from the Definition and Proposition 3.13, we know that principal null hypersurfaces are spherically symmetric and indeed null. Arguing as in the Proof of Proposition 3.14, one sees that a maximal principal null hypersurface will be maximal in the sense that it is not contained in any strictly larger principal null hypersurface. In particular, if one generating null geodesic is maximal, then all of them are. Finally, principal null hypersurfaces are connected by definition.

With these considerations at hand, let us prove the following partitioning property of the null section of the phase space of any spacetime of class $\mathcal{S}$. To express this, we will canonically lift the null bundles over the involved spherically symmetric photon surfaces and principal null hypersurfaces to the null section of the phase space. Recall that maximal spherically symmetric photon surfaces are connected by definition.

Proposition 3.18. Let $(\mathbb{R} \times M^n, \mathfrak{g}) \in \mathcal{S}$. Then, the null section of the phase space of $(\mathbb{R} \times M^n, \mathfrak{g})$, $\mathcal{N} := \{\omega \in T^*(\mathbb{R} \times M^n) \,|\, \mathfrak{g}(\omega^\#, \omega^\#) = 0\}$, is partitioned by the canonical lifts $\mathcal{N}(P^n) \subset \mathcal{N}$ of the null bundles over all maximal spherically symmetric photon surfaces P^n and the canonical lifts $\mathcal{N}(H^n) \subset \mathcal{N}$ of the null bundles over all maximal principal null hypersurfaces H^n.

Proof. Consider $\omega \in \mathcal{N}$ and let $\zeta : J \ni 0 \to \mathbb{R} \times M^n$ be the unique maximal null geodesic satisfying $\dot{\zeta}(0) = \omega^\#$. Let H^n_ζ be the hypersurface of $\mathbb{R} \times M^n$ generated by ζ, and note that $\omega \in \mathcal{N}(H^n_\zeta)$ holds by construction. By Definition and Propositions 3.13 and 3.14 and Definition 3.17, we know that H^n_ζ is a maximal photon surface if the angular momentum ℓ of ζ is positive and a maximal principal null hypersurface if $\ell = 0$ vanishes. As $\ell \geq 0$, this shows that ω is either contained in at least one canonical lift of a maximal photon surface or contained in at least one canonical lift of a maximal principal null hypersurface. Furthermore, ω cannot lie in the canonical lifts of two different maximal spherically symmetric photon surfaces because these would both be generated by ζ and hence coincide by Proposition 3.14. Finally, ω cannot lie in the canonical lifts of two different maximal principal null hypersurfaces, which can be seen by repeating the arguments in the Proof of Proposition 3.14. □

IV. A RIGIDITY RESULT FOR PHOTON SURFACES WITH EQUIPOTENTIAL TIME SLICES

As discussed in Sec. III (see also Fig. 1), the Schwarzschild spacetime of mass $m > 0$ in $n+1$ dimensions possesses not only the well-known photon sphere at $r = (nm)^{\frac{1}{n-2}}$ but also many other photon surfaces. Except for the planar ones, all these Schwarzschild photon surfaces are spherically symmetric and thus particularly equipotential, as defined in Sec. II. In this section, we will prove the following theorem that can be considered complementary to Corollary 3.9 in the context of static, vacuum, and asymptotically flat spacetimes.

Theorem 4.1. *Let $(\mathfrak{L}^{n+1}, \mathfrak{g})$ be a static, vacuum, and asymptotically isotropic spacetime of mass m. Assume that $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is geodesically complete up to its inner boundary $\partial\mathfrak{L}$, which is assumed to be a (possibly disconnected) photon surface, $\partial\mathfrak{L} =: P^n$. Assume, in addition, that P^n is equipotential, outward directed, and has compact time slices $\Sigma^{n-1}(t) = P^n \cap M^n(t)$. Then, $(\mathfrak{L}^{n+1}, \mathfrak{g})$ is isometric to a suitable piece of the Schwarzschild spacetime of mass m, and in fact, $m > 0$. In particular, P^n is connected and is (necessarily) a spherically symmetric photon surface in Schwarzschild spacetime.*

The proof relies on the following theorem by the first author.

Theorem 4.2 (Ref. 23). *Assume $n \geq 3$ and let M^n be a smooth connected n-dimensional manifold with non-empty, possibly disconnected, smooth compact inner boundary $\partial M = \dot{\cup}_{i=1}^{I} \Sigma_i^{n-1}$. Let g be a smooth Riemannian metric on M^n. Assume that (M^n, g) has a non-negative scalar curvature,*

$$R \geq 0,$$

and that it is geodesically complete up to its inner boundary ∂M. Assume, in addition, that (M^n, g) is asymptotically isotropic with one end of mass $m \in \mathbb{R}$. Assume that the inner boundary ∂M is umbilic in (M^n, g) and that each component Σ_i^{n-1} has constant mean curvature H_i with respect to the outward pointing unit normal ν_i. Furthermore, assume that there exists a function $u : M^n \to \mathbb{R}$ with $u > 0$ away from ∂M, which is smooth and harmonic on (M^n, g),

$$\triangle_g u = 0.$$

We ask that u is asymptotically isotropic of the same mass m such that $u|_{\Sigma_i^{n-1}} =: u_i$ and the normal derivative of u across Σ_i^{n-1}, $\nu_i(u)|_{\Sigma_i^{n-1}} =: \nu(u)_i$, are constant on each Σ_i^{n-1}. Finally, we assume that for each $i = 1, \ldots, I$, we are either in the semi-static horizon case,

$$H_i = 0, \quad u_i = 0, \quad \nu(u)_i \neq 0, \tag{4.1}$$

or in the true CMC *case $H_i > 0$, $u_i > 0$, and there exists $c_i > \frac{n-2}{n-1}$ such that*

$$R_{\sigma_i} = c_i H_i^2, \tag{4.2}$$

$$2\nu(u)_i = \left(c_i - \frac{n-2}{n-1}\right) H_i u_i, \tag{4.3}$$

where R_{σ_i} denotes the scalar curvature of Σ_i^{n-1} with respect to its induced metric σ_i.

Then, $m > 0$ and (M^n, g) is isometric to a suitable piece $(\widetilde{M}_m^n \backslash B_S(0), \widetilde{g}_m)$ of the (isotropic) Schwarzschild manifold of mass m with $S \geq s_m$. Moreover, u coincides with the restriction of $\widetilde{u}_m$ (up to the isometry), and the isometry is smooth.

Remark 4.3 (Generalization). Our Proof of Theorem 4.1 makes use of the static vacuum Einstein equations (2.4), (2.5), and (2.6). In fact, as we will see in the proof in the following, it is sufficient to ask that the vacuum Einstein equations hold in a neighborhood of P^n; outside this neighborhood, it suffices that $\triangle_g N = 0$ and that the dominant energy condition $R \geq 0$ holds.

Remark 4.4 (Multiple ends). As Theorem 4.2 generalizes to multiple ends (see Ref. 23), Theorem 4.1 also readily applies in the case of multiple ends satisfying the decay conditions (2.14) and (2.17) with potentially different masses m_i in each end E_i. Note that, in each end E_i, it is necessary that both g_{ij} and N have the same mass m_i in their expansions.

Remark 4.5 [Discussion of $\eta(N) > 0$]. *The assumption that P^n is outward directed, $\eta(N) > 0$ (hence $dN \neq 0$) along P^n, can be removed if instead, one assumes that $m > 0$ a priori and that P^n is connected. Using the Laplace equation $\triangle_g N = 0$ and the divergence theorem as well as the asymptotics (2.14) and (2.17), one computes*

$$\frac{1}{\omega_{n-1}} \int_{\Sigma^{n-1}} \nu(N)\, dA = m > 0,$$

where $\Sigma^{n-1} := P^n \cap \{t = const\}$ and ω_{n-1} is the volume of $(\mathbb{S}^{n-1}, \Omega)$ (see Definition 4.2.1 of Ref. 7 for the $n = 3$ case); the argument is identical in higher dimensions. From this and connectedness of Σ^{n-1}, we can deduce that $\nu(N) > 0$ and thus, particularly, $\eta(N) > 0$ at least in an open subset of Σ^{n-1}. However, we will see in the Proof of Theorem 4.1 that this necessarily implies that $\nu(N) \equiv \text{const} > 0$ in this open neighborhood (noting that all computations performed there are purely local). As Σ^{n-1} is connected, we obtain $\nu(N) \equiv \text{const} > 0$, and thus, in particular, $\eta(N) > 0$ everywhere on Σ^{n-1} [see Eq. (4.6) in the following].

Proof. We write $M^n(t)$ for the time slice $\{t\} \times M$ (cf. Remark 2.4) and consider each connected component P_i^n, $i = 1, \dots, I$ of P^n separately. For the component of P_i^n under consideration, let $\Sigma_i^{n-1}(t) := P_i^n \cap M^n(t)$. We will drop the explicit reference to i in the following and only start reusing it toward the end of the proof, where we bring in global arguments.

Let ν denote the outward unit normal to $\Sigma^{n-1}(t) \hookrightarrow (M^n(t), g)$, pointing to the asymptotically isotropic end. Let η denote the outward unit normal of $P^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$. As $N =: u(t)$ on $\Sigma^{n-1}(t)$ and because we assumed $\eta(N) > 0$ and hence $\nu(N) > 0$ [see (4.6) below] on P^n, we have

$$\nu = \frac{\operatorname{grad} N}{|\operatorname{grad} N|} = \frac{\operatorname{grad} N}{|dN|} = \frac{\operatorname{grad} N}{\nu(N)}. \tag{4.4}$$

Now, let $\mu(s) = (s, x(s))$ be a curve in P^n, i.e., $N \circ \mu(s) = u(s)$. This implies by chain rule that $dN(\dot{\mu}) = \dot{u}$. If $\dot{\mu}(t) \perp \Sigma^{n-1}(t)$, the tangent vector of μ can be computed explicitly as

$$Z := \dot{\mu} = \partial_t + \dot{x} = \partial_t + \frac{\dot{u}}{\nu(N)}\nu \in \Gamma(TP^n). \tag{4.5}$$

Expressed in words, Z is the vector field going "straight up" along P^n.

The explicit formula (4.5) for Z allows us to explicitly compute the spacetime unit normal η to P^n too: It has to be perpendicular to Z and its projection onto M^n has to be proportional to ν. From this, we find

$$\eta = \frac{\nu + \frac{\dot{u}}{u^2\nu(N)^2}\partial_t}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}. \tag{4.6}$$

From umbilicity of $P^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$, it follows that the corresponding second fundamental form $\mathfrak{h}$ of $P^n \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$ satisfies

$$\mathfrak{h} = \frac{1}{n}\mathfrak{H}\, p, \tag{4.7}$$

where p is the induced metric on P^n and $\mathfrak{H} := tr_p \mathfrak{h}$. From Proposition 3.3 in Ref. 9, we know that $\mathfrak{H} \equiv \text{const}$. Equations (4.7) implies particularly that, for any tangent vector fields $X, Y \in \Gamma(T\Sigma^{n-1}(t))$, we have

$$\mathfrak{h}(X, Y) = \frac{1}{n}\mathfrak{H}\,\sigma(X, Y), \tag{4.8}$$

where σ denotes the induced metric on $\Sigma^{n-1}(t)$. Now, extend X, Y arbitrarily smoothly along P^n such that they remain tangent to Σ_t^{n-1}. We compute

$$\begin{aligned}
\mathfrak{h}(X, Y) &= -\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X Y, \eta\right) \\
&= -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}\,\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X Y, \nu + \frac{\dot{u}}{u^2\nu(N)^2}\partial_t\right) \\
&= -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}\left\{\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X Y, \nu\right) + \frac{\dot{u}}{u\nu(N)^2}\,\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X Y, \frac{1}{u}\partial_t\right)\right\} \\
&= -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}\left\{\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X Y, \nu\right) - \frac{\dot{u}}{u\nu(N)^2}K(X, Y)\right\} \\
&\overset{K=0}{=} -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}\,\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X Y, \nu\right) \\
&\overset{\mathfrak{g}\ \text{static}}{=} -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}\,g\left({}^{g}\nabla_X Y, \nu\right) \\
&= \frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}}\,h(X, Y),
\end{aligned} \tag{4.9}$$

where $K = 0$ denotes the second fundamental form of $M^n(t) \hookrightarrow (\mathbb{R} \times M^n, \mathfrak{g})$ and h denotes the second fundamental form of $\Sigma^{n-1}(t) \hookrightarrow (\{t\} \times M^n, g)$. In particular, $\Sigma^{n-1}(t) \hookrightarrow (\{t\} \times M^n, g)$ is umbilic and its mean curvature H inside M^n can be computed as

$$H = \frac{n-1}{n}\,\mathfrak{H}\sqrt{1-\frac{\dot{u}^2}{u^2 v(N)^2}} \tag{4.10}$$

when combining (4.8) with (4.9). Of course, then $h = \frac{1}{n-1}H\sigma$. We will now proceed to show that H and $v(N)$ are constant for each fixed t. Consider first the contracted Codazzi equation for $\Sigma^{n-1}(t) \hookrightarrow (\{t\}\times M^n, g)$. It gives

$$\mathrm{Ric}(X,\nu) = \frac{n-2}{n-1}X(H). \tag{4.11}$$

On the other hand, using the static equation gives

$$\begin{aligned} X(v(N)) &\overset{N\equiv u(t)}{=} X(v(N)) - {}^{g}\nabla_X v(N) \\ &= {}^{g}\nabla^2 N(X,\nu) \\ &= N\mathrm{Ric}(X,\nu) \\ &\overset{(4.11)}{=} \frac{n-2}{n-1}\,u\,X(H). \end{aligned} \tag{4.12}$$

Furthermore, (4.10) allows us to compute

$$\begin{aligned} X(H) &= \frac{(n-1)\,\mathfrak{H}}{n\sqrt{1-\frac{\dot{u}^2}{u^2 v(N)^2}}}\,\frac{\dot{u}^2}{u^2\,v(N)^3}\,X(v(N)) \\ &\overset{(4.12)}{=} \frac{(n-2)\,\dot{u}^2\,\mathfrak{H}}{nu\,v(N)^3\sqrt{1-\frac{\dot{u}^2}{u^2 v(N)^2}}}\,X(H). \end{aligned} \tag{4.13}$$

Assume that $X(H) \neq 0$ in some open subset $U \subset \Sigma^{n-1}(t)$. Then, in U, we have

$$\begin{aligned} & nu\,v(N)^3\sqrt{1-\frac{\dot{u}^2}{u^2 v(N)^2}} = (n-2)\,\dot{u}^2\,\mathfrak{H} \\ \Leftrightarrow\quad & v(N)^6\left(1-\frac{\dot{u}^2}{u^2 v(N)^2}\right) = \frac{(n-2)^2\,\dot{u}^4\,\mathfrak{H}^2}{n^2\,u^2} \\ \Leftrightarrow\quad & v(N)^6 - \frac{\dot{u}^2}{u^2}\,v(N)^4 - \frac{(n-2)^2\,\dot{u}^4\,\mathfrak{h}^2}{n^2\,u^2} = 0. \end{aligned}$$

This is a polynomial equation for $v(N)$ with coefficients that only depend on t. As a consequence, $v(N)$ has to be constant in U. However, from (4.12), we know that then, also H has to be constant in U, a contradiction to $X(H) \neq 0$ in U. Thus, H and, by (4.12), also $v(N)$ are constants along $\Sigma^{n-1}(t)$ and only depend on t. From now on, we will drop the explicit reference to t and also go back to using N instead of $u(t)$ as the remaining part of the proof applies to each t separately.

Thus, each $\Sigma^{n-1} = \Sigma^{n-1}(t)$ is an umbilic, CMC, equipotential surface in M^n with $v(N)$ being constant too. From the usual decomposition of the Laplacian on functions and the static vacuum Eq. (2.6), we find

$$0 = \triangle_g N = \triangle_\sigma N + N\mathrm{Ric}(\nu,\nu) + Hv(N) = N\mathrm{Ric}(\nu,\nu) + Hv(N)$$

so that $\mathrm{Ric}(\nu,\nu) = -\frac{Hv(N)}{N}$ must also be constant along Σ^{n-1}.

Plugging this into the contracted Gauß equation and using $R = 0$, we obtain

$$\begin{aligned} & -2\mathrm{Ric}(\nu,\nu) = R_\sigma - \frac{n-2}{n-1}H^2 \\ \Leftrightarrow\quad & R_\sigma = \frac{2Hv(N)}{N} + \frac{n-2}{n-1}H^2. \end{aligned} \tag{4.14}$$

This shows that (Σ^{n-1}, σ) also has constant scalar curvature. Now, define the constant $c > \frac{n-2}{n-1}$ by

$$c := \frac{n-2}{n-1} + \frac{2v(N)}{NH}. \tag{4.15}$$

Together with (4.14), this definition of c ensures that

$$R_\sigma = cH^2, \tag{4.16}$$

$$2v(N) = \left(c - \frac{n-2}{n-1}\right)HN. \tag{4.17}$$

Let us summarize as follows: Fix t and keep dropping the explicit reference to it. Then, each component of $(\Sigma^{n-1}, \sigma) \hookrightarrow (M^n, g)$ is umbilic, CMC, equipotential and has constant scalar curvature and constant $v(N)$, and all these constants together satisfy Eqs. (4.16) and (4.17) with constant c given by (4.15), which is potentially different for each component of Σ^{n-1}. Recall the assumption that (M^n, g) is geodesically complete up to its inner boundary, so in particular, $(M^n \backslash K, g)$ is geodesically complete up to Σ^{n-1}, where K is the compact set such that $\Sigma^{n-1} = \partial(M^n \backslash K)$. Moreover, $(M^n \backslash K, g)$ satisfies the static vacuum equations. Altogether, these facts ensure that Theorem 4.2 applies. Thus, $(M^n \backslash K, g)$ is isometric to a spherically symmetric piece of the spatial Schwarzschild manifold of mass m given by the asymptotics (2.14) and (2.17), and N corresponds to the Schwarzschild lapse function of the same mass m under this isometry. The area radius r of the inner boundary Σ^{n-1} in the spatial Schwarzschild manifold is determined by $R_\sigma = \frac{(n-1)(n-2)}{r^2}$.

Thus, recalling the dependence on t, the manifold $[\{t\} \times (M^n \backslash K(t)), g]$ outside the photon surface time slice $\Sigma^{n-1}(t)$ is isometric to a piece of the spatial Schwarzschild manifold of mass m with inner boundary area radius $r(t)$. In particular, $\Sigma^{n-1}(t)$ is a connected sphere and we find $m > 0$ (as per Theorem 1.1 in Ref. 23). Recombining the time slices $\Sigma^{n-1}(t)$ to the photon surface P^n, this shows that the part of the spacetime $(\mathbb{R} \times M^n, \mathfrak{g})$ lying outside the photon surface P^n is isometric to a piece of the Schwarzschild spacetime of mass m and $m > 0$ necessarily. Moreover, P^n is connected and its isometric image in the Schwarzschild spacetime is spherically symmetric with the radius profile $r(t)$. □

We point out in connection with Remarks 4.4 and 4.3 that the assumptions of Theorem 4.2 keep being met if we start with several ends and the static vacuum equations only holding near Σ^{n-1}, with $\triangle_g N = 0$ everywhere in M^n as all the above computations and arguments were purely local near P^n.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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