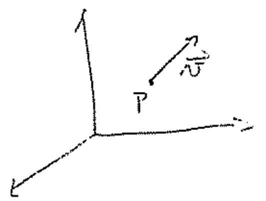
Chapter 6. Geodesics and the Gauss-Bonnet Theorem

Geodesics in surfaces

Straight lines in $\mathbb{R}^2$ or $\mathbb{R}^3$

1) Kinematical description: lines are curves of zero acceleration



$$\sigma = p + tv$$

$$\frac{d\sigma}{dt} = v$$

$$\frac{d^2\sigma}{dt^2} = 0$$

2) lines are curves of zero curvature

$s \to \sigma(s)$

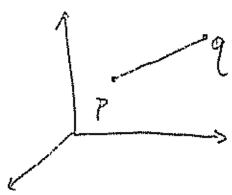
$T = \frac{d\sigma}{ds}$

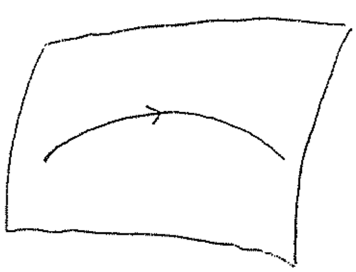
$\kappa = |T'| \qquad \left(' = \frac{d}{ds} \right)$

$\kappa = |\sigma''|$

$\kappa = 0 \Rightarrow \sigma'' = 0$

3) lines are shortest curves





$t \to \sigma(t)$

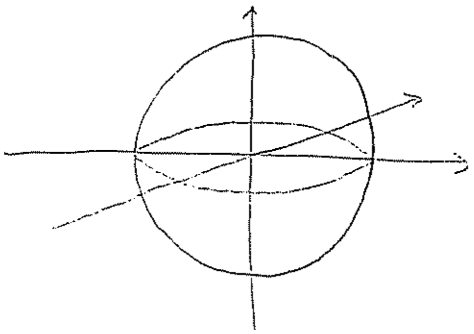
What does it mean for this curve to be a geodesic?

$\frac{d^2\sigma}{dt^2} = 0$ can't be our definition because great circles on the sphere are geodesics, but they have acceleration

$\frac{d^2\sigma}{dt^2} \perp M$ at each point is the correct idea

Def: A geodesic in M is a curve $t \to \sigma(t)$ in M s.t. $\frac{d^2\sigma}{dt^2} \perp M$ at each point of σ

ex. $M = S^2 : x^2+y^2+z^2=1$



Equator: $x^2+y^2=1,\ z=0$

$x = \cos t$

$y = \sin t$

$z = 0$

$\sigma(t) = (\cos t, \sin t, 0)$

$\frac{d\sigma}{dt} = (-\sin t, \cos t, 0)$

$\frac{d^2\sigma}{dt^2} = (-\cos t, -\sin t, 0)$

$\frac{d^2\sigma}{dt^2} = -\sigma \perp S^2$

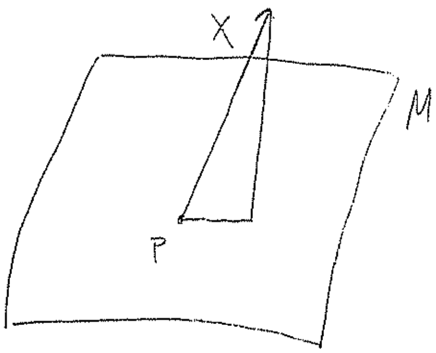
radius vector

$\frac{d^2\sigma}{dt^2} \perp S^2$

$\Rightarrow \sigma(t) = (\cos t, \sin t, 0)$ geodesic

11/15-3

Def (Rephrased):

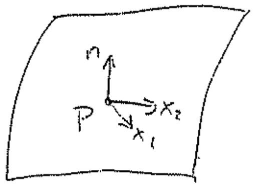


$$X = X^T + X^\perp$$

(X can be decomposed into two vectors, one tangential & one perpendicular to the plane)

$\tan\ T_p\mathbb{R}^3 \longrightarrow T_pM$ (tan is just an operation)

$\tan(X) = X^T$



$$\{\bar{x}_1, \bar{x}_2, n\}$$

$$X = X^1\bar{x}_1 + X^2\bar{x}_2 + X^3 n$$

$$\tan X = X^1\bar{x}_1 + X^2\bar{x}_2$$

Covariant acceleration of motion of $t \to \sigma(t)$ in M:

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right)$$ (this is a projection of the acceleration)

Fact: $t \to \sigma(t)$ is a geodesic iff its covariant acceleration vanishes, i.e., iff $\underbrace{\frac{D}{dt}\frac{d\sigma}{dt} = 0}_{\text{geodesic eqn}}$

$$\frac{D}{dt}\frac{d\sigma}{dt} = 0 \Leftrightarrow \tan\left(\frac{d^2\sigma}{dt^2}\right) = 0 \Longleftrightarrow \frac{d^2\sigma}{dt^2} \perp M$$

Claim A geodesic $t \to \sigma(t)$ in M is necessarily a curve of constant speed

$$\left|\frac{d\sigma}{dt}\right| = \text{constant}$$

Proof: $\left|\frac{d\sigma}{dt}\right| = \sqrt{\left\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \right\rangle}$

WTS: $\left\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \right\rangle = \text{constant}$

$$\frac{d}{dt}\left\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \right\rangle = \left\langle \frac{d^2\sigma}{dt^2}, \frac{d\sigma}{dt} \right\rangle + \left\langle \frac{d\sigma}{dt}, \frac{d^2\sigma}{dt^2} \right\rangle = 2\left\langle \frac{d^2\sigma}{dt^2}, \frac{d\sigma}{dt} \right\rangle = 0$$

because $\frac{d^2\sigma}{dt^2}$ is $\perp$ to M & $\frac{d\sigma}{dt}$ is tangential to M

$$\Rightarrow \left|\frac{d\sigma}{dt}\right| = c = \frac{ds}{dt} \Rightarrow s = ct$$

a) $c = 0$

$$\left|\frac{d\sigma}{dt}\right| \equiv 0 \Rightarrow \frac{d\sigma}{dt} \equiv 0 \Rightarrow \sigma(t) = p \quad \text{(trivial geodesic)}$$

b) $c \neq 0$ $(c > 0)$

$t \longrightarrow \sigma(t)$ regular curve

$s \longrightarrow \sigma(s)$

$$\frac{d\sigma}{dt} = \frac{d\sigma}{ds}\frac{ds}{dt} = c\frac{d\sigma}{ds}$$

$$\frac{d\sigma}{dt} = c\frac{d\sigma}{ds}$$

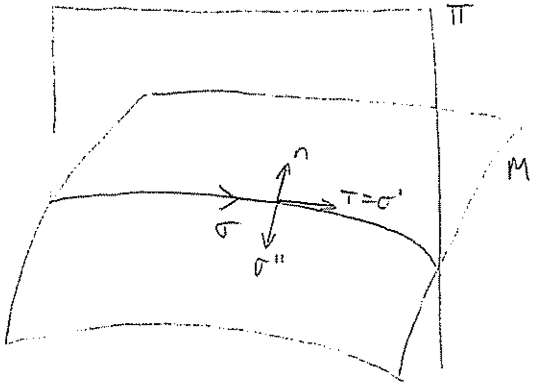
$$\frac{d^2\sigma}{ds^2} = c^2\frac{d^2\sigma}{ds^2}$$

Fact: A unit speed curve $s \to \sigma(s)$ in M is a geodesic iff $\sigma'' \perp M$

A simple way to identify geodesics (this condition is sufficient but not necessary)

$s \longrightarrow \sigma(s)$

Fact. Suppose Π is a plane that intersects M orthogonally at every point of intersection. Then the curve of intersection, $s \longrightarrow \sigma(s)$, is a geodesic



WTS: $\sigma'' \perp M$

$\sigma'' \propto n$

$s \longrightarrow \sigma(s)$

$\sigma \subset \Pi$

$T = \sigma'$ tangent to Π

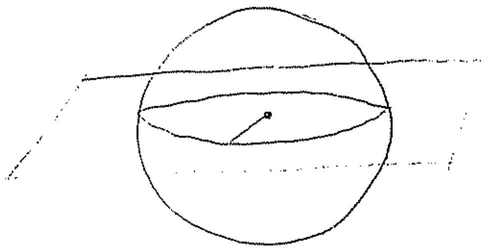
$T' = \sigma''$ tangent to Π

$\sigma'' = T' \perp T$

$\Rightarrow \sigma'' = \lambda n$

$\sigma'' \perp M$

ex $M = S^2_R$

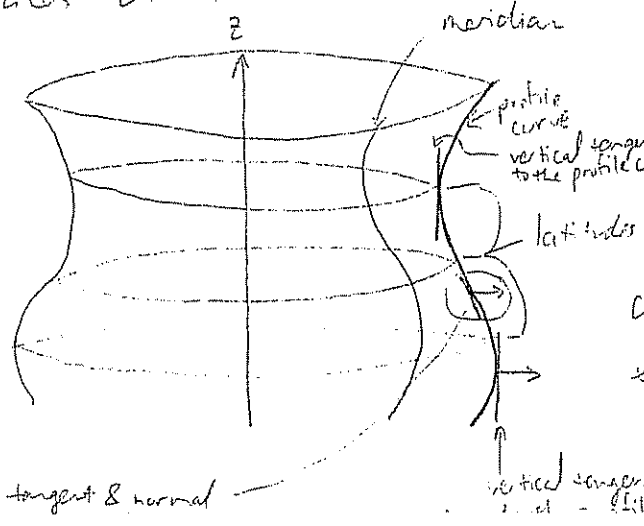


A great circle is the intersection of a plane passing through the origin & a sphere. The plane meets the sphere orthogonally.

Fact above $\Rightarrow$ all great circles geodesics

Note circles of latitudes are not geodesics (except for the equator)

Surfaces of Revolution:



σ: $x = r(t)$

$z = z(t)$ profile curve

$\sigma'(t) = (r'(t), z'(t))$

θ = constant

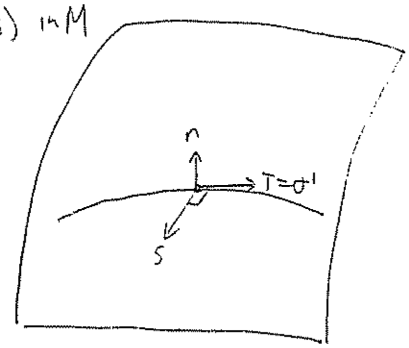
Claim 1) every meridian is a geodesic

every meridian is the intersection of the surface of revolution w/ a plane through the z-axis

2) Circles of latitude are geodesics at points where the profile curve has vertical tangents ($\Leftrightarrow r'(t)=0$)

Geodesic curvature

$s \to \sigma(s)$ in M



T = unit tangent to σ

n = unit normal to σ

S = intrinsic (surface) normal

$$S = T \times n \quad (|S|=1)$$

(S is unique up to sign)

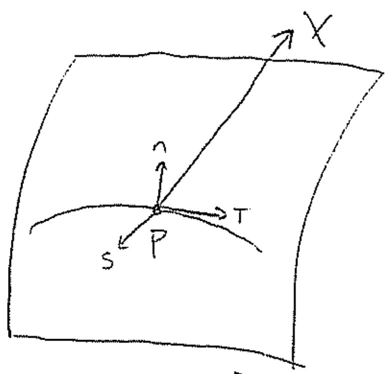
(in his notes, he has $S = n \times T$ but here he switched to make pictures easier to draw)

$T = T(s)$

$S = S(s)$

$n = n(s)$

$\{T(s), S(s), n(s)\}$ orthonormal frame of vectors based at $\sigma(s)$



$X \in T_p\mathbb{R}^3$

$$X = \langle X, T\rangle T + \langle X, S\rangle S + \langle X, n\rangle n$$

$$X = \sigma''$$

$$\sigma'' = \langle \sigma'', T\rangle T + \langle \sigma'', S\rangle S + \langle \sigma'', n\rangle n$$

$$\langle \sigma'', T\rangle = \langle T', T\rangle = 0$$

Recall: κ_n = normal component of σ''

$$\kappa_n = \langle \sigma'', n \rangle$$

$$\left(\kappa_n = \mathcal{L}(T,T) \right)$$

$$\langle \sigma'', n \rangle = \kappa_n$$

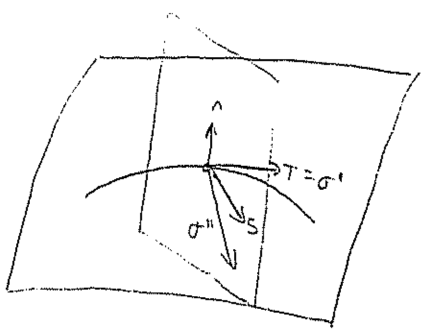
$$\sigma'' = \underbrace{\langle \sigma'', S \rangle}_{\kappa_g} S + \kappa_n n$$

κ_g = geodesic curvature

= component of the curvature vector tangent to M

$$\kappa_g = \langle \sigma'', S \rangle$$

$$\sigma'' = \kappa_g S + \kappa_n n$$



$$\sigma'' = \underbrace{\kappa_g S}_{\text{tangential part}} + \underbrace{\kappa_n n}_{\text{normal part}}$$

<u>Proposition:</u> $s \to \sigma(s)$ in M. σ is a geodesic iff $\kappa_g \equiv 0$.

<u>Proof.</u> $s \to \sigma(s)$ is a geodesic $\iff \sigma'' \perp M \iff \sigma'' \propto n \iff \kappa_g = 0$

Relationship between $\kappa, \kappa_n, \kappa_g$

$$\langle \sigma'', \sigma'' \rangle = \langle \kappa_g S + \kappa_n n, \kappa_g S + \kappa_n n \rangle = \kappa_g^2 \overset{=1}{\langle S, S \rangle} + \kappa_n^2 \overset{=1}{\langle n, n \rangle} + 2\kappa_g \kappa_n \overset{=0}{\langle S, n \rangle} = |\sigma''|^2$$

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

$s \to \sigma(s)$ be a unit speed curve in M

$$\frac{D}{ds}\frac{d\sigma}{ds} = \tan\left(\frac{d^2\sigma}{ds^2}\right)$$

$$= \tan\left(\kappa_g S + \kappa_n n\right)$$

$$\frac{D}{ds}\frac{d\sigma}{ds} = \kappa_g S$$

$$\left|\frac{D}{ds}\frac{d\sigma}{ds}\right| = \left|\kappa_g S\right| = \left|\kappa_g\right|$$

$$\left|\kappa_g\right| = \left|\frac{D}{ds}\frac{d\sigma}{ds}\right|$$

↑ we will show next time that this in intrinsic

Geodesics

Def. $t \to \sigma(t)$ in M. σ is a geodesic iff $\frac{d^2\sigma}{dt^2} \perp M$ $\left(\frac{d^2\sigma}{dt^2} = \lambda n\right)$ at each point of σ.

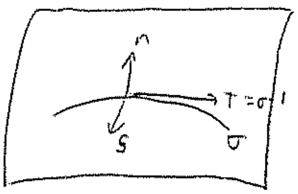
Covariant acceleration: an intrinsic quantity

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right)$$

$t \to \sigma(t)$ is a geodesic iff $\underbrace{\frac{D}{dt}\frac{d\sigma}{dt} = 0}_{\text{geodesic equation}}$

Observed: Geodesics are necessarily constant speed curves. $\left|\frac{d\sigma}{dt}\right| = \text{const} \neq 0$.

A unit speed curve $s \to \sigma(s)$ is a geodesic iff its curvature vector σ'', $' = \frac{d}{ds}$, is perpendicular to M at each point of σ



$S = T \times n$

$$\sigma'' = \kappa_g S + \kappa_n n$$

κ_n = normal curvature

κ_g = geodesic curvature

Fact: $s \to \sigma(s)$ is a geodesic iff $\kappa_g \equiv 0$.

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

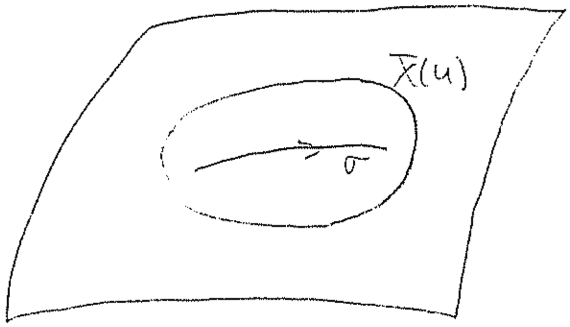
Geodesics in Coordinates

$$\frac{D}{dt}\frac{d\sigma}{dt} = 0$$

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right)$$

proper patch: $\bar{x} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$

$\bar{x} = \bar{x}(u^1, u^2)$



$$\sigma(t) = \overline{X}(u^1(t), u^2(t))$$

$$\frac{d\sigma}{dt} = \frac{\partial \overline{X}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \overline{X}}{\partial u^2}\frac{du^2}{dt}$$

$$= \frac{du^1}{dt}\overline{X}_1 + \frac{du^2}{dt}\overline{X}_2$$

$$\frac{d\sigma}{dt} = \sum_{i=1}^{2} \frac{du^i}{dt}\overline{X}_i$$

$$\frac{d^2\sigma}{dt^2} = \sum_{i=1}^{2} \frac{d}{dt}\left(\frac{du^i}{dt}\overline{X}_i\right)$$

$$= \sum_i \left(\frac{d^2u^i}{dt^2}\overline{X}_i + \frac{du^i}{dt}\frac{d}{dt}\overline{X}_i\right)$$

$$= \sum_i \frac{d^2u^i}{dt^2}\overline{X}_i + \sum_i \frac{du^i}{dt}\frac{d}{dt}\overline{X}_i$$

change dummy variable from $i \to k$

$$\overline{X}_i = \overline{X}_i(u^1, u^2), \quad u^1 = u^1(t), \; u^2 = u^2(t)$$

$$\frac{d\overline{X}_i}{dt} = \frac{\partial \overline{X}_i}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \overline{X}_i}{\partial u^2}\frac{du^2}{dt}$$

$$= \sum_{j=1}^{2} \frac{\partial \overline{X}_i}{\partial u^j}\frac{du^j}{dt}$$

$$= \sum_j \frac{du^j}{dt}\overline{X}_{ij}$$

$$\frac{d^2\sigma}{dt^2} = \sum_{k=1}^{2} \frac{d^2u^k}{dt^2}\overline{X}_k + \sum_{i,j} \overline{X}_{ij}\frac{du^i}{dt}\frac{du^j}{dt}$$

Recall: $\overline{X}_{ij} = \Gamma_{ij}^k \overline{X}_k + L_{ij} n$

$$\Gamma_{ij}^k = \frac{1}{2} g^{lk}\left[g_{il,j} + g_{jl,i} - g_{ij,l}\right]$$

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

So, $$\frac{d^2\sigma}{dt^2} = \sum_{k=1}^{2} \frac{d^2u^k}{dt^2}\bar{X}_k + \sum_{i,j} \bar{X}_{ij}\frac{du^i}{dt}\frac{du^j}{dt}$$

$$= \sum_k \frac{d^2u^k}{dt^2}\bar{X}_k + \sum_k\left(\sum_{i,j}\Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt}\right)\bar{X}_k + \left(\sum_{i,j} L_{ij}\frac{du^i}{dt}\frac{du^j}{dt}\right)n$$

$$= \sum_k\left[\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt}\right]\bar{X}_k + L\left(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\right)n$$

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right) = \sum_k\left[\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt}\right]\bar{X}_k$$

$t \to \sigma(t)$ is a geodesic iff $\frac{D}{dt}\frac{d\sigma}{dt} = 0$

$$\Rightarrow \frac{d^2u^k}{dt^2} + \sum_{i,j=1}^{2}\Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0, \quad k = 1,2$$

geodesic equations

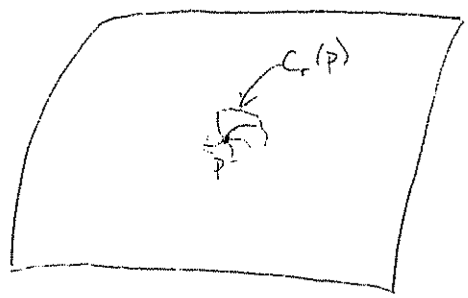
$\sigma(t) = \bar{x}(u^1(t), u^2(t))$

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{11}\left(\frac{du^1}{dt}\right)^2 + 2\Gamma^k_{12}\frac{du^1}{dt}\frac{du^2}{dt} + \Gamma^k_{22}\left(\frac{du^2}{dt}\right)^2 = 0, \quad k = 1,2$$

Basic existence & uniqueness theorems for systems of ODE's imply the following:

<u>Proposition</u>: Given $p \in M$, $X \in T_pM$, $\exists$ a unique geodesic $t \to \sigma(t)$ satisfying $\sigma(0) = p$, $\frac{d\sigma}{dt}(0) = X$

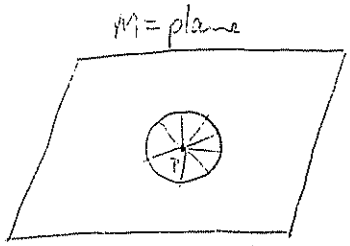
$C_r(p)$ = geodesic circle of radius r, centered at p



<u>Fact</u>: Provided r is small enough, $C_r(p)$ will be a smooth curve

$D_r(p)$ = geodesic disk of radius r centered at p

= region bounded by $C_r(p)$



$C_r(p)$ = Euclidean circle of radius r

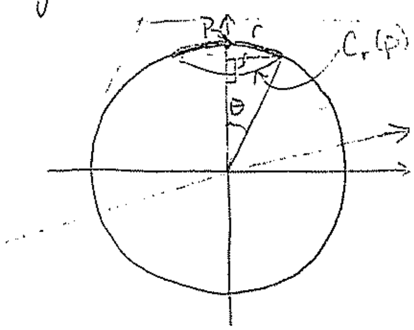
$D_r(p)$ = Euclidean disk of radius r

$L(C_r(p)) = 2\pi r$ (if we are in the plane)

$A(D_r(p)) = \pi r^2$ (if we are in the plane)

S_R = sphere of radius R

$x^2+y^2+z^2=R^2$



$C_r(p)$ = latitude

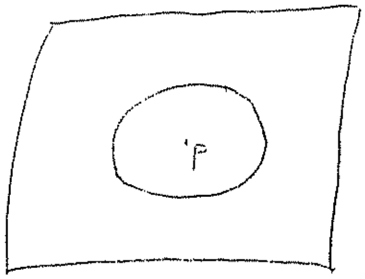
$L(C_r(p)) = 2\pi\rho = 2\pi R\sin\theta = 2\pi R\sin\left(\frac{r}{R}\right)$

Mac Series: $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\left(\frac{r}{R}\right) = \frac{r}{R} - \frac{1}{6}\frac{r^3}{R^3} + \ldots$

$L(C_r(p)) = 2\pi R\left(\frac{r}{R} - \frac{1}{6}\frac{r^3}{R^3} + \ldots\right) = 2\pi r - \frac{\pi}{3}\underbrace{\left(\frac{1}{R^2}\right)}_{K = \text{gaussian curvature}} r^3 + \text{HOT in } r$

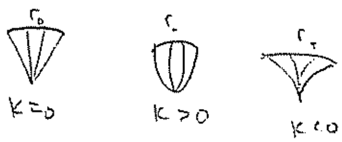
In general,



$$L(C_r(p)) = 2\pi r - \frac{\pi}{3} K(p) r^3 + \text{HOT}$$

$$A(D_r(p)) = \pi r^2 - \frac{\pi}{12} K(p) r^4 + \text{HOT} \qquad (*)$$

Exercise: Verify $(*)$ for S^2_R



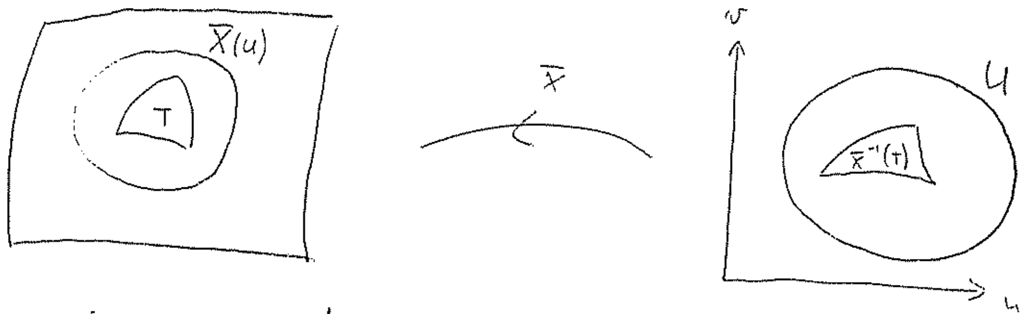
$r_- < r_0 < r_+$

11/30-

Gauss-Bonnet Theorem

Angle Excess Theorem

Def. A triangle T in M is a simple region in M bounded by 3 smooth curve segments



Simple: it sits inside a proper patch (bounded by a simple (non-intersecting) closed curve)

Def: A geodesic triangle is a triangle whose sides are geodesics

Comment: A geodesic triangle in $\mathbb{R}$ is a Euclidean triangle where the sides are straight lines. For a Euclidean triangle, $A+B+C=\pi$

Angle Excess Theorem: For a geodesic triangle,

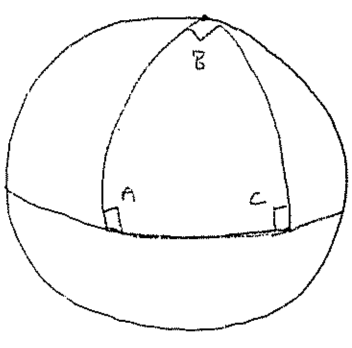
$$A+B+C=\pi+\iint_T K\,dS$$

ex. $M=$ x-y plane

$K=0$

$A+B+C=\pi$

ex $M=S^2_R$



$A+B+C=\frac{3}{2}\pi$ (guess from picture)

$$\iint_T K\,dS=\iint_T \frac{1}{R^2}\,dA=\frac{1}{R^2}\iint_T dA=\frac{1}{R^2}A(T)$$

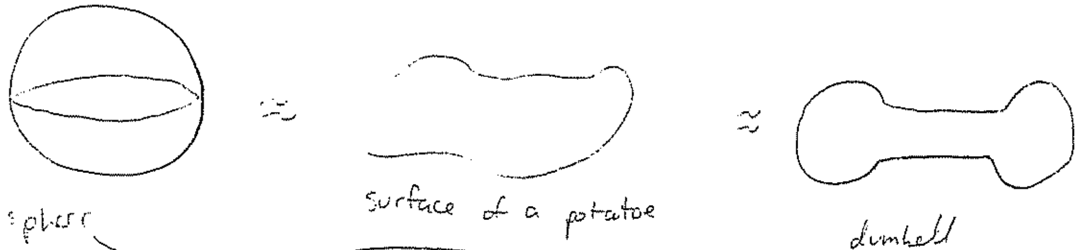
$$=\frac{1}{R^2}\cdot\frac{4\pi R^2}{\cancel{8}\,2}=\frac{\pi}{2}$$

$\Rightarrow A+B+C=\pi+\frac{\pi}{2}=\frac{3\pi}{2}$, just like we guessed

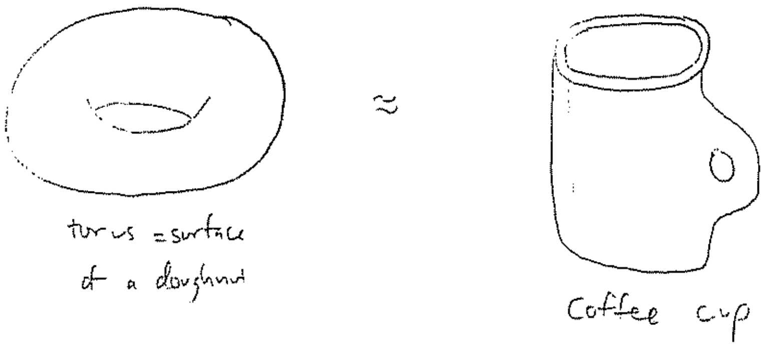
$K>0 \Rightarrow A+B+C > \pi \Rightarrow$ fat triangles

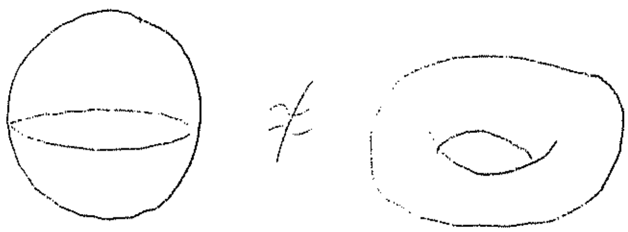
$K<0 \Rightarrow A+B+C < \pi \Rightarrow$ skinny triangles

Topology of Surfaces



From the point of view of topology, these are all equivalent surfaces





Euler Number (characteristic)

This number determines if two surfaces are of the same topological type (determines if they are diffeomorphisms)

If two surfaces have the same Euler number then they are topologically equivalent

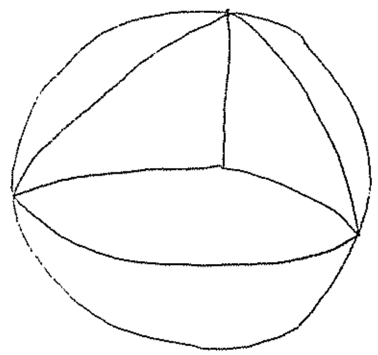
Def: A triangulation of M is a decomposition of M into a finite number of triangles $T_1, T_2, \dots, T_n$ s.t

1) $\bigcup_{i=1}^{n} T = M$

2) $T_i \cap T_j \neq 0$ then $T_i \cap T_j$ is either a common edge or a vertex

Fact: Every compact (closed & bounded) surface can be triangulated

ex



consider a tetrahedron

$\chi(M)$ = Euler number

$= V - E + F$

V = # of vertices

E = # of edges

F = # of faces

ex For the sphere above,

$F = 4$, $E = 6$, $F = 4$

$\chi(M) = 4 - 6 + 4 = 2$

Fact: The Euler number does not depend on the particular triangulation

$\chi(M)$ = topological invariant

Fact: Two surfaces are diffeomorphic (have the same topological type) iff they have the same Euler characteristic ex.

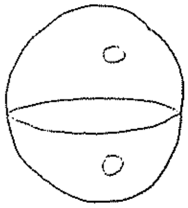
T^2 = torus

$\chi(T^2)=0$ (we can try to compute this on our own or just trust him)

$\chi(T^2) \neq \chi(S^2)$

genus & classification of compact surfaces in $\mathbb{R}^3$

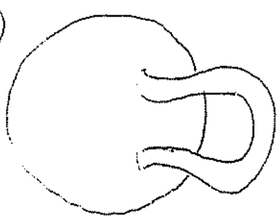
1)



S^2

add a handle  = hollow cylinder = tube & assume it's flexible & glue ends of tube to holes in the sphere. This gives 2)

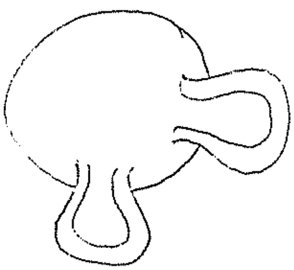
2)



sphere with one handle

add a handle to get 3)

3)



sphere with two handles

continuing in this way we can construct a sphere with g handles, where g = genus = # of handles attached

g is always a non-negative integer

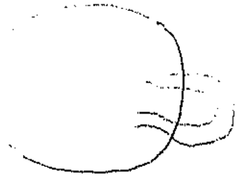
This is a list of all possible topological types

Theorem: Every compact surface in $\mathbb{R}^3$ is diffeomorphic to a sphere with g handles, for some g

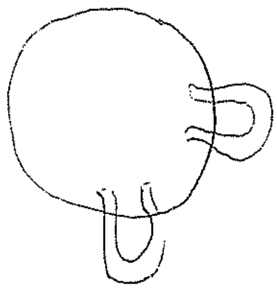
ex 1)  = sphere with zero handles

$g=0$

surface of a potatoe

2)  $\approx$ 

$g=1$

3)  $\approx$ 

2-holed torus

Fact: Let M^2 be a compact surface of genus g. Then $\chi(M) = 2(1-g)$

ex $M = S^2$

$\chi(M) = 2$

$g=0$

ex $M = T^2$

$\chi(M) = 0$

$g = 1$

Gauss-Bonnet Theorem: If M is a compact surface in $\mathbb{R}^3$ then

$$\iint_M K\,dA = 2\pi\chi = 4\pi(1-g)$$

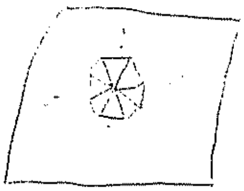
Corollary: Suppose M has everywhere positive Gaussian curvature, $K>0$. Then $\iint_M K\,dA > 0$ $\Rightarrow 4\pi(1-g) > 0 \Rightarrow g=0$. Therefore, M is diffeomorphic to a sphere.

Proof (of Gauss-Bonnet): Consequence of the angle excess theorem.

T = geodesic triangle

$$A+B+C = \pi + \iint_T K\,dS$$

Triangulate M into geodesic triangles $T_1, T_2, \ , T_n$



$$\iint_M K\,dS = \sum_{i=1}^{n} \iint_{T_i} K\,dS = \sum_{i=1}^{n} \left[A_i + B_i + C_i - \pi \right] = \sum_{i=1}^{n} (A_i + B_i + C_i) - n\pi$$

$F = n$

$$\iint_M K\,ds = \sum_{i=1}^{n} (A_i + B_i + C_i) - \pi F$$

Claim: $\sum_{i=1}^{n} A_i + B_i + C_i = 2\pi V$

$\sum_{i=1}^{n} A_i + B_i + C_i$ = sum of all angles over all triangles

sum of all the angles around this vertex is $2\pi \Rightarrow \sum_{i=1}^{n} A_i + B_i + C_i = 2\pi$

$$\iint_M K\,dS = 2\pi V - \pi F = 2\pi V - 3\pi F + 2\pi F$$

Claim: $3F = 2E$

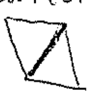
Count edges:

1) First try

$E = 3F$

but each edge belongs to each edge

2) Correct



$E = \frac{3F}{2} \Rightarrow 2E = 3F$

$$\iint_M K\,dS = 2\pi V - 3\pi F + 2\pi F = 2\pi V - 2\pi E + 2\pi F = 2\pi(V - E + F)$$

$$= 2\pi\,\chi(M)$$