On the topology of black holes and beyond

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- What about higher dimensional black holes?

Outline of the talk

- Begin with some basic background on General Relativity and black holes.
- Discuss Hawking's theorem on the topology of black holes in $3+1$ dimensions (and its connection to black hole uniqueness)
- Present a generalization of Hawking's Theorem to higher dimensions (work with R. Schoen).
- Describe some recent work (with M. Eichmair and D. Pollack) about the topology of the region of space exterior to black holes. (This is connected to the notion of *topological censorship*.)
- These results rely on the theory of *marginally outer trapped surfaces*, which are natural spacetime analogues of minimal surfaces in Riemannian geometry.

In GR the space of events is represented by a Lorentzian manifold, i.e. smooth manifold M^{n+1} equipped with a metric g of Lorentzian signature. Thus, at each $p \in M$,

$$g : T_pM \times T_pM \to \mathbb{R}$$

is a scalar product of signature $(-1, +1, ..., +1)$, With respect to an o.n. basis $\{e_0, e_1, ..., e_n\}$, as a matrix,

$$[g(e_i, e_j)] = \mathrm{diag}(-1, +1, ..., +1)\,.$$

Example: Minkowski space, the spacetime of Special Relativity. Minkowski space is $\mathbb{R}^{n+1}$, equipped with the Minkowski metric η: For vectors $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ at p,

$$\eta(X, Y) = -X^0 Y^0 + \sum_{i=1}^{n} X^i Y^i\,.$$

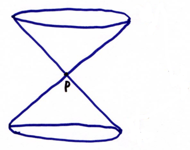
Thus, a Lorentzian manifold is locally modeled on Minkowski space.

Aspects of GR

At any point p in a Lorentzian manifold M, we have a classification of vectors $X \in T_pM$ as follows,

$$X \text{ is } \begin{cases} \text{timelike} & \text{if } g(X,X) < 0 \\ \text{null} & \text{if } g(X,X) = 0 \\ \text{spacelike} & \text{if } g(X,X) > 0 \ . \end{cases}$$

The set of null vectors at p form a double cone in T_pM:



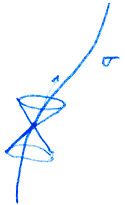
If the assignment of a past and future cone can be made in a continuous manner, say that M is time-orientable.

$$\boxed{\text{spacetime} = \text{time-oriented Lorentian manifold}}$$

Aspects of GR

Curves.

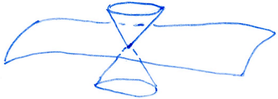
A curve $t \to \sigma(t)$ is timelike provide each tangent vector $\sigma'(t)$ is timelike.



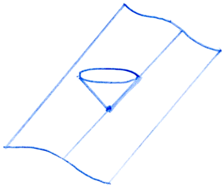
Similarly for null, spacelike, causal curves.

Hypersurfaces.

A spacelike hypersurface is a hypersurface all of whose tangent vectors are spacelike.



A null hypersurface is a hypersurface such that the null cone at each of its points is tangent to it.



Aspects of GR

- Let ∇ be the Levi-Civita connection associated to the Lorentz metric g. On a coordinate neighborhood M, ∇ is determined by the Christoffel symbols,
$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k\,\partial_k, \qquad (\partial_i = \frac{\partial}{\partial x^i}, etc.)$$
- *Geodesics.* Geodesics are curves $t \to \sigma(t)$ of zero acceleration,
$$\nabla_{\sigma'(t)}\sigma'(t) = 0\,.$$
Timelike geodesics correspond to *free falling* observers.
- The Riemann curvature tensor is defined by,
$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$
The components ${R^\ell}_{kij}$ are determined by,
$$R(\partial_i,\partial_j)\partial_k = \sum_\ell {R^\ell}_{kij}\partial_\ell$$
- The Ricci tensor Ric and scalar curvature R are obtained by taking traces,
$$R_{ij} = \sum_\ell {R^\ell}_{i\ell j} \quad \text{and} \quad R = \sum_{i,j} g^{ij}R_{ij}$$

Aspects of GR

The Einstein equations are given by:

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij} \,,$$

where T_{ij} is the energy-momentum tensor.

"Mass tells space-time how to curve, and space-time tells mass how to move"

Comments:

- Einstein equations are a system of second order quasi-linear equations for the g_{ij}'s. May be viewed as a generalization of Poisson's equation in Newtonian gravity.
- The *vacuum* Einstein equations are obtained by setting $T_{ij} = 0$. Equivalent to setting $R_{ij} = 0$.
- A spacetime (M, g) which satisfies the Einstein equations is said to obey the dominant energy condition provided the energy-momentum tensor satisfies,

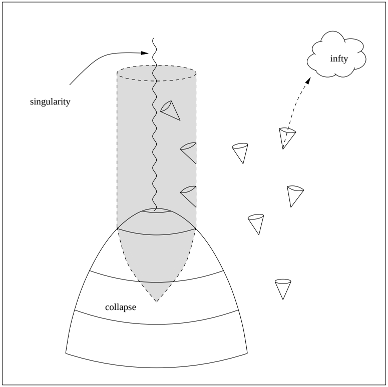
$$T(X, Y) = \sum_{i,j} T_{ij} X^i Y^j \geq 0$$

for all future directed causal vectors X, Y.

Black holes

Black holes are certainly one of the most remarkable predictions of General Relativity.

The following depicts the process of gravitational collapse and formation of a black hole.



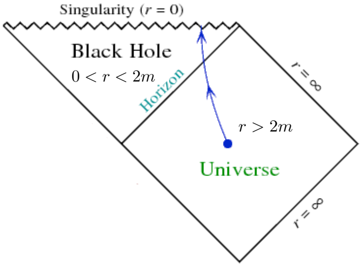
The shaded region is the black hole region. The boundary of this region is the black hole event horizon.

Black holes

Ex. The Schwarzshild solution (1916) Static (time-independent, nonrotating) spherically symmetric, vacuum solution to the Einstein equations.

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The region $0 < r < 2m$ is the black hole region; $r = 2m$ corresponds to the event horizon. This metric represents the region outside a (collapsing) spherically symmetric star.



Ex. The Kerr solution (1963) Stationary (time-independent, rotating), axisymmetric, vacuum solution.

$$\begin{aligned} \mathrm{d}s^2 \;&=\; -\left[1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right]\mathrm{d}t^2 - \frac{4mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\,\mathrm{d}t\,\mathrm{d}\phi \\ &\quad + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}\right]\mathrm{d}r^2 + (r^2 + a^2\cos^2\theta)\,\mathrm{d}\theta^2 \\ &\quad + \left[r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right]\sin^2\theta\,\mathrm{d}\phi^2. \end{aligned}$$

The Kerr solution is determined by two parameters: mass m and angular momentum parameter a.

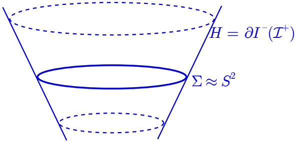
It is believed that “true” astrophysical black holes “settle down” to a Kerr solution. This belief is largely based on results that establish the uniqueness of Kerr among all stationary, asymptotically flat (AF) vacuum solutions of the Einstein equations.

Black holes

A basic step in the proof of the uniqueness of the Kerr solution is Hawking's theorem on the topology of black holes in $3+1$ dimensions.

Theorem (Hawking's black hole topology theorem)

Suppose (M, g) *is a* $(3+1)$*-dimensional AF stationary black hole spacetime obeying the DEC. Then cross sections of the event horizon are topologically 2-spheres.*



Comment on the proof: The proof is variational in nature. Using the Gauss-Bonnet theorem, it is shown that if Σ has genus ≥ 1 then Σ can be deformed outward to an outer trapped surface.

Black holes in higher dimensional spacetimes

- String theory, and various related developments (e.g., the AdS/CFT correspondence, braneword scenarios, entropy calculations) have generated a great deal of interest in gravity in higher dimensions, and in particular, in higher dimensional black holes.
- One of the first questions to arise was:

  Does black hole uniqueness hold in higher dimensions?

- With impetus coming from the development of string theory, in 1986, Myers and Perry constructed natural higher dimensional generalizations of the Kerr solution, which, in particular, have spherical horizon topology.

Black holes in higher dimensional spacetimes

- But in 2002, Emparan and Reall discovered a remarkable example of a $4+1$ dimensional AF stationary vacuum black hole spacetime with horizon topology $S^2 \times S^1$ (the black ring):

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A Rotating Black Ring Solution in Five Dimensions

Roberto Emparan[1,*] and Harvey S. Reall[2]
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(Received 8 November 2001; published 21 February 2002)

Thus in higher dimensions, black hole uniqueness does not hold and horizon topology need not be spherical.

This caused a great surge of activity in the study of higher dimensional black holes.

- *Question:* What horizon topologies are allowed in higher dimensions? What restrictions are there?

Marginally Outer Trapped Surfaces

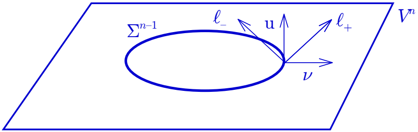
Want to describe a generalization of Hawking's theorem to higher dimensions. This will be based on properties of marginally outer trapped surfaces (MOTSs).

$$(M^{n+1}, g) = \text{spacetime of dimension n+1, } n \geq 3$$

$$(V^n, h, K) = \text{initial data set in } (M^{n+1}, g)$$

$$\Sigma^{n-1} = \text{closed 2-sided hypersurface in } V^n$$

Σ admits a smooth unit normal field ν in V.



$\ell_+ = u + \nu$ f.d. outward null normal

$\ell_- = u - \nu$ f.d. inward null normal

MOTSs

Null second fundamental forms: χ_+, χ_-

$$\chi_\pm(X,Y) = g(\nabla_X \ell_\pm, Y) \qquad X, Y \in T_p\Sigma$$

Null expansion scalars: θ_+, θ_-

$$\theta_\pm = \mathrm{tr}_\gamma \chi_\pm = \gamma^{AB}(\chi_\pm)_{AB} = \mathrm{div}_\Sigma \ell_\pm$$

Physically, θ_+ measures the divergence of the outgoing light rays from Σ.

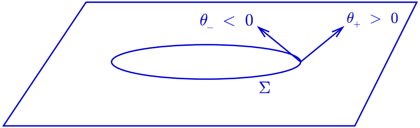
In terms of initial data (V^n, h, K),

$$\theta_\pm = \mathrm{tr}_\gamma K \pm H$$

where $H =$ mean curvature of Σ within V.

MOTSs

For round spheres in Euclidean slices in Minkowski space (and, more generally, large "radial" spheres in AF spacelike hypersurfaces),



However, in a strong gravitational field one can have both,

$$\theta_- < 0 \qquad \text{and} \qquad \theta_+ < 0\,,$$

in which case Σ is called a trapped surface.

Under appropriate energy and causality conditions, the occurrence of a trapped surface signals the onset of gravitational collapse and the formation of a black hole (Penrose, Hawking).

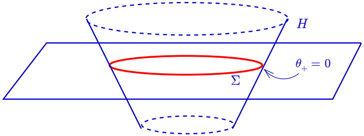
MOTSs

Focusing attention on the outward null normal only,

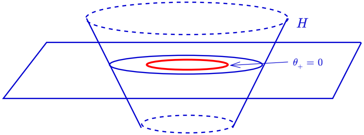
- If $\theta_+ < 0$ - we say Σ is outer trapped
- If $\theta_+ = 0$ - we say Σ is a marginally outer trapped surface (MOTS)

MOTSs arise naturally in several situations.

- In stationary black hole spacetimes - cross sections of the event horizon are MOTS.

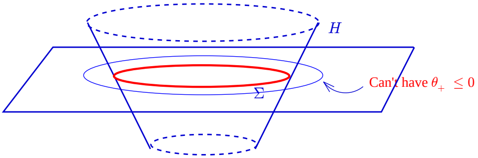


- In dynamical black hole spacetimes - MOTS typically occur inside the event horizon:



Generalization of Hawking's theorem

- **Definition** A MOTS Σ in (V, h, K) is said to be outermost provided there are no outer trapped ($\theta_+ < 0$) or marginally outer trapped ($\theta_+ = 0$) surfaces outside of and homologous to Σ.
- **Fact:** Cross sections of the event horizon in AF stationary black hole spacetimes obeying the DEC are outermost MOTSs.



- More generally, outermost MOTSs can arise as the boundary of the "trapped region" (Andersson and Metzger, Eichmair).

Generalization of Hawking's theorem

Theorem (G. and Schoen, 2006)

Let (V^n, g, K), $n \geq 3$, be an initial data set in a spacetime obeying the DEC. If Σ^{n-1} is an outermost MOTS in V^n then (apart from certain exceptional circumstances) Σ^{n-1} must be of positive Yamabe type.

Comments:

- Σ is of positive Yamabe type means that Σ admits a metric of positive scalar curvature.
- Exceptional circumstances: Various geometric quantities vanish, e.g. $\chi_+ = 0$, $T(u, \ell_+)|_\Sigma = 0$, $\mathrm{Ric}_\Sigma = 0$.

Thus, apart from these exceptional circumstances, Σ is of positive Yamabe type.

Σ being positive Yamabe implies many well-known restrictions on the topology.

Some topological restrictions

- $\dim \Sigma = 2$ ($\dim M = 3 + 1$): Then $\Sigma \approx S^2$ by Gauss-Bonnet, and one recovers Hawking's theorem.

- $\dim \Sigma = 3$ ($\dim M = 4 + 1$): We have,

  Theorem (Gromov-Lawson, Schoen-Yau) If Σ is a closed orientable 3-manifold of positive Yamabe type then Σ must be diffeomorphic to:

  - a spherical space, or
  - $S^2 \times S^1$, or
  - a connected sum of the above two types.

  Comment on the proof: Apply the prime decomposition theorem, and use the fact that since Σ is positive Yamabe, it cannot contain any $K(\pi, 1)$'s in its prime decomposition.

  Thus, the basic horizon topologies in $\dim \Sigma = 3$ case are S^3 and $S^2 \times S^1$.

Meta-comments on the proof:

- MOTSs admit a notion of stability (Andersson, Mars, Simon) somewhat analogous to that for minimal surfaces.
- Variations $\delta\theta_+$ of θ_+ can be expressed in terms of the MOTS stability operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ (2nd order linear elliptic)
- Σ outermost $\implies$ Σ is stable i.e., $\lambda_1(L) \geq 0$.
- Key step: $\lambda_1(L_0) \geq \lambda_1(L)$, where L_0 is a certain simplified "symmetric version" of L. Thus, $\lambda_1(L_0) \geq 0$.
- This, in essence, reduces the situation to the time-symmetric (Riemannian) case where known methods apply (Schoen-Yau, 1979).
- Make the conformal change: $\tilde{h} = \phi^{\frac{2}{n-2}} h$, where ϕ is a positive eigenfunction corresponding to $\lambda_1(L_0)$. A computation shows $S(\tilde{h}) \geq 0$.

The borderline case

A drawback of the theorem is that it allows certain possibilities that one would like to rule out. E.g., the theorem does not rule out the possibility of a vacuum black hole spacetime with toroidal horizon topology.

In subsequent work I was able to rule out such possibilities.

Theorem (G., 2008)

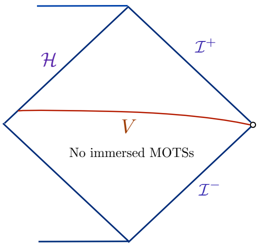
Let (V^n, g, K), $n \geq 3$, be an initial data set in a spacetime obeying the DEC. If a MOTS Σ^{n-1} is not of positive Yamabe type then it cannot be outermost.

We actually prove a rigidity result: An outer neighborhood of Σ must be foliated by MOTSs.

The topology of the region exterior to black holes

According to the Principle of Topological Censorship, the region exterior to black holes (and white holes) should be simple.

An aim in our recent work with M. Eichmair and D. Pollack was to establish a result supportive of this principle at the initial data level.



- Should think of the initial data manifold V as representing an asymptotically flat spacelike slice in the DOC whose boundary ∂V corresponds to a cross section of the event horizon.
- At the initial data level, we represent this cross section by a MOTS.
- We assume there are no (immersed) MOTSs in $V \setminus \partial V$.

The topology of the region exterior to black holes

Theorem (Eichmair, G., Pollack)

Let (V, h, K) be a 3-dimensional asymptotically flat initial data set such that V is a manifold-with-boundary, whose boundary ∂V is a compact MOTS. If there are no immersed MOTS in $V \setminus \partial V$, then V is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls.

Remarks

- The proof makes use of powerful existence results for MOTSs (Schoen, Andersson and Metzger, Eichmair).
- The proof also makes use of an important consequence of geometrization, namely that the fundamental group of every closed 3-manifold is residually finite.
- Dominant energy condition not required!
- MOTSs detect nontrivial topology (somewhat reminiscent of how minimal surfaces are used in Riemannian geometry to detect nontrivial topology, cf., Meeks-Simon-Yau).

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