Chapter 5. The Second Fundamental Form

Directional Derivatives in $\mathbb{R}^3$.

Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function defined on an open subset of $\mathbb{R}^3$. Fix $p \in U$ and $X \in T_p \mathbb{R}^3$. The directional derivative of $f$ at $p$ in the direction $X$, denoted $D_X f$ is defined as follows. Let $\sigma : \mathbb{R} \to \mathbb{R}^3$ be the parameterized straight line, $\sigma(t) = p + tX$. Note $\sigma(0) = p$ and $\sigma'(0) = X$. Then,

$$D_X f = \frac{d}{dt} f \circ \sigma(t) \big|_{t=0}$$

$$= \frac{d}{dt} f(p + tX) \big|_{t=0}$$

$$= \lim_{t \to 0} \frac{f(p + tX) - f(p)}{t}.$$

Fact: The directional derivative is given by the following formula,

$$D_X f = X \cdot \nabla f(p)$$

$$= (X^1, X^2, X^3) \cdot \left( \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \frac{\partial f}{\partial x_3}(p) \right)$$

$$= X^1 \frac{\partial f}{\partial x_1}(p) + X^2 \frac{\partial f}{\partial x_2}(p) + X^3 \frac{\partial f}{\partial x_3}(p)$$

$$= \sum_{i=1}^{3} X^i \frac{\partial f}{\partial x_i}(p).$$

Proof Chain rule!

Vector Fields on $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a rule which assigns to each point of $\mathbb{R}^3$ a vector at the point,

$$x \in \mathbb{R}^3 \to Y(x) \in T_x \mathbb{R}^3$$
Analytically, a vector field is described by a mapping of the form,

\[ Y: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \]
\[ Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \in T_x \mathbb{R}^3. \]

Components of \( Y \):
\[ Y^i: U \rightarrow \mathbb{R}, \quad i = 1, 2, 3. \]

Ex. \( Y: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \ Y(x, y, z) = (y + z, z + x, x + y) \). E.g., \( Y(1, 2, 3) = (5, 4, 3) \), etc.
\( Y^1 = y + z, \ Y^2 = z + x, \) and \( Y^3 = x + y \).

The directional derivative of a vector field is defined in a manner similar to the directional derivative of a function: Fix \( p \in U, \ X \in T_p \mathbb{R}^3 \). Let \( \sigma: \mathbb{R} \rightarrow \mathbb{R}^3 \) be the parameterized line \( \sigma(t) = p + tX, \quad (\sigma(0) = p, \sigma'(0) = X) \). Then \( t \rightarrow Y \circ \sigma(t) \) is a vector field along \( \sigma \) in the sense of the definition in Chapter 2. Then, the directional derivative of \( Y \) in the direction \( X \) at \( p \), is defined as,

\[ D_X Y = \frac{d}{dt} Y \circ \sigma(t) \big|_{t=0} \]

I.e., to compute \( D_X Y \), restrict \( Y \) to \( \sigma \) to obtain a vector valued function of \( t \), and differentiate with respect to \( t \).

In terms of components, \( Y = (Y^1, Y^2, Y^3) \),

\[ D_X Y = \frac{d}{dt} (Y^1 \circ \sigma(t), Y^2 \circ \sigma(t), Y^3 \circ \sigma(t)) \big|_{t=0} \]
\[ = \left( \frac{d}{dt} Y^1 \circ \sigma(t) \big|_{t=0}, \frac{d}{dt} Y^2 \circ \sigma(t) \big|_{t=0}, \frac{d}{dt} Y^3 \circ \sigma(t) \big|_{t=0} \right) \]
\[ = (D_X Y^1, D_X Y^2, D_X Y^3). \]
Directional derivatives on surfaces.

Let $M$ be a surface, and let $f : M \to \mathbb{R}$ be a smooth function on $M$. Recall, this means that $\hat{f} = f \circ \mathbf{x}$ is smooth for all proper patches $\mathbf{x} : U \to M$ in $M$.

**Def.** For $p \in M$, $X \in T_p M$, the directional derivative of $f$ at $p$ in the direction $X$, denoted $\nabla_X f$, is defined as follows. Let $\sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3$ be any smooth curve in $M$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then,

$$\nabla_X f = \frac{d}{dt} f \circ \sigma(t) \big|_{t=0}$$

I.e., to compute $\nabla_X f$, restrict $f$ to $\sigma$ and differentiate with respect to parameter $t$.

**Proposition.** The directional derivative is well-defined, i.e. independent of the particular choice of $\sigma$.

**Proof.** Let $\mathbf{x} : U \to M$ be a proper patch containing $p$. Express $\sigma$ in terms of coordinates in the usual manner,

$$\sigma(t) = \mathbf{x}(u^1(t), u^2(t)).$$

By the chain rule,

$$\frac{d\sigma}{dt} = \sum \frac{du_i}{dt} \mathbf{x}_i \quad \left( \mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u_i} \right)$$

$X \in T_p M \Rightarrow X = \sum X^i \mathbf{x}_i$. The initial condition, $\frac{d\sigma}{dt}(0) = X$ then implies

$$\frac{du^i}{dt}(0) = X^i, \quad i = 1, 2.$$

Now,

$$f \circ \sigma(t) = f(\sigma(t)) = f(\mathbf{x}(u^1(t), u^2(t))) = f \circ \mathbf{x}(u^1(t), u^2(t)) = \hat{f}(u^1(t), u^2(t)).$$
Hence, by the chain rule,
\[
\frac{d}{dt} f \circ \sigma(t) = \frac{\partial \hat{f}}{\partial u^1} \frac{du^1}{dt} + \frac{\partial \hat{f}}{\partial u^2} \frac{du^2}{dt} \quad = \sum \frac{\partial \hat{f}}{\partial u^i} \frac{du^i}{dt} = \sum \frac{du^i}{dt} \frac{\partial \hat{f}}{\partial u^i}.
\]

Therefore,
\[
\nabla_X f = \left. \frac{d}{dt} f \circ \sigma(t) \right|_{t=0} = \sum \frac{du^i}{dt}(0) \frac{\partial \hat{f}}{\partial u^i}(u^1, u^2), \quad (p = x(u^1, u^2))
\]
\[
\nabla_X f = \sum X^i \frac{\partial \hat{f}}{\partial u^i}(u^1, u^2),
\]
or simply,
\[
\nabla_X f = \sum X^i \frac{\partial \hat{f}}{\partial u^i}
\]
\[
= X^1 \frac{\partial \hat{f}}{\partial u^1} + X^2 \frac{\partial \hat{f}}{\partial u^2}.
\quad (*)
\]

**Ex.** Let \( X = x_1 \). Since \( x_1 = 1 \cdot x_1 + 0 \cdot x_2 \), \( X^1 = 1 \) and \( X^2 = 0 \). Hence the above equation implies, \( \nabla_{x_1} f = \frac{\partial \hat{f}}{\partial u^1} \). Similarly, \( \nabla_{x_2} f = \frac{\partial \hat{f}}{\partial u^2} \). I.e.,
\[
\nabla_{x_i} f = \frac{\partial \hat{f}}{\partial u^i}, \quad i = 1, 2.
\]

The following proposition summarizes some basic properties of directional derivatives in surfaces.

**Proposition**

1. \( \nabla_{(aX+bY)} f = a \nabla_X f + b \nabla_Y f \)
2. \( \nabla_X (f + g) = \nabla_X f + \nabla_X g \)
3. \( \nabla_X fg = \nabla_X f g + f \nabla_X g \)

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Exercise 5.1. Prove this proposition.

Vector fields along a surface.

A vector field along a surface $M$ is a rule which assigns to each point of $M$ a vector at that point,

$$x \in M \rightarrow Y(x) \in T_x \mathbb{R}^3$$

N.B. $Y(x)$ need not be tangent to $M$.

Analytically vector fields along a surface $M$ are described by mappings.

$$Y : M \rightarrow \mathbb{R}^3$$

$$Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \in T_x \mathbb{R}^3$$

Components of $Y$: $Y^i : M \rightarrow \mathbb{R}$, $i = 1, 2, 3$. We say $Y$ is smooth if its component functions are smooth.

The directional derivative of a vector field along $M$ is defined in a manner similar to the directional derivative of a function defined on $M$.

Given a vector field along $M$, $Y : M \rightarrow \mathbb{R}^3$, for $p \in M, X \in T_p M$, the directional derivative of $Y$ in the direction $X$, denoted $\nabla_X Y$, is defined as,

$$\nabla_X Y = \left. \frac{d}{dt} Y \circ \sigma(t) \right|_{t=0}$$

where $\sigma : (-\epsilon, \epsilon) \rightarrow M$ is a smooth curve in $M$ such that $\sigma(0) = p$ and $\frac{d\sigma}{dt}(0) = X$. 

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I.e. to compute $\nabla_X Y$, restrict $Y$ to $\sigma$ to obtain a vector valued function of $t$ - then differentiate with respect to $t$.

**Fact** If $Y(x) = (Y^1(x), Y^2(x), Y^3(x))$ then,

$$\nabla_X Y = (\nabla_X Y^1, \nabla_X Y^2, \nabla_X Y^3)$$

**Proof**: Exercise.

**Surface Coordinate Expression.** Let $x : U \rightarrow M$ be a proper patch in $M$ containing $p$. Let $X \in T_p M$, $X = \sum_i x^i x_i$. An argument like that for functions on $M$ shows,

$$\nabla_X Y = \sum_{i=1}^2 X^i \frac{\partial \hat{Y}}{\partial u^i}(u^1, u^2), \quad (p = x(u^1, u^2))$$

where $\hat{Y} = Y \circ x : U \rightarrow \mathbb{R}^3$ is $Y$ expressed in terms of coordinates.

**Exercise 5.2.** Derive the expression above for $\nabla_X Y$. In particular, show

$$\nabla_x Y = \frac{\partial \hat{Y}}{\partial u^i}, \quad i = 1, 2.$$ 

Some basic properties are described in the following proposition.

**Proposition**

1. $\nabla_{aX + bY} Z = a \nabla_X Z + b \nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X (fY) = (\nabla_X f)Y + f \nabla_X Y$
4. $\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$
The Weingarten Map and the 2nd Fundamental Form.

We are interested in studying the shape of surfaces in $\mathbb{R}^3$. Our approach (essentially due to Gauss) is to study how the unit normal to the surface “wiggles” along the surface.

The objects which describe the shape of $M$ are:

1. The Weingarten Map, or shape operator. For each $p \in M$ this is a certain linear transformation $L : T_p M \to T_p M$.

2. The second fundamental form. This is a certain bilinear form $\mathcal{L} : T_p M \times T_p M \to \mathbb{R}$ associated in a natural way with the Weingarten map.

We now describe the Weingarten map. Fix $p \in M$. Let $n : W \to \mathbb{R}^3$, $p \in W \to n(p) \in T_p \mathbb{R}^3$, be a smooth unit normal vector field defined along a neighborhood $W$ of $p$.

Remarks

1. $n$ can always be constructed by introducing a proper patch $\mathbf{x} : U \to M$, $\mathbf{x} = \mathbf{x}(u^1, u^2)$ containing $p$:

   \[ \hat{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}, \]

   $\hat{n} : U \to \mathbb{R}^3$, $\hat{n} = \hat{n}(u^1, u^2)$. Then, $n = \hat{n} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbb{R}^3$ is a smooth unit normal v.f. along $\mathbf{x}(U)$.

2. The choice of $n$ is not quite unique: $n \rightarrow -n$; choice of $n$ is unique “up to sign”
3. A smooth unit normal field $n$ always exists in a neighborhood of any given point $p$, but it may not be possible to extend $n$ to all of $M$. This depends on whether or not $M$ is an orientable surface.

**Ex.** Möbius band.

![Diagram of a Möbius band]

**Lemma.** Let $M$ be a surface, $p \in M$, and $n$ be a smooth unit normal vector field defined along a neighborhood $W \subset M$ of $p$. Then for any $X \in T_p M$, $\nabla_X n \in T_p M$.

**Proof.** It suffices to show that $\nabla_X n$ is perpendicular to $n$. $|n| = 1 \Rightarrow \langle n, n \rangle = 1 \Rightarrow$

$$\nabla_X \langle n, n \rangle = \nabla_X 1$$
$$\langle \nabla_X n, n \rangle + \langle n, \nabla_X n \rangle = 0$$
$$2\langle \nabla_X n, n \rangle = 0$$

and hence $\nabla_X n \perp n$.

**Def.** Let $M$ be a surface, $p \in M$, and $n$ be a smooth unit normal v.f. defined along a nbd $W \subset M$ of $p$. The Weingarten Map (or shape operator) is the map $L : T_p M \to T_p M$ defined by,

$$L(X) = -\nabla_X n.$$

**Remarks**

1. The minus sign is a convention – will explain later.
2. $L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0}$
Lemma: \( L : T_p M \to T_p M \) is a linear map, i.e.,
\[
L(aX + bY) = aL(X) + bL(Y)
\]
for all \( X, Y, \in T_p M, \ a, b \in \mathbb{R} \).

Proof. Follows from properties of directional derivative,
\[
L(aX + bY) = -\nabla_{aX+bY}n
= -[a\nabla_X n + b\nabla_Y n]
= a(-\nabla_X n) + b(-\nabla_Y n)
= aL(X) + bL(Y).
\]

Ex. Let \( M \) be a plane in \( \mathbb{R}^3 \):
\[
M : ax + by + cz = d
\]
Determine the Weingarten Map at each point of \( M \). Well,
\[
n = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} = \left( \frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda} \right),
\]
where \( \lambda = \sqrt{a^2 + b^2 + c^2} \). Hence,
\[
L(X) = -\nabla_X n = -\nabla_X \left( \frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda} \right)
= -\left( \nabla_X \frac{a}{\lambda}, \nabla_X \frac{b}{\lambda}, \nabla_X \frac{c}{\lambda} \right) = 0.
\]
Therefore \( L(X) = 0 \ \forall \ X \in T_p M \), i.e. \( L \equiv 0 \).

Ex. Let \( M = S^2_r \) be the sphere of radius \( r \), let \( n \) be the outward pointing unit normal. Determine the Weingarten map at each point of \( M \).
Fix $p \in S^2_r$, and let $X \in T_pS^2_r$. Let $\sigma : (-\varepsilon, \varepsilon) \to S^2_r$ be a curve in $S^2_r$ such that $\sigma(0) = p$, $\frac{d\sigma}{dt}(0) = X$.

Then,

$$L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0}.$$ 

But note, $n \circ \sigma(t) = n(\sigma(t)) = \frac{\sigma(t)}{||\sigma(t)||}$. Hence,

$$L(X) = -\frac{d}{dt} \frac{\sigma(t)}{r}|_{t=0} = -\frac{1}{r} \frac{d\sigma}{dt}|_{t=0}$$

$$L(X) = -\frac{1}{r} X.$$ 

for all $X \in T_pM$. Hence, $L = -\frac{1}{r} id$, where $id : T_pM \to T_pM$ is the identity map, $id(X) = X$.

**Remark.** If we had taken the inward pointing normal then $L = \frac{1}{r} id$.

**Def.** For each $p \in M$, the *second fundamental form* is the bilinear form $L = T_pM \times T_pM \to \mathbb{R}$ defined by,

$$L(X, Y) = \langle L(X), Y \rangle$$

$$= -\langle \nabla_X n, Y \rangle.$$

$L$ is indeed *bilinear*, e.g.,

$$L(aX + bY, Z) = \langle L(aX + bY), Z \rangle$$

$$= \langle aL(X) + bL(Y), Z \rangle$$

$$= a\langle L(X), Z \rangle + b\langle L(Y), Z \rangle$$

$$= aL(X, Z) + bL(Y, Z).$$

**Ex.** $M = \text{plane}, \ L \equiv 0$:

$$L(X, Y) = \langle L(X), Y \rangle = \langle 0, Y \rangle = 0.$$ 

**Ex.** The sphere $S^2_r$ of radius $r$, $L : T_pS^2_r \times T_pS^2_r \to \mathbb{R}$,

$$L(X, Y) = \langle L(X), Y \rangle$$

$$= \langle -\frac{1}{r} X, Y \rangle$$

$$= -\frac{1}{r} \langle X, Y \rangle$$
Hence, $L = -\frac{1}{r}\langle \cdot, \cdot \rangle$. Multiple of the first fundamental form!

**Coordinate expressions**

Let $x : U \to M$ be a patch containing $p \in M$. Then $\{x_1, x_2\}$ is a basis for $T_p M$. We express $L : T_p M \to T_p M$ and $\mathcal{L} : T_p M \times T_p M \to \mathbb{R}$ with respect to this basis. Since $L(x_j) \in T_p M$, we have,

$$L(x_j) = L^1_j x_1 + L^2_j x_2, \quad j = 1, 2$$

$$= \sum_{i=1}^2 L^i_j x_i.$$

The numbers $L^i_j$, $1 \leq i, j \leq 2$, are called the components of $L$ with respect to the coordinate basis $\{x_1, x_2\}$. The $2 \times 2$ matrix $[L^i_j]$ is the matrix representing the linear map $L$ with respect to the basis $\{x_1, x_2\}$.

**Exercise 5.3** Let $X \in T_p M$ and let $Y = L(X)$. In terms of components, $X = \sum_j X^j x_j$ and $Y = \sum_i Y^i x_i$. Show that

$$Y^i = \sum_j L^i_j X^j, \quad i = 1, 2,$$

which in turn implies the matrix equation,

$$\begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} = [L^i_j] \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}.$$

This is the Weingarten map expressed as a matrix equation.

Introduce the unit normal field along $W = x(U)$ with respect to the patch $x : U \to M$,

$$\hat{n} = \frac{x_1 \times x_2}{|x_1 \times x_2|}, \quad \hat{n} = \hat{n}(u^1, u^2),$$

$$n = \hat{n} \circ x^{-1} : W \to \mathbb{R}.$$

Then by Exercise 5.2,

$$L(x_j) = -\nabla_{x_j} n = -\frac{\partial \hat{n}}{\partial u^\sigma}.$$ 

Setting $n_j = \frac{\partial n}{\partial u^\sigma}$ we have

$$n_j = -L(x_j)$$

$$n_j = -\sum_i L^i_j x_i, \quad j = 1, 2 \quad \text{(The Weingarten equations.)}$$

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These equations can be used to compute the components of the Weingarten map. However, in practice it turns out to be more useful to have a formula for computing the components of the second fundamental form.

**Components of \( \mathcal{L} \):**

The components of \( \mathcal{L} \) with respect to \( \{x_1, x_2\} \) are defined as,

\[
L_{ij} = \mathcal{L}(x_i, x_j), \quad 1 \leq i, j \leq 2.
\]

By bilinearity, the components completely determine \( \mathcal{L} \),

\[
\mathcal{L}(X, Y) = \mathcal{L}(\sum_i X^i x_i, \sum_j Y^j x_j) = \sum_{i,j} X^i Y^j \mathcal{L}(x_i, x_j) = \sum_{i,j} L_{ij} X^i Y^j.
\]

The following proposition provides a useful formula for computing the \( L_{ij} \)’s.

**Proposition.** The components \( L_{ij} \) of \( \mathcal{L} \) are given by,

\[
L_{ij} = \langle \hat{n}, x_{ij} \rangle,
\]

where \( x_{ij} = \frac{\partial^2 x}{\partial u^j \partial u^i} \).

**Remark.** Henceforth we no longer distinguish between \( n \) and \( \hat{n} \), i.e., lets agree to drop the “ \(^\wedge\) ”, then,

\[
L_{ij} = \langle n, x_{ij} \rangle,
\]

**Proof:**
Along $x(U)$ we have, $\langle n, \frac{\partial x}{\partial u} \rangle = 0$, and hence,

\[
\frac{\partial}{\partial w} \langle n, \frac{\partial x}{\partial u} \rangle = 0
\]

\[
\langle \frac{\partial n}{\partial u}, \frac{\partial x}{\partial u} \rangle + \langle n, \frac{\partial^2 x}{\partial u \partial u} \rangle = 0
\]

\[
\langle \frac{\partial n}{\partial u}, \frac{\partial x}{\partial u} \rangle = -\langle n, \frac{\partial^2 x}{\partial u \partial u} \rangle
\]

or, using shorthand notation,

\[
\langle n, x \rangle = -\langle n, x \rangle.
\]

But,

\[
L_{ij} = L(x_i, x_j) = \langle L(x_i), x_j \rangle = -\langle n, x_j \rangle,
\]

and hence $L_{ij} = \langle n, x \rangle$.

Observe,

\[
L_{ij} = \langle n, x \rangle = \langle n, x \rangle \quad \text{(mixed partials equal!)}
\]

\[
L_{ij} = L_{ji}, \quad 1 \leq i, j \leq 2.
\]

In other words, $L(x_i, x_j) = L(x_j, x_i)$.

**Proposition.** The second fundamental form $L : T_p M \times T_p M \to \mathbb{R}$ is symmetric, i.e.

\[
L(X, Y) = L(Y, X) \quad \forall \ X, Y \in T_p M.
\]

**Exercise 5.4** Prove this proposition by showing $L$ is symmetric iff $L_{ij} = L_{ji}$ for all $1 \leq i, j \leq 2$.

**Relationship between $L^i_j$ and $L_{ij}$**

\[
L_{ij} = L(x_i, x_j) = L(x_j, x_i)
\]

\[
= \langle L(x_j), x_i \rangle = \sum_k L^k_{ij} x_k, x_i, \}
\]

\[
= \sum_k L^k_{ij} x_k, x_i
\]

\[
L_{ij} = \sum_k g_{ik} L^k_{ij}, \quad 1 \leq i, j \leq 2
\]
Classical tensor jargon: $L_{ij}$ obtained from $L^k_j$ by “lowering the index $k$ with the metric”. The equation above implies the matrix equation
\[
[L_{ij}] = [g_{ij}][L^j_i].
\]

Geometric Interpretation of the 2nd Fundamental Form

Normal Curvature. Let $s \to \sigma(s)$ be a unit speed curve lying in a surface $M$. Let $p$ be a point on $\sigma$, and let $n$ be a smooth unit normal v.f. defined in a nbd $W$ of $p$. The normal curvature of $\sigma$ at $p$, denoted $\kappa_n$, is defined to be the component of the curvature vector $\sigma'' = T'$ along $n$, i.e.,
\[
\kappa_n = \text{normal component of the curvature vector} = \langle \sigma'', n \rangle = \langle T', n \rangle = |T'| |n| \cos \theta = \kappa \cos \theta,
\]
where $\theta$ is the angle between the curvature vector $T'$ and the surface normal $n$. If $\kappa \neq 0$ then, recall, we can introduce the principal normal $N$ to $\sigma$, by the equation, $T' = \kappa N$; in this case $\theta$ is the angle between $N$ and $n$.

Remark: $\kappa_n$ gives a measure of how much $\sigma$ is bending in the direction perpendicular to the surface; it neglects the amount of bending tangent to the surface.

Proposition. Let $M$ be a surface, $p \in M$. Let $X \in T_p M$, $|X| = 1$ (i.e. $X$ is a unit tangent vector). Let $s \to \sigma(s)$ be any unit speed curve in $M$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then
\[
\mathcal{L}(X, X) = \text{normal curvature of } \sigma \text{ at } p = \langle \sigma'', n \rangle.
\]
Proof. Along $\sigma$,

$$\langle \sigma'(s), n \circ \sigma(s) \rangle = 0, \quad \text{for all } s$$
$$\frac{d}{ds} \langle \sigma', n \circ \sigma \rangle = 0$$
$$\langle \sigma'', n \circ \sigma \rangle + \langle \sigma', \frac{d}{ds} n \circ \sigma \rangle = 0.$$

At $s = 0$,

$$\langle \sigma'', n \rangle + \langle X, \nabla_X n \rangle = 0$$
$$\langle \sigma'', n \rangle = -\langle X, \nabla_X n \rangle$$
$$\kappa_n = \langle X, L(X) \rangle$$
$$\kappa_n = \langle L(X), X \rangle$$
$$= \mathcal{L}(X, X).$$

Remark: the sign convention used in the definition of the Weingarten map ensures that $\mathcal{L}(X, X) = +\kappa_n$ (rather than $-\kappa_n$).

**Corollary.** All unit speed curves lying in a surface $M$ which pass through $p \in M$ and have the same unit tangent vector $X$ at $p$, have the same normal curvature at $p$. That is, the normal curvature depends only on the tangent direction $X$.

Thus it makes sense to say:

$\mathcal{L}(X, X)$ is the normal curvature in the direction $X$.

Given a unit tangent vector $X \in T_p M$, there is a distinguished curve in $M$, called the normal section at $p$ in the direction $X$. Let,

$$\Pi = \text{plane through } p \text{ spanned by } n \text{ and } X.$$ 

$\Pi$ cuts $M$ in a curve $\sigma$. Parameterize $\sigma$ wrt arc length, $s \to \sigma(s)$, such that $\sigma(0) = p$ and $\frac{d\sigma}{ds}(0) = X$:
By definition, $\sigma$ is the normal section at $p$ in the direction $X$. By the previous proposition,

$$\mathcal{L}(X, X) = \text{normal curvature of the normal section } \sigma$$

$$= \langle \sigma'' n \rangle = \langle T', n \rangle$$

$$= \kappa \cos \theta,$$

where $\theta$ is the angle between $n$ and $T'$. Since $\sigma$ lies in $\Pi$, $T'$ is tangent to $\Pi$, and since $T'$ is also perpendicular to $X$, it follows that $T'$ is a multiple of $n$. Hence, $\theta = 0$ or $\pi$, which implies that $\mathcal{L}(X, X) = \pm \kappa$.

Thus we conclude that,

$$\mathcal{L}(X, X) = \text{signed curvature of the normal section at } p \text{ in the direction } X.$$

Principal Curvatures.

The set of unit tangent vectors at $p$, $X \in T_p M$, $|X| = 1$, forms a circle in the tangent plane to $M$ at $p$. Consider the function from this circle into the reals,

$$X \rightarrow \text{normal curvature in direction } X$$

$$X \rightarrow \mathcal{L}(X, X).$$

The principal curvatures of $M$ at $p$, $\kappa_1 = \kappa_1(p)$ and $\kappa_2 = \kappa_2(p)$, are defined as follows,

$$\kappa_1 = \text{the maximum normal curvature at } p$$

$$= \max_{|X|=1} \mathcal{L}(X, X)$$

$$\kappa_2 = \text{the minimum normal curvature at } p$$

$$= \min_{|X|=1} \mathcal{L}(X, X)$$

This is the geometric characterization of principal curvatures. There is also an important algebraic characterization.
Some Linear Algebra

Let $V$ be a vector space over the reals, and let $\langle \, , \, \rangle : V \times V \to \mathbb{R}$ be an inner product on $V$; hence $V$ is an inner product space. Let $L : V \to V$ be a linear transformation. Our main application will be to the case: $V = T_p M, \langle \, , \, \rangle =$ induced metric, and $L =$ Weingarten map.

$L$ is said to be **self adjoint** provided

$$\langle L(v), w \rangle = \langle v, L(w) \rangle \quad \forall \ v, w \in V.$$  

**Remark.** Let $V = \mathbb{R}^n$, with the usual dot product, and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $[L^i_j] = $ matrix representing $L$ with respect to the *standard* basis, $e_1 : (1, 0, \ldots, 0)$, etc. Then $L$ is self-adjoint if and only if $[L^i_j] =$ symmetric $[L^i_j] = [L^j_i]$.

**Proposition.** The Weingarten map $L : T_p M \to T_p M$ is self adjoint, i.e.

$$\langle L(X), Y \rangle = \langle X, L(Y) \rangle \quad \forall \ X, Y \in T_p M,$$

where $\langle \, , \, \rangle =$ 1st fundamental form.

**Proof.** We have,

$$\langle L(X), Y \rangle = \mathcal{L}(X,Y) = \mathcal{L}(Y,X)$$

$$= \langle L(Y), X \rangle = \langle X, L(Y) \rangle.$$  

Self adjoint linear transformations have very nice properties, as we now discuss. For this discussion, we restrict attention to 2-dimensional vector spaces, $\dim V = 2$.

A vector $v \in V, \ v \neq 0$, is called an *eigenvector* of $L$ if there is a real number $\lambda$ such that,

$$L(v) = \lambda v.$$  

$\lambda$ is called an *eigenvalue* of $L$. The eigenvalues of $L$ can be determined by solving

$$\det(A - \lambda I) = 0 \quad \quad \quad (*)$$

where $A$ is a matrix representing $L$ and $I =$ identity matrix. The equation $(*)$ is a quadratic equation in $\lambda$, and hence has at most 2 real roots; it may have no real roots.

**Theorem.** (Fundamental Theorem of Self Adjoint Operators) Let $V$ be a 2-dimensional inner product space. Let $L : V \to V$ be a self-adjoint linear map. Then $V$ admits an orthonormal basis consisting of eigenvectors of $L$. That is, there exists an orthonormal basis $\{e_1, e_2\}$ of $V$ and real numbers $\lambda_1, \lambda_2, \ \lambda_1 \geq \lambda_2$ such that

$$L(e_1) = \lambda_1 e_1, \quad L(e_2) = \lambda_2 e_2,$$

$$\mathcal{L}(e_1, e_2) = \mathcal{L}(e_2, e_1) = 0,$$

$$\mathcal{L}(e_1, e_1) = \langle e_1, e_1 \rangle = 1, \quad \mathcal{L}(e_2, e_2) = \langle e_2, e_2 \rangle = 1.$$
i.e., $e_1$ and $e_2$ are eigenvectors of $L$ and $\lambda_1, \lambda_2$ are the corresponding eigenvalues. Moreover the eigenvalues are given by

$$\lambda_1 = \max_{|v|=1} \langle L(v), v \rangle$$
$$\lambda_2 = \min_{|v|=1} \langle L(v), v \rangle.$$

**Proof.** See handout from Do Carmo.

**Remark on orthogonality of eigenvectors.** Let $e_1, e_2$ be eigenvectors with eigenvalues $\lambda_1, \lambda_2$. If $\lambda_1 \neq \lambda_2$, then $e_1$ and $e_2$ are necessarily orthogonal, as seen by the following,

$$\lambda_1 \langle e_1, e_2 \rangle = \langle L(e_1), e_2 \rangle = \langle e_1, L(e_2) \rangle = \lambda_2 \langle e_1, e_2 \rangle,$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle e_1, e_2 \rangle = 0 \Rightarrow \langle e_1, e_2 \rangle = 0.$$ On the other hand, if $\lambda_1 = \lambda_2 = \lambda$ then $L(v) = \lambda v$ for all $v$. Hence any o.n. basis is a basis of eigenvectors.

We now apply these facts to the Weingarten map,

$$L : T_pM \rightarrow T_pM,$$

$$\mathcal{L} : T_pM \times T_pM \rightarrow \mathbb{R}, \quad \mathcal{L}(X, Y) = \langle L(X), Y \rangle.$$ Since $L$ is self adjoint, and, by definition,

$$\kappa_1 = \max_{|X|=1} \mathcal{L}(X, X) = \max_{|X|=1} \langle L(X), X \rangle$$

$$\kappa_2 = \min_{|X|=1} \mathcal{L}(X, X) = \min_{|X|=1} \langle L(X), X \rangle,$$

we obtain the following.

**Theorem.** The principal curvatures $\kappa_1, \kappa_2$ of $M$ at $p$ are the eigenvalues of the Weingarten map $L : T_pM \rightarrow T_pM$. There exists an orthonormal basis $\{e_1, e_2\}$ of $T_pM$ such that

$$L(e_1) = \kappa_1 e_1, \quad L(e_2) = \kappa_2 e_2,$$

i.e., $e_1, e_2$ are eigenvectors of $L$ associated with the eigenvalues $\kappa_1, \kappa_2$, respectively. The eigenvectors $e_1$ and $e_2$ are called **principal directions**.

Observe that,

$$\kappa_1 = \kappa_1 \langle e_1, e_1 \rangle = \langle L(e_1), e_1 \rangle = \mathcal{L}(e_1, e_1)$$

$$\kappa_2 = \kappa_2 \langle e_2, e_2 \rangle = \langle L(e_2), e_2 \rangle = \mathcal{L}(e_2, e_2),$$

i.e., the principal curvature $\kappa_1$ is the normal curvature in the principal direction $e_1$, and similarly for $\kappa_2$. 18
Now, let $A$ be the matrix associated to the Weingarten map $L$ with respect to the orthonormal basis $\{e_1, e_2\}$; thus,

\[
L(e_1) = \kappa_1 e_1 + 0 e_2 \\
L(e_2) = 0 e_1 + \kappa_2 e_2
\]

which implies,

\[
A = \begin{bmatrix}
\kappa_1 & 0 \\
0 & \kappa_2
\end{bmatrix}.
\]

Then,

\[
\det L = \det A = \kappa_1 \kappa_2 \\
\text{tr} L = \text{tr} A = \kappa_1 + \kappa_2.
\]

**Definition.** The *Gaussian curvature* of $M$ at $p$, $K = K(p)$, and the *mean curvature* of $M$ at $p$, $H = H(p)$ are defined as follows,

\[
K = \det L = \kappa_1 \kappa_2 \\
H = \text{tr} L = \kappa_1 + \kappa_2.
\]

**Remarks.** The Gaussian curvature is the more important of the two curvatures; it is what is meant by the *curvature* of a surface. A famous discovery by Gauss is that it is intrinsic – in fact can be computed in terms of the $g_{ij}$’s (This is not obvious!). The *mean curvature* (which has to do with minimal surface theory) is *not* intrinsic. This can be easily seen as follows. Changing the normal $n \to -n$ changes the sign of the Weingarten map,

\[
L_{-n} = -L_n.
\]

This in turn changes the sign of the principal curvatures, hence $H = \kappa_1 + \kappa_2$ changes sign, but $K = \kappa_1 \kappa_2$ does not change sign.
Some Examples

**Ex.** For \( S^2_r \) = sphere of radius \( r \), compute \( \kappa_1, \kappa_2, K, H \) (Use outward normal).

Geometrically: \( p \in S^2_r, \ X \in T_p M, \ |X| = 1, \)

\[
\mathcal{L}(X, X) = \pm \text{curvature of normal section in direction } X \\
= -\text{curvature of great circle} \\
= -\frac{1}{r}.
\]

Therefore

\[
\kappa_1 = \max_{|X|=1} \mathcal{L}(X, X) = -\frac{1}{r},
\]

\[
\kappa_2 = \min_{|X|=1} \mathcal{L}(X, X) = -\frac{1}{r},
\]

\[
K = \kappa_1 \kappa_2 = \frac{1}{r^2} > 0, \quad H = \kappa_1 + \kappa_2 = -\frac{2}{r}.
\]

Algebraically: Find eigenvalues of Weingarten map: \( L : T_p M \rightarrow T_p M \). We showed previously,

\[
L = -\frac{1}{r} \text{id}, \quad \text{i.e., } \]

\[
L(X) = -\frac{1}{r} X \quad \text{for all } X \in T_p M.
\]

Thus, with respect to \textit{any} orthonormal basis \( \{e_1, e_2\} \) of \( T_p M \),

\[
L(e_i) = -\frac{1}{r} e_i \quad i = 1, 2.
\]
Therefore, $\kappa_1 = \kappa_2 = -\frac{1}{r}$, $K = \frac{1}{r^2}$, $H = -\frac{2}{r}$.

**Ex.** Let $M$ be the cylinder of radius $a$: $x^2 + y^2 = a^2$. Compute $\kappa_1, \kappa_2, K, H$. (Use the inward pointing normal)

Geometrically:

\[
\mathcal{L}(X_1, X_1) = \pm \text{curvature of normal section in direction } X_1
\]
\[
= \pm \text{curvature of circle of radius } a
\]
\[
= \pm \frac{1}{a},
\]
\[
\mathcal{L}(X_2, X_2) = \pm \text{curvature of normal section in direction } X_2
\]
\[
= \text{curvature of line}
\]
\[
= 0.
\]

In general, for $X \neq X_1, X_2$,

\[
\mathcal{L}(X, X) = \text{curvature of ellipse through } p.
\]

The curvature is between 0 and $\frac{1}{a}$, and thus,

\[
0 \leq \mathcal{L}(X, X) \leq \frac{1}{a}.
\]

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We conclude that,

\[ \kappa_1 = \max_{|X|=1} \mathcal{L}(X, X) = \mathcal{L}(X_1, X_1) = \frac{1}{a}, \]

\[ \kappa_2 = \min_{|X|=1} \mathcal{L}(X, X) = \mathcal{L}(X_2, X_2) = 0. \]

Thus, \( K = 0 \) (cylinder is flat!) and \( H = \frac{1}{a} \).

Algebraically: Determine the eigenvalues of the Weingarten map. By a rotation and translation we may take \( p \) to be the point \( p = (a, 0, 0) \). Let \( e_1, e_2 \in T_pM \) be the tangent vectors \( e_1 = (0, 1, 0) \) and \( e_2 = (0, 0, 1) \).

To compute \( \mathcal{L}(e_1) \), consider the circle,

\[ \sigma(s) = (a \cos(\frac{s}{a}), a \sin(\frac{s}{a}), 0) \]

Note that \( \sigma(0) = p \) and \( \sigma'(0) = e_1 \). Thus,

\[ \mathcal{L}(e_1) = -\nabla_{e_1} n \]

\[ = -\frac{d}{ds} n(\sigma(s))|_{s=0} \]

But,

\[ n(\sigma(s)) = n(\sigma(s)) = \frac{\sigma(s)}{|\sigma(s)|} = \frac{\sigma(s)}{a} \]

\[ = -\left( \cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right), 0 \right) \]
Therefore, 
\[ L(e_1) = \frac{d}{ds} \left( \cos \left( \frac{s}{a} \right), \sin \left( \frac{s}{a} \right), 0 \right) |_{s=0} \]
\[ = \frac{1}{a} \left( -\sin \left( \frac{s}{a} \right), \cos \left( \frac{s}{a} \right), 0 \right) |_{s=0} \]
\[ = \frac{1}{a} (0, 1, 0) \]

\[ L(e_1) = \frac{1}{a} e_1 \]

Thus, \( e_1 \) is an eigenvector with eigenvalue \( \frac{1}{a} \). Similarly (exercise!),

\[ L(e_2) = 0 = 0 \cdot e_2 \]

i.e., \( e_2 \) is an eigenvector with eigenvalue 0. (Note; \( e_2 \) is tangent to a vertical line in the surface, along which \( n \) is constant.)

We conclude that, \( \kappa_1 = \frac{1}{a}, \kappa_2 = 0, K = 0, H = \frac{1}{a} \).

**Ex.** Consider the saddle surface, \( M \): \( z = y^2 - x^2 \), Compute \( \kappa_1, \kappa_2, K, H \) at \( p = (0, 0, 0) \).

\[ \mathcal{L}(e_1, e_1) = \pm \text{curvature of normal section in direction of } e_1 \]
\[ = + \text{curvature of } z = y^2 \]

The curvature is given by,

\[ \kappa = \frac{\left| \frac{d^2 z}{dy^2} \right|}{\left[ 1 + \left( \frac{dz}{dy} \right)^2 \right]^{3/2}} = 2 \]
and so, $\mathcal{L}(e_1, e_1) = 2$. Similarly, $\mathcal{L}(e_2, e_2) = -2$. Observe,

$$\mathcal{L}(e_2, e_2) \leq \mathcal{L}(X, X) \leq \mathcal{L}(e_1, e_1)$$

Therefore, $\kappa_1 = 2$, $\kappa_2 = -2$, $K = -4$, and $H = 0$ at $(0, 0, 0)$.

**Exercise 5.5.** For the saddle surface $M$ above, consider the Weingarten map $L : T_pM \to T_pM$ at $p = (0, 0, 0)$. Compute $L(e_1)$ and $L(e_2)$ directly from the definition of the Weingarten map to show,

$$L(e_1) = 2e_1 \text{ and } L(e_2) = -2e_2.$$  

Hence, $-2$ and $2$ are the eigenvalues of $L$, which means $\kappa_1 = 2$ and $\kappa_2 = -2$.

**Remark.** We have computed the quantities $\kappa_1$, $\kappa_2$, $K$, and $H$ of the saddle surface only at a single point. To compute these quantities at all points, we will need to develop better computational tools.

---

**Significance of the sign of Gaussian Curvature**

We have,

$$K = \det L = \kappa_1 \kappa_2.$$  

1. $K > 0 \iff \kappa_1$ and $\kappa_2$ have the same sign $\iff$ the normal sections in the principal directions $e_1, e_2$ both bend in the same direction,

![Image](image_url)

**Ex.** $z = ax^2 + by^2$, $a, b$ have the same sign (elliptic paraboloid). At $p = (0, 0, 0)$, $K = 4ab > 0$.

2. $K < 0 \iff \kappa_1$ and $\kappa_2$ have opposite signs $\iff$ normal sections in principle directions $e_1$ and $e_2$ bend in opposite directions,

![Image](image_url)

**Ex.** $z = ax^2 + by^2$, $a, b$ have opposite sign (hyperbolic paraboloid). At $p = (0, 0, 0)$, $K = 4ab < 0$. 

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Thus, roughly speaking,

\[ K > 0 \text{ at } p \Rightarrow \text{surface is “bowl-shaped” near } p \]
\[ K < 0 \text{ at } p \Rightarrow \text{surface is “saddle-shaped” near } p \]

This rough observation can be made more precise, as we now show. Let \( M \) be a surface, \( p \in M \). Let \( e_1, e_2 \) be principal directions at \( p \). Choose \( e_1, e_2 \) so that \( \{e_1, e_2, n\} \) is a positively oriented orthonormal basis.

By a translation and rotation of the surface, we can assume, (see the figure),

1. \( p = (0, 0, 0) \)
2. \( e_1 = (1, 0, 0), \ e_2 = (0, 1, 0), \ n = (0, 0, 1) \) at \( p \)
3. Near \( p = (0, 0, 0) \), the surface can be described by an equation of form, \( z = f(x, y) \), where \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \) is smooth and \( f(0, 0) = 0 \).

Claim:

\[ z = \frac{1}{2} \kappa_1 x^2 + \frac{1}{2} \kappa_2 y^2 + \text{higher order terms} \]

Proof. Consider the Taylor series about \((0, 0)\) for functions of two variables,

\[ z = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2 + \text{higher order terms} \]

We must compute 1st and 2nd order partial derivatives of \( f \) at \((0, 0)\). Introduce the Monge patch,

\[ x = u \]
\[ y = v \]
\[ z = f(u, v) \]

i.e. \( \mathbf{x}(u, v) = (u, v, f(u, v)) \).
We have,
\[
\begin{align*}
x_1 &= x_u = (1, 0, f_u), \\
x_2 &= x_v = (0, 1, f_v), \\
n &= \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{x_u \times x_v}{|x_u \times x_v|} \\
&= \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}.
\end{align*}
\]

At \((u, v) = (0, 0)\): \(n = (0, 0, 1) \Rightarrow f_u = f_v = 0, \Rightarrow x_1 = (1, 0, 0) = e_1 \) and \(x_2 = (0, 1, 0) = e_2\).

Recall, the components of the 2nd fundamental form \(L_{ij} = \mathcal{L}(x_i, x_j)\) may be computed from the formula,
\[
L_{ij} = \langle n, x_{ij} \rangle, \quad x_{ij} = \frac{\partial^2 x}{\partial u^i \partial u^j}.
\]

In particular, \(L_{11} = \langle n, x_{11} \rangle\), where \(x_{11} = x_{uu} = (0, 0, f_{uu})\).

At \((u, v) = (0, 0)\): \(L_{11} = \langle n, x_{11} \rangle = (0, 0, 1) \cdot (0, 0, f_{uu}(0, 0)) = f_{uu}(0, 0)\).
Therefore, \(f_{uu}(0, 0) = L_{11} = \mathcal{L}(x_1, x_1) = \mathcal{L}(e_1, e_1) = \kappa_1\). Similarly,
\[
\begin{align*}
f_{uv}(0, 0) &= \mathcal{L}(e_2, e_2) = \kappa_2 \\
f_{vu}(0, 0) &= \mathcal{L}(e_1, e_2) = \langle \mathcal{L}(e_1), e_2 \rangle = \lambda_1(e_1, e_2) = 0.
\end{align*}
\]

Thus, setting \(x = u, \ y = v\), we have shown,
\[
\begin{align*}
f_x(0, 0) &= f_y(0, 0) = 0 \\
f_{xx}(0, 0) &= \kappa_1, \quad f_{yy}(0, 0) = \kappa_2, \quad f_{xy}(0, 0) = 0,
\end{align*}
\]
which, substituting in the Taylor expansion, implies,
\[
z = \frac{1}{2} \kappa_1 x^2 + \frac{1}{2} \kappa_2 y^2 + \text{higher order terms}.
\]

**Computational Formula for Gaussian Curvature.**

We have,
\[
K = \text{Gaussian curvature} = \det L = \det[L^i_j].
\]
From the equation at the top of p. 14,
\[
[L_{ij}] = [g_{ij}][L^i_j],
\]
\[
\det[L_{ij}] = \det[g_{ij}] \det[L^i_j]
= \det[g_{ij}] \cdot K
\]
Hence,
\[
K = \frac{\det[L_{ij}]}{\det[g_{ij}]}, \quad g_{ij} = \langle x_i, x_j \rangle, \quad L_{ij} = \langle n, x_{ij} \rangle.
\]

Further,
\[
\det[L_{ij}] = \det \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}
= L_{11}L_{22} - L_{12}^2,
\]
since \(L_{12} = L_{21}\), and similarly,
\[
\det[g_{ij}] = g_{11}, g_{22} - g_{12}^2.
\]
Thus,
\[
K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}.
\]

**Ex.** Compute the Gaussian curvature of the saddle surface \(z = y^2 - x^2\).

Introduce the Monge patch, \(x(u, v) = (u, v, v^2 - u^2)\).

Compute metric components \(g_{ij}\):

\[
x_u = (1, 0, -2u), \quad x_v = (0, 1, 2v),
\]
\[
g_{uu} = \langle x_u, x_u \rangle = (1, 0, -2u) \cdot (1, 0, -2u)
= 1 + 4u^2.
\]

Similarly,
\[
g_{vv} = \langle x_v, x_v \rangle = 1 + 4v^2,
\]
\[
g_{uv} = \langle x_u, x_v \rangle = -4uv.
\]

Thus,
\[
\det[g_{ij}] = g_{uu}g_{vv} - g_{uv}^2
= (1 + 4u^2)(1 + 4v^2) - 16u^2v^2
= 1 + 4u^2 + 4v^2.
\]
Compute the second fundamental form components $L_{ij}$:

We use, $L_{ij} = \langle n, x_{ij} \rangle$. We have,

$$n = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{(2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}},$$

and,

$$x_{uu} = (0, 0, -2), \quad x_{vv} = (0, 0, 2), \quad x_{uv} = (0, 0, 0).$$

Then,

$$L_{uu} = \langle n, x_{uu} \rangle = \frac{-2}{\sqrt{1 + 4u^2 + 4v^2}},$$

$$L_{vv} = \langle n, x_{vv} \rangle = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}},$$

$$L_{uv} = \langle n, x_{uv} \rangle = 0.$$

Thus,

$$\det[L_{ij}] = L_{uu}L_{vv} - L_{uv}^2 = \frac{-4}{1 + 4u^2 + 4v^2},$$

and therefore,

$$K(u, v) = \frac{\det[L_{ij}]}{\det[g_{ij}]} = \frac{-4}{1 + 4u^2 + 4v^2} \cdot \frac{1}{1 + 4u^2 + 4v^2}.$$

Hence the saddle surface $z = y^2 - x^2$ has Gaussian curvature function,

$$K(x, y) = \frac{-4}{(1 + 4x^2 + 4y^2)^2}.$$

Observe that $K < 0$ and, $K = \frac{-4}{(1 + 4r^2)^2} \sim \frac{1}{r^4}$, where $r = \sqrt{x^2 + y^2}$ is the distance from the $z$-axis. As $r \to \infty$, $K \to 0$ rapidly.

**Exercise 5.6.** Consider the surface $M$ which is the graph of $z = f(x, y)$. Show that the Gaussian curvature $K = K(x, y)$ is given by,

$$K(x, y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

where $f_x = \frac{\partial f}{\partial x}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, etc.
Exercise 5.7. Let $M$ be the torus of large radius $R$ and small radius $r$ described in Exercise 3.3. Using the parameterization, 
\[ x(t, \theta) = ((R + r \cos t) \cos \theta, (R + r \cos t) \sin \theta, r \sin t) \]
show that the Gaussian curvature $K = K(t, \theta)$ is given by,
\[ K = \frac{\cos t}{r(R + r \cos t)}. \]
Where on the torus is the Gaussian curvature negative? Where is it positive?

Exercise 5.8. Derive the following expression for the mean curvature $H$,
\[ H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11}}{g_{11}g_{22} - g_{12}^2}. \]

The principal curvatures $\kappa_1$ and $\kappa_2$ at a point $p \in M$ are the normal curvatures in the principal directions $e_1$ and $e_2$. The normal curvature in any direction $X$ is determined by $\kappa_1$ and $\kappa_2$ as follows.

If $X \in T_p M$, $|X| = 1$ then $X$ can be expressed as (see the figure),
\[ X = \cos \theta e_1 + \sin \theta e_2. \]

Proposition (Euler’s formula). The normal curvature in the direction $X$ is given by,
\[ L(X, X) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta, \]
where $\kappa_1, \kappa_2$ are the principal curvatures, and $\theta$ is the angle between $X$ and the principal direction $e_1$.

Proof. Use the shorthand, $c = \cos \theta$, $s = \sin \theta$. Then $X = ce_1 + se_2$, and
\[ L(X) = L(ce_1 + se_2) \]
\[ = cL(e_1) + sL(e_2) \]
\[ = c\kappa_1 e_1 + s\kappa_2 e_2. \]
Therefore,
\[ L(X, X) = \langle L(X), X \rangle \]
\[ = \langle c\kappa_1 e_1 + s\kappa_2 e_2, ce_1 + se_2 \rangle \]
\[ = c^2\kappa_1 + s^2\kappa_2. \]

**Exercise 5.9.** Assuming \( \kappa_1 > \kappa_2 \), determine where (i.e., for which values of \( \theta \)) the function,
\[ \kappa(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta, \quad 0 \leq \theta \leq 2\pi \]
achieves its maximum and minimum. The answer shows that the principal directions \( e_1, e_2 \) are unique, up to sign, in this case.

**Gauss Theorema Egregium**

The Weingarten map,
\[ L(X) = -\nabla_X n \]
is an extrinsically defined object - it involves the normal to the surface. There is no reason to suspect that the determinant of \( L \), the Gaussian curvature, is intrinsic, i.e. can be computed from measurements taken in the surface. But Gauss carried out some courageous computations and made the extraordinary discovery that, in fact, the Gaussian curvature \( K \) is intrinsic - i.e., can be computed from the \( g_{ij} \)'s. This is the most important result in the subject - albeit not the prettiest! If this result were not true then the subject of differential geometry, as we know it, would not exist.

We now embark on the same path - courageously carrying out the same computation.

**Some notation.** Introduce the “inverse” metric components, \( g^{ij} \), \( 1 \leq i, j \leq 2 \), by
\[ [g^{ij}] = [g_{ij}]^{-1}, \]
i.e. \( g^{ij} \) is the \( i-j \)th entry of the inverse of the matrix \( [g_{ij}] \). Using the formula,
\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]
we can express \( g^{ij} \) explicitly in terms of the \( g_{ij} \), e.g.
\[ g^{11} = \frac{g_{22}}{g_{11}g_{22} - g_{12}^2}, \quad \text{etc.} \]

Note, in an orthogonal coordinate system, i.e., a proper patch in which \( g_{12} = \langle x_1, x_2 \rangle = 0 \), we have simply,
\[ g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{12} = g^{21} = 0. \]
By definition of inverse, we have

\[ [g_{ij}] [g^{ij}] = I \]

where \( I \) = identity matrix = \( [\delta^i_j] \), where \( \delta^i_j \) is the Kronecker delta (cf., Chapter 1),

\[ \delta^i_j = \begin{cases} 
0 & , i \neq j \\
1 & , i = j 
\end{cases} \]

and so,

\[ [g_{ij}] [g^{ij}] = [\delta^i_j] \, . \]

The product formula for matrices then implies,

\[ \sum_k g_{ik} g^{kj} = \delta^i_j \]

or, by the Einstein summation convention,

\[ g_{ik} g^{kj} = \delta^i_j \, . \]

Now, let \( M \) be a surface and \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow M \subset \mathbb{R}^3 \) be any proper patch in \( M \). Then,

\[ \mathbf{x} = \mathbf{x}(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2)) \, , \]

\[ \mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u^i} = \left( \frac{\partial x}{\partial u^i}, \frac{\partial y}{\partial u^i}, \frac{\partial z}{\partial u^i} \right) \, , \]

\[ \mathbf{x}_{ij} = \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i} = \left( \frac{\partial^2 x}{\partial u^j \partial u^i}, \frac{\partial^2 y}{\partial u^j \partial u^i}, \frac{\partial^2 z}{\partial u^j \partial u^i} \right) \, . \]

We seek useful expressions for these second derivatives. At any point \( p \in \mathbf{x}(U) \), \{\( \mathbf{x}_1, \mathbf{x}_2, n \)\} form a basis for \( T_p \mathbb{R}^3 \). Since at \( p \), \( \mathbf{x}_{ij} \in T_p \mathbb{R}^3 \), we can write,

\[ \mathbf{x}_{ij} = \Gamma^1_{ij} \mathbf{x}_1 + \Gamma^2_{ij} \mathbf{x}_2 + \lambda_{ij} n \, , \]

\[ \mathbf{x}_{ij} = \sum_{\ell=1}^2 \Gamma^\ell_{ij} \mathbf{x}_\ell + \lambda_{ij} n \, . \]

or, making use of the Einstein summation convention,

\[ \mathbf{x}_{ij} = \Gamma^\ell_{ij} \mathbf{x}_\ell + \lambda_{ij} n \, . \quad (\ast) \]
We obtain expressions for $\lambda_{ij}, \Gamma_{ij}^\ell$. Dotting (*) with $n$ gives,

$$\langle x_{ij}, n \rangle = \Gamma_{ij}^\ell \langle x_\ell, n \rangle + \lambda_{ij} \langle n, n \rangle$$

$$\Rightarrow \lambda_{ij} = \langle x_{ij}, n \rangle = \langle n, x_{ij} \rangle$$

$$\lambda_{ij} = L_{ij}.$$ 

Dotting (*) with $x_k$ gives,

$$\langle x_{ij}, x_k \rangle = \Gamma_{ij}^\ell \langle x_\ell, x_k \rangle + \lambda_{ij} \langle n, x_k \rangle$$

$$\langle x_{ij}, x_k \rangle = \Gamma_{ij}^\ell g_{\ell k}.$$ 

Solving for $\Gamma_{ij}^\ell$,

$$\langle x_{ij}, x_k \rangle g^{km} = \Gamma_{ij}^\ell g_{\ell k} g^{km}$$

$$= \Gamma_{ij}^\ell \delta_m^\ell$$

$$\langle x_{ij}, x_k \rangle g^{km} = \Gamma_m^\ell \delta_{ij}$$

Thus,

$$\Gamma_{ij}^\ell = g^{\ell k} \langle x_{ij}, x_k \rangle$$ 

**Claim.** The quantity $\langle x_{ij}, x_k \rangle$ is given by,

$$\langle x_{ij}, x_k \rangle = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

$$= \frac{1}{2} (g_{ik,j} + g_{jk,i} - g_{ij,k})$$ 

**Proof of Claim.** We use Gauss’ trick of permuting indices.

$$g_{ij,k} = \frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k} \langle x_i, x_j \rangle$$

$$= \langle \frac{\partial x_i}{\partial u^k}, x_j \rangle + \langle x_i, \frac{\partial x_j}{\partial u^k} \rangle$$

(1) $g_{ij,k} = \langle x_{ik}, x_j \rangle + \langle x_i, x_{jk} \rangle$

(2) $g_{ik,j} = \langle x_{ij}, x_k \rangle + \langle x_i, x_{kj} \rangle$

(3) $g_{jk,i} = \langle x_{ji}, x_k \rangle + \langle x_j, x_{ki} \rangle$
Then \((2) + (3) - (1)\) gives:

\[
g_{ik,j} + g_{jk,i} - g_{ij,k} = 2 \langle x_{ij}, x_k \rangle
\]

Thus,

\[
\Gamma^\ell_{ij} = \frac{1}{2} g^{k\ell} (g_{ik,j} + g_{jk,i} - g_{ij,k}).
\]

**Remark.** These are known as the **Christoffel symbols.**

Summarizing, we have,

\[
x_{ij} = \Gamma^\ell_{ij} x_\ell + L_{ij} n,
\]

**Gauss Formula**

where \(L_{ij}\) are the components of the 2nd fundamental form and \(\Gamma^\ell_{ij}\) are the Christoffel symbols as given above. Let us also recall the **Weingarten equations** (p. 11),

\[
n_j = -L^i_{j,i} x_i
\]

**Remark.** The vector fields \(x_1, x_2, n\), play a role in surface theory roughly analogous to the Frenet frame for curves. The two formulas above for the partial derivatives of \(x_1, x_2, n\) then play a role roughly analogous to the Frenet formulas.

Now, Gauss takes things one step further and computes the 3rd derivatives,

\[
x_{ijk} = \frac{\partial}{\partial u^k} x_{ij}:
\]

\[
x_{ijk} = \frac{\partial}{\partial u^k} \left( \Gamma^\ell_{ij} x_\ell + L_{ij} n \right) = \frac{\partial}{\partial u^k} \Gamma^\ell_{ij} x_\ell + \frac{\partial}{\partial u^k} L_{ij} n
\]

\[
= \Gamma^\ell_{ij,k} x_\ell + \Gamma^\ell_{ij} x_{\ell k} + L_{ij,k} n + L_{ij} n_k
\]

\[
= \Gamma^\ell_{ij,k} x_\ell + \Gamma^\ell_{ij} \left( \Gamma^m_{\ell k} x_m + L_{\ell k} n \right) + L_{ij,k} n + L_{ij} \left( -L^\ell_k x_\ell \right)
\]

\[
= \Gamma^\ell_{ij,k} x_\ell + \Gamma^\ell_{ij} \Gamma^m_{\ell k} x_m + \Gamma^\ell_{ij} L_{\ell k} n + L_{ij,k} n - L_{ij} L^\ell_k x_\ell
\]

Thus,

\[
x_{ijk} = \left( \Gamma^\ell_{ij,k} + \Gamma^m_{ij} \Gamma^\ell_{mk} - L_{ij,k} L^\ell_k \right) x_\ell + \left( L_{ij,k} + \Gamma^\ell_{ij} L_{\ell k} \right) n,
\]

and interchanging indices \((j \leftrightarrow k)\),

\[
x_{ikj} = \left( \Gamma^\ell_{ik,j} + \Gamma^m_{ik} \Gamma^\ell_{mj} - L_{ik,j} L^\ell_j \right) x_\ell + \left( L_{ik,j} + \Gamma^\ell_{ik} L_{\ell j} \right) n.
\]

Now, \(x_{ikj} = x_{ijk}\) implies

\[
\Gamma^\ell_{ik,j} + \Gamma^m_{ik} \Gamma^\ell_{mj} - L_{ik,j} L^\ell_j = \Gamma^\ell_{ij,k} + \Gamma^m_{ij} \Gamma^\ell_{mk} - L_{ij} L^\ell_k =
\]
or,

\[
\Gamma_{ik,j}^\ell - \Gamma_{ij,k}^\ell + \Gamma_{ik}^m \Gamma_{mj}^\ell - \Gamma_{ij}^m \Gamma_{mk}^\ell = L_{ik} L_j^\ell - L_{ij} L_k^\ell .
\]

These are the components of the famous Riemann curvature tensor. Observe: \( R_{ijk}^\ell \) are intrinsic, i.e. can be computed from the \( g_{ij} \)'s (involve 1st and 2nd derivatives of the \( g_{ij} \)'s).

We arrive at,

\[
R_{ijk}^\ell = L_{ik} L_j^\ell - L_{ij} L_k^\ell \quad \text{The Gauss Equations.}
\]

**Gauss’ Theorem Egregium.** The Gaussian curvature of a surface is intrinsic, i.e. can be computed in terms of the \( g_{ij} \)'s.

**Proof.** This follows from the Gauss equations. Multiply both sides by \( g_{m\ell} \),

\[
g_{m\ell} R_{ijk}^\ell = L_{ik} g_{m\ell} L_j^\ell - L_{ij} g_{m\ell} L_k^\ell .
\]

But recall (see p. 13),

\[
L_{mj} = g_{m\ell} L_j^\ell .
\]

Hence,

\[
g_{m\ell} R_{ijk}^\ell = L_{ik} L_{mj} - L_{ij} L_{mk} .
\]

Setting \( i = k = 1, m = j = 2 \) we obtain,

\[
g_{2\ell} R_{121}^\ell = L_{11} L_{22} - L_{12} L_{21}
\]

\[
= \det[L_{ij}] .
\]

Thus,

\[
K = \frac{\det[L_{ij}]}{\det[g_{ij}]}
\]

\[
K = \frac{g_{2\ell} R_{121}^\ell}{g}, \quad g = \det[g_{ij}]
\]

**Comment.** Gauss’ Theorema Egregium can be interpreted in a slightly different way in terms of isometries. We discuss this point here very briefly and very informally.

Let \( M \) and \( N \) be two surfaces. A one-to-one, onto map \( f : M \to N \) that preserves lengths and angles is called an isometry. (This may be understood at the level of tangent vectors: \( f \) takes curves to curves, and hence velocity vectors to velocity vectors. \( f \) is an isometry \( \iff \) \( f \) preserves angle between velocity vectors and preserve length of velocity vectors.)
Ex. The process of wrapping a piece of paper into a cylinder is an isometry.

**Theorem** Gaussian curvature is a bending invariant, i.e. is invariant under isometries, by which we mean: if $f : M \to N$ is an isometry then

$$K_N(f(p)) = K_M(p),$$

i.e., the Gaussian curvature is the same at corresponding points.

**Proof** $f$ preserves lengths and angles. Hence, in appropriate coordinate systems, the metric components for $M$ and $N$ are the same. By the formula for $K$ above, the Gaussian curvature will be the same at corresponding points.

**Application 1.** The cylinder has Gaussian curvature $K = 0$ (because a plane has zero Gaussian curvature).

**Application 2.** No piece of a plane can be bent into a piece of a sphere without distorting lengths (because $K_{\text{plane}} = 0$, $K_{\text{sphere}} = \frac{1}{r^2}$, $r =$radius).

**Theorem (Riemann).** Let $M$ be a surface with vanishing Gaussian curvature, $K = 0$. Then each $p \in M$ has a neighborhood which is isometric to an open set in the Euclidean plane.

**Exercise 5.10.** Although the Gaussian curvature $K$ is a “bending invariant”, show that the principal curvatures $\kappa_1, \kappa_2$ are not. I.e., show that the principle curvatures are not in general invariant under an isometry. (Hint: Consider the bending of a rectangle into a cylinder).