Chapter 3. Surfaces

We all understand intuitively what a surface is. In calculus we encounter surfaces in several ways.

1. As graphs of functions of two variables, \( z = f(x, y) \).
   
   Ex. \( z = x^2 + y^2 \)

2. As level surfaces of functions of three variables, \( F(x, y, z) = c \).
   
   Ex. \( x^2 + y^2 + z^2 = 1 \)

3. As surfaces of revolution.
   
   Ex. Torus: surface of a doughnut. This surface is not of type (1) or (2).

We will need to be fairly precise about what we mean by a surface. Our definition will need to cover all these cases. The key is to describe surfaces parametrically. Very roughly speaking, a surface for us is going to be a subset of \( \mathbb{R}^3 \) which can be broken up into overlapping pieces such that each piece is described parametrically, i.e. described by a 2-parameter map.

Hence, the starting point is the notion of parameterized surfaces.

Def. A smooth parameterized surface in \( \mathbb{R}^3 \) is a smooth map \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), \( (u, v) \rightarrow \mathbf{x}(u, v) \).

As \( (u, v) \) varies over \( U \), \( \mathbf{x}(u, v) \in \mathbb{R}^3 \) traces out a “surface” in \( \mathbb{R}^3 \).
In terms of components, \( \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \),

\[
\begin{align*}
  x &= x(u, v) \\
  \mathbf{x} : y &= y(u, v) \quad (u, v) \in U \\
  z &= z(u, v)
\end{align*}
\]

An effective way to see what gets traced out is to look at the “\( u \)-curves” and “\( v \)-curves”.

(1) if \( v \) is held constant, \( v = v_0 \) and \( u \) varies,

\[
  u \rightarrow \mathbf{x}(u, v_0) \quad \text{“} u \text{-curve”}
\]

(2) if \( u \) is held constant, \( u = u_0 \) and \( v \) varies,

\[
  v \rightarrow \mathbf{x}(u_0, v) \quad \text{“} v \text{-curve”}
\]

One way to examine a parameterized surface is to plot many “coordinate” curves, \( u = \text{const}, v = \text{const} \). This is how e.g., Mathematica plots parameterized surfaces.

Ex. \( \mathbf{x} : U \to \mathbb{R}^3, U = \{(u, v) : 0 < u < 2\pi, 0 < v < 3\} \), \( \mathbf{x}(u, v) = (2 \cos u, 2 \sin u, v) \),

\[
\begin{align*}
  x &= 2 \cos u \\
  \mathbf{x} : y &= 2 \sin u \quad 0 < u < 2\pi, \quad 0 < v < 3 \\
  z &= v
\end{align*}
\]

For this example it is convenient to consider closed rectangle \( \overline{U} : 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 3 \). We plot some \( u \)-curves and \( v \)-curves:

\[
\begin{align*}
  v = 0 : \quad x &= 2 \cos u \\
  y &= 2 \sin u \quad 0 \leq u \leq 2\pi \quad \text{circle in } z = 0 \\
  z &= 0
\end{align*}
\]

\[
\begin{align*}
  v = 1 : \quad x &= 2 \cos u \\
  y &= 2 \sin u \quad 0 \leq u \leq 2\pi \quad \text{circle in } z = 1 \\
  z &= 1
\end{align*}
\]

etc.
\[ u = 0 : \begin{align*} x &= 2 \\ y &= 0 \\ z &= v \end{align*} \quad \text{vertical line} \]

\[ z = 0 \quad 0 \leq v \leq 3 \]

\[ u = \pi/2 : \begin{align*} x &= 0 \\ y &= 2 \\ z &= v \end{align*} \quad \text{vertical line} \]

\[ x = 0 \quad 0 \leq v \leq 3 \]

etc.

This parameterized surface describes a cylinder. Note that the coordinate functions satisfy:

\[ x^2 + y^2 = 4, \quad 0 \leq z \leq 3 \]

**Note:** On the original domain \( U \), \( x \) is 1-1. We will restrict attention to parameterized surfaces \( x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) which are 1-1. Cylinder of radius \( a \):

\[ x(u, v) = (a \cos u, a \sin u, v) \]

**Coordinate Vector Fields.** Given a smooth surface,

\[ x(u, v) = (x(u, v), y(u, v), z(u, v)) \]

we can differentiate wrt \( u \) and \( v \),

\[
\frac{\partial x}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} \frac{\partial z}{\partial v} \end{pmatrix}
\]

These partial derivatives have natural interpretations,

\[
\frac{\partial x}{\partial u}(u_0, v_0) = \text{tangent vector to } u\text{-curve} \quad u \rightarrow x(u, v_0) \text{ at } x(u_0, v_0)
\]

\[
\frac{\partial x}{\partial v}(u_0, v_0) = \text{tangent vector to } v\text{-curve} \quad v \rightarrow x(u_0, v) \text{ at } x(u_0, v_0)
\]
Hence,
\[ \frac{\partial x}{\partial u} = \text{velocity vector field to } u\text{-curves} \]
\[ \frac{\partial x}{\partial v} = \text{velocity vector field to } v\text{-curves}. \]

**Remark.** The coordinate curves \( u = u_0, \ v = v_0 \) lie in the surface. Hence the coordinate vectors \( \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial x}{\partial v}(u_0, v_0) \) are *tangent* vectors to the surface at \( x(u_0, v_0) \).

**Standard Picture:** Grid of horizontal and vertical lines in \( U \subset \mathbb{R}^2 \) gives rise to a grid of curves - the coordinate curves on \( x(U) \). This amounts to introducing coordinates on \( x(U) \).

**Shorthand Notation:** \( x_u = \frac{\partial x}{\partial u}, \ x_v = \frac{\partial x}{\partial v}. \)

Actually, to insure that the image of a parameterized surface \( x \) looks like a surface (i.e. smooth 2-dimensional object), we need a *regularity* condition, akin to the regularity condition for parameterized curves (\( \sigma'(t) \neq 0 \)).

**Ex.** \( x: \mathbb{R}^2 \to \mathbb{R}^3, \ x(u, v) = (0, 0, 0) \ \forall (u, v). \) Image a single point! \( \frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = \mathbf{0}. \)

**Ex.** \( x: \mathbb{R}^2 \to \mathbb{R}^3, \ x(u, v) = (\cos(u + v^2), \sin(u + v^2), 1) \)

\[ x = \cos(u + v^2) \]
\[ \mathbf{x}: \ y = \sin(u + v^2) \]
\[ z = 1 \]

Image: \( x^2 + y^2 = 1, z = 1, \) a circle!

Compute: \( \frac{\partial x}{\partial v} = -2v \frac{\partial x}{\partial u}, \) i.e. \( \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \) are linearly *dependent* (at every point).

To avoid this type of “degeneracy” must require that \( \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \) be linearly *independent*. There are several ways to characterize this independence.

Consider a parameterized surface, \( x: U \subset \mathbb{R}^2 \to \mathbb{R}^3, \ x(u, v) = (x(u, v), y(u, v), z(u, v)) \),

\[ x = x(u, v) \]
\[ \mathbf{x}: \ y = y(u, v) \]
\[ z = z(u, v) \]
$Dx = \text{Jacobian matrix of } x, \text{ is the } 2 \times 3 \text{ matrix:}

$$Dx = \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{bmatrix}$$

Recall, the rank of a matrix = no. of linearly independent rows = no. of linearly independent columns

**Prop.** Let $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface. Then the following conditions are equivalent.

1. $Dx$ has rank 2.
2. $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$ are linearly independent.
3. $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \neq 0$.

**Proof.**

$Dx$ has rank 2 $\iff$ columns lin. indep.

$\iff \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$ lin. indep.

$\iff$ one is not a multiple of the other

$\iff \frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \neq 0$.

**Def.** A **regular (parameterized) surface** in $\mathbb{R}^3$ is a smooth parameterized surface $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \neq 0$ for all $(u, v) \in U$. 
A coordinate patch is a regular surface $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ which is one-to-one. Roughly speaking, a surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is covered by coordinate patches.

If the regularity condition is not satisfied, the image of $x$ can degenerate to a point, or curve – or something that does not look like a smooth surface (surface with “folds” or “cusps”). If, however, the regularity condition is satisfied, then the image of $x$ will look like a smooth surface. This is made rigorous in the following proposition.

**Proposition.** Let $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface. Then for each $(u_0, v_0) \in U$ there is a neighborhood $V \subset U$ of $(u_0, v_0)$ such that the image $x(V) \subset \mathbb{R}^3$ coincides with the graph of an equation of the form,

$$z = f(x, y) \quad \text{or} \quad y = g(x, z) \quad \text{or} \quad x = h(y, z),$$

where $f, g, h$ are smooth functions of two variables.

**Proof.** The proof is an application of the Inverse Function theorem. We have $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $x(u, v) = (x(u, v), y(u, v), z(u, v))$, and

$$\frac{\partial x}{\partial u} = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right), \quad \frac{\partial x}{\partial v} = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

Then,

$$\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} i - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} j + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} k.$$

Since, by regularity, $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \neq 0$ at $(u_0, v_0)$, one of the components must be nonzero, say,

$$\left| \begin{array}{ll} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array} \right| \neq 0 \quad \text{at} \quad (u_0, v_0).$$

Now, consider the map $\Phi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by, $\Phi(u, v) = (x(u, v), y(u, v))$,

$$\Phi : \begin{align*} x &= x(u, v) \\ y &= y(u, v) \end{align*}$$
Φ has Jacobian matrix,

$$D\Phi = \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{bmatrix}.$$ 

Hence $\det D\Phi \neq 0$ at $(u_0, v_0)$, i.e. $D\Phi$ is nonsingular at $(u_0, v_0)$. By the IFT there exists a neighborhood $V$ of $(u_0, v_0)$ such that $W = \Phi(V)$ is open in $\mathbb{R}^2$ and $\Phi : V \subset \mathbb{R}^2 \to W \subset \mathbb{R}^2$ is a diffeomorphism, i.e., $\Phi^{-1} : W \subset \mathbb{R}^2 \to V \subset \mathbb{R}^2$ is smooth. In terms of components, $\Phi^{-1}(x, y) = (u(x, y), v(x, y))$,

$$\Phi^{-1} : \begin{cases}
u = u(x, y) \\
v = v(x, y)
\end{cases}, (x, y) \in W$$

Now, let $f = z \circ \Phi^{-1}$, $f : W \subset \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = z(\Phi^{-1}(x, y)) = z(u(x, y), v(x, y)).$$

The graph of $f$ is the set of points in $\mathbb{R}^3$,

$$\text{graph } f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in W\}$$

Claim. $x(V) = \text{graph } f$:

$f = z \circ \Phi^{-1} \Rightarrow z = f \circ \Phi$, hence $z(u, v) = f(\Phi(u, v)) = f(x(u, v), y(u, v))$. Thus,

$$x(u, v) = (x(u, v), y(u, v), z(u, v))$$

$$= (x(u, v), y(u, v), f(x(u, v), y(u, v))) \in \text{graph } f.$$
Some Parameterized Surfaces

1. Graphs of functions of two variables, \( z = f(x, y) \), \( f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) smooth function,
\[
\text{graph } f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in U\}.
\]

Ex. \( f(x, y) = x^2 + y^2 \) graph \( f \): all \((x, y, z)\) such that \( z = x^2 + y^2 \)

Standard parameterization: \( x : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3, x(u, v) = (u, v, f(u, v)) \),
\[
\begin{align*}
x & = u \\
y & = v \\
z & = f(u, v)
\end{align*}
\]
\( x \) is a regular surface (in fact, a coordinate patch). Check regularity condition:
\[
\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = \left( -\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \right) \neq 0.
\]
\( x \) is called the Monge patch associated to \( f \).

Ex. \( f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = \sqrt{1 - x^2 - y^2}, U = \{(x, y) : x^2 + y^2 < 1\} \),

graph \( f : z = \sqrt{1 - x^2 - y^2} \), a hemisphere. Associated Monge patch:
\[
\begin{align*}
x & = u \\
y & = v \\
z & = \sqrt{1 - u^2 - v^2}
\end{align*}
\]
i.e., \( x(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \), \( x : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \).

2. Geographical Coordinates on a sphere.

\( S^2_R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\} \)

\( \theta = \) colatitude, \( 0 \leq \theta \leq \pi \)
\[
\phi = \) longitude, \( 0 \leq \phi \leq \pi \) .
By spherical coordinates,
\[ x = R \sin \theta \cos \phi \]
\[ y = R \sin \theta \sin \phi \]
\[ z = R \cos \theta . \]

Let \( U = \{ (\theta, \phi) : 0 < \theta < \pi, \ 0 < \phi < 2\pi \} \). Define \( \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) by,
\[ \mathbf{x}(\theta, \phi) = (R \sin \theta \cos \phi, \ R \sin \theta \sin \phi, \ R \cos \theta) . \]

\( \mathbf{x} \) is clearly a smooth parameterized surface.

Coordinate curves:

\( \theta \)-curves: \( \phi = \text{const} - \) longitudes (meridians)
\( \phi \)-curves: \( \theta = \text{const} - \) circles of latitude

Coordinate vector fields:
\[ \frac{\partial \mathbf{x}}{\partial \theta} = (R \cos \theta \cos \phi, \ R \cos \theta \sin \phi, -R \sin \theta) \]
\[ \frac{\partial \mathbf{x}}{\partial \phi} = (-R \sin \theta \sin \phi, \ R \sin \theta \cos \phi, \ 0) . \]

E.g., at \( (\theta, \phi) = (\pi/2, \ \pi/2) \),
\[ \frac{\partial \mathbf{x}}{\partial \theta} = (0, 0, -R), \quad \frac{\partial \mathbf{x}}{\partial \phi} = (-R, 0, 0) \]

**Exercise 3.1.** Show by computation that \( \left| \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi} \right| = R^2 \sin \theta > 0, \ 0 < \theta < \pi. \)

Hence, \( \mathbf{x} \) is regular surface (in fact, a coordinate patch).


Consider a regular curve \( \sigma \) in the \( x-z \) plane, \( \sigma(t) = (r(t), 0, z(t)) \), i.e,
\[ x = r(t) \]
\[ \sigma : \ y = 0 \quad a < t < b . \]
\[ z = z(t) \]

(Assume \( \sigma \) does not meet the \( z \)-axis.) Now rotate \( \sigma \) about the \( z \)-axis to generate a surface of revolution:
Parameterize as follows: Let $U = (a, b) \times (-\pi, \pi)$. Define $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by,

$$
\mathbf{x}(t, \theta) = (r(t) \cos \theta, r(t) \sin \theta, z(t)).
$$

This gives a parametric description of the surface of revolution; $t$ measures position along $\sigma$ and $\theta$ measure how far $\sigma$ has been rotated.

$t$-curves: $\theta=\text{const}$, longitudes (meridians)

$\theta$-curves: $t = \text{const}$, circles of latitude (parallels).

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**Exercise 3.2.** Show that $\mathbf{x}$ as defined above is a regular surface (in fact a coordinate patch provided $\sigma$ is 1-1).

**Exercise 3.3.** Rotate the circle pictured below about the $z$-axis to obtain a torus.

Show that the torus is parameterized by the following map: $U = (0, 2\pi) \times (-\pi, \pi)$,

$$
\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3,
$$

$$
\mathbf{x}(t, \theta) = ((R + r \cos t) \cos \theta, (R + r \cos t) \sin \theta, r \sin t).
$$
Hint: Parameterize the circle appropriately.

Reparameterizations.

**Def.** Let \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) be a regular surface. Let \( f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \) be a diffeomorphism. Then \( y = x \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3 \) is called a reparameterization.

**Proposition.** Given a regular surface \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) and a diffeomorphism \( f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \), the map \( y = x \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3 \) is a regular surface.

**Proof.** \( y = x \circ f \) is smooth. Show that \( y \) satisfies the regularity condition. To do this we first show how the two sets of coordinate vectors \( \{ \frac{\partial x}{\partial u^i} \} \) and \( \{ \frac{\partial y}{\partial v^i} \} \) are related. Some notation:

\[
\begin{align*}
  f &: V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \\
  f(v_1, v_2) &= (u_1, u_2) = (f^1(v_1', v_2'), f^2(v_1', v_2')) \\
  f &: u^1 = f^1(v_1', v_2') \\
  &u^2 = f^2(v_1', v_2') \\
  Df &= \left[ \frac{\partial u^i}{\partial v^k} \right]_{2 \times 2}
\end{align*}
\]

Then,

\[
\begin{align*}
  y(v_1', v_2') &= x \circ f(v_1', v_2') = x(f(v_1, v_2)) \\
  &= x(f^1(v_1', v_2'), f^2(v_1', v_2')) \\
  &= x(u^1, u^2)
\end{align*}
\]

i.e.

\[
y = x(u^1, u^2) \quad \text{where} \quad f : \begin{cases} u^1 = f^1(v_1', v_2') \\ u^2 = f^2(v_1', v_2') \end{cases}
\]
Hence, by the chain rule,
\[
\frac{\partial y}{\partial v^k} = \frac{\partial x}{\partial u^1} \frac{\partial u^1}{\partial v^k} + \frac{\partial x}{\partial u^2} \frac{\partial u^2}{\partial v^k}, \quad k = 1, 2
\]
\[
= \sum_{j=1}^{2} \frac{\partial x}{\partial w^j} \frac{\partial w^j}{\partial v^k}
\]
\[
\frac{\partial y}{\partial v^k} = \sum_{j} \frac{\partial w^j}{\partial v^k} \frac{\partial x}{\partial w^j}, \quad k = 1, 2.
\]

**Exercise 3.4** Show that,
\[
\frac{\partial y}{\partial v^1} \times \frac{\partial y}{\partial v^2} = \det Df \cdot \frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2}
\]
\[
= \frac{\partial (u^1, u^2)}{\partial (v^1, v^2)} \cdot \frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2} \quad (\neq 0)
\]

Hence, \( y \) is regular if \( x \) is.

**Terminology:** The reparameterization map \( f \) is called a coordinate transformation, and describes a change of coordinates on the surface.

**Surfaces.**

We now want to make the transition from the notion of a parameterized surface to that of a surface. Roughly speaking, a surface in \( \mathbb{R}^3 \) is a subset of \( \mathbb{R}^3 \) which is covered by coordinate patches. For example, the sphere
\[
S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}
\]
will, by our definition, be a surface. It can be covered by several coordinate patches – but not by a single coordinate patch.

Before giving the definition, we need to refine the notion of a coordinate patch a little. Consider a coordinate patch, i.e. a 1-1 regular parameterized surface, \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \). Then,
\[
x : U \to x(U)
\]
is a continuous, 1-1 and onto map. Hence we can consider the inverse,
\[
x^{-1} : x(U) \to U.
\]
The inverse need not be continuous:
$p, q$ close, as points in $\mathbb{R}^3$, but $x^{-1}(p), x^{-1}(q)$ are not close.

**Def.** A coordinate patch $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is *proper* provided the inverse $x^{-1} : x(U) \rightarrow U$ is continuous.

**Terminology:** proper patch = proper coordinate patch. Note: $x$ is proper iff $x : U \rightarrow x(U)$ is a homeomorphism.

**Subspace topology.** Any subset $M \subset \mathbb{R}^3$ of $\mathbb{R}^3$ inherits a natural topology - collection of open sets:

$$W \subset M \text{ is open iff } W = U \cap M$$

for some open set $U$ in $\mathbb{R}^3$.

**Def.** A subset $M \subset \mathbb{R}^3$ is a *smooth surface* provided each point of $M$ is contained in a proper patch, i.e. provided for each $p \in M$ there exists a proper patch $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, such that (i) $p \in x(U)$, and (ii) $x(U)$ is an open subset of $M$.

Equivalently, $M$ is a smooth surface provided $M$ is *covered* by proper patches (i.e. there is a collection of proper patches whose images are open sets in $M$, and the union of which equals $M$).

**Ex:** Consider the sphere,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$  

In this example the sphere is covered by six proper patches: $z^+, z^-, y^+, y^-, x^+, x^-$, each a parameterized hemisphere.
$z^+$: upper hemisphere: $z = \sqrt{1 - x^2 - y^2}$, with domain $D : x^2 + y^2 < 1$. Associated Monge patch: $z^+ : U \to \mathbb{R}^3$, $U = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$,

\[
\begin{align*}
x &= u \\
z^+ : y &= v \\
z &= \sqrt{1 - u^2 - v^2},
\end{align*}
\]

i.e. $z^+(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$.

**Claim.** $z^+$ is a proper patch in $S^2$.

1. $z^+$ is a coordinate patch (Monge patch)
2. $z^+(U)$ is an open subset of $S^2$:
   \[
   z^+(U) = \{(x, y, z) \in S^2 : z > 0\} = S^2 \cap \{z > 0\} \text{ open in } \mathbb{R}^3.
   \]
3. $(z^+)^{-1} : z^+(U) \to U$ is continuous:

   \[
   (z^+)^{-1}(x, y, z) = (x, y) - \text{projection onto the first two coordinates, which is continuous.}
   \]

$z^-$: lower hemisphere: $z = -\sqrt{1 - x^2 - y^2}$; associated Monge patch: $z^- : U \to \mathbb{R}^3$, $z^-(u, v) = (u, v, -\sqrt{1 - u^2 - v^2})$.

**Other hemispheres.**

\[
\begin{align*}
y^+ &= \text{Monge patch associated with hemisphere } S^2 \cap \{y > 0\} \quad (y = \sqrt{1 - x^2 - z^2}) \\
y^- &= \ldots \quad S^2 \cap \{y < 0\} \\
x^+ &= \ldots \quad S^2 \cap \{x > 0\} \\
x^- &= \ldots \quad S^2 \cap \{x < 0\}.
\end{align*}
\]
Proposition. (Smooth overlap property) Let $M$ be a surface. Let $x : U \to \mathbb{R}^3$ and $y : V \to \mathbb{R}^3$ be two proper patches in $M$ which overlap, $W := x(U) \cap x(V) \neq \emptyset$ Then,

$$y^{-1} \circ x : x^{-1}(W) \subset \mathbb{R}^2 \to y^{-1}(W) \subset \mathbb{R}^2$$

is a diffeomorphism.

Proof. Inverse function theorem! (See, e.g., DoCarmo, p. 70, Prop. 1.)

Ex. In sphere example consider the overlapping patches $z^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $z^+(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ and $y^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $y^+(u, v) = (u, \sqrt{1 - u^2 - v^2}, v)$. Observe, $y^+(U) = S^2 \cap \{y > 0\}$ and $z^+(U) = S^2 \cap \{z > 0\}$. Then,

$$W := y^+(U) \cap z^+(U) = S^2 \cap \{y > 0\} \cap \{z > 0\} \neq \emptyset.$$

Consider $(z^+)^{-1} \circ y^+ : (y^+)^{-1}(W) \to (z^+)^{-1}(W)$. Note, $(y^+)^{-1}(W) =$ half-disk $= U \cap \{v > 0\}$. Now,

$$y^+(u, v) = (u, \sqrt{1 - u^2 - v^2}, v)$$

$$(z^+)^{-1}(x, y, z) = (x, y)$$

and hence

$$(z^+)^{-1} \circ y^+(u, v) = (z^+)^{-1}(y^+(u, v)) = (z^+)^{-1}(u, \sqrt{1 - u^2 - v^2}, v)$$

$$= (u, \sqrt{1 - u^2 - v^2}),$$
which is smooth on \( U \cap \{ v > 0 \} \)!

**Remark.** In above proposition, let \( g = y^{-1} \circ x \). Then, \( x = y \circ g \) on \( x^{-1}(W) \). Thus, \( x|_{x^{-1}(W)} \) is a reparameterization of \( y|_{y^{-1}(W)} \). Also, \( y = x \circ g^{-1} \), so \( y|_{y^{-1}(W)} \) is also a reparameterization of \( x|_{x^{-1}(W)} \).

The smooth overlap property is the key ingredient used to generalize the notion of surfaces in \( \mathbb{R}^3 \) to *differentiable manifolds*. That this property holds for surfaces is important. For example, it is used to show that certain properties which are defined in terms of coordinate charts (proper charts), don’t really depend on the specific coordinate charts chosen. We give an illustration.

Consider a function \( f : M \rightarrow \mathbb{R} \), where \( M \) is a surface. What does it mean for \( f \) to be smooth?

**Def.** \( f : M \rightarrow \mathbb{R} \) is smooth provided for each \( p \in M \) there exists a proper patch \( x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) in \( M \) containing \( p \) (\( p \in x(U) \subset M \)) such that \( f \circ x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is smooth.

\[ \hat{f} = f \circ x \] expressed in coordinates.

Equivalently, \( f \) is smooth provided there exists a collection of charts covering \( M \) such that each coordinate expression \( \hat{f} \) is smooth.

This definition of smoothness does not depend on the particular choice of proper charts covering \( M \).

**Exercise 3.5.** If \( x \) and \( y \) are any two overlapping proper patches in \( M \) then on the overlap, \( f \circ x \) is smooth iff \( f \circ y \). (*Hint:* Smooth overlap property.)

The following proposition identifies a large and important class of surfaces.

**Proposition.** Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be a smooth function. Consider the level set

\[ M = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\} \]

If \( \nabla f = (f_x, f_y, f_z) \neq 0 \) at each point of \( M \) then \( M \) is a surface.
**Ex.** The sphere.

\[
S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}
= \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\},
\]

where \(f(x, y, z) = x^2 + y^2 + z^2 - 1\). Now,

\[
\nabla f = (2x, 2y, 2z),
\]

and so \(\nabla f \neq 0\) except at \((x, y, z) = (0, 0, 0) \notin S^2\). Hence, \(\nabla f \neq 0\) at each point of \(S^2\). Therefore \(S^2\) is a surface.

**Ex.** Double Cone.

\[
M = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\}
= \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\},
\]

where \(f(x, y, z) = x^2 + y^2 - z^2\). Then,

\[
\nabla f = (2x, 2y, 2z) \neq 0
\]

except at \((0, 0, 0)\). But this time \((0, 0, 0) \in M\). So the proposition doesn’t guarantee that \(M\) is a surface, and in fact it is not. The origin is not contained in a proper patch. In general, however, away from points where the gradient vanishes we do get a surface.

**Remarks**

1. \(Df = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \sim (f_x, f_y, f_z) = \nabla f\). I.e. the gradient of \(f\) essentially corresponds to the Jacobian of \(f\).

2. \(M = f^{-1}(0)\). For this reason, the proposition is often referred to as the **inverse image theorem**.

**Sketch of proof.** Uses the inverse function theorem (actually the implicit function theorem, which is a consequence of the IFT; see DoCarmo, p. 59, Prop. 2 for complete details).

We want to show \(M\) is covered by proper patches. Choose \(p_0 = (x_0, y_0, z_0) \in M\); \(\nabla f|_{p_0} \neq 0\). Suppose then that \(\frac{\partial f}{\partial z}(p_0) \neq 0\).

\[
M : \ f(x, y, z) = 0 \quad (\star)
\]
By the IFT, near \( p_0 = (x_0, y_0, z_0) \), (*) can be solved smoothly for \( z \) in terms of \( x \) and \( y \),

\[
z = h(x, y),
\]
i.e., there exists a nbd \( U \) of \((x_0, y_0)\) and a smooth function \( h : U \subset \mathbb{R}^2 \to \mathbb{R} \) such that

\[(x, y, h(x, y)) \text{ satisfies (*) for all } (x, y) \text{ in } U, \]

\[
f(x, y, h(x, y)) = 0 \quad \forall (x, y) \in U.
\]

Hence,

\[(x, y, h(x, y)) \in M \text{ for all } (x, y) \in U.\]

Now consider the Monge patch associated to \( h \), \( x : U \to \mathbb{R}^3 \),

\[
x(u, v) = (u, v, h(u, v)).
\]

Then \( x \) is a proper patch in \( M \) which contains \( p_0 \).

**Tangent Vectors to a Surface.**

**Def.** Let \( M \) be a surface, and \( p \in M \). \( X \) is a tangent vector to \( M \) at \( p \) provided \( X \) is the velocity vector at \( p \) of some smooth curve \( \sigma \) which lies in \( M \), i.e. provided there exists a smooth curve \( \sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3 \) such that \( \sigma(0) = p \) and \( \sigma'(0) = X \).

**Remark.** This definition is independent of coordinate patches – coordinate free concept. But for computational purposes it's convenient to introduce coordinates.
Let $x : U \rightarrow M \subset \mathbb{R}^3$ be a proper patch in $M$ which contains $p$, $p = x(u_0, v_0)$.

Observe: $\frac{\partial x}{\partial u}(u_0, v_0)$ and $\frac{\partial x}{\partial v}(u_0, v_0)$ are tangent vectors to $M$ at $p = x(u_0, v_0)$, according to the definition:

$$x_u(u_0, v_0) = \text{velocity vector to } u \rightarrow x(u, v_0) \text{ at } x(u_0, v_0), \text{ and},$$

$$x_v(u_0, v_0) = \text{velocity vector to } v \rightarrow x(u_0, v) \text{ at } x(u_0, v_0)$$

Notation/Terminology.

$T_p \mathbb{R}^3 :=$ tangent space of $\mathbb{R}^3$ at $p$

$= \text{set of all vectors in } \mathbb{R}^3 \text{ based at } p.$

$T_p \mathbb{R}^3$ is a 3-dimensional vector space. For $M$ a surface, $p \in M$,

$$T_p M := \text{tangent space of } M \text{ at } p$$

$= \text{set of all tangent vectors to } M \text{ at } p.$

In the following proposition we show that $T_p M$ is a 2-dimensional subspace of $T_p \mathbb{R}^3$ spanned by $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$.

**Proposition.** Let $M$ be a surface, $p \in M$. Let $x : U \rightarrow M \subset \mathbb{R}^3$ be a proper patch in $M$ containing $p$, $p = x(u_0, v_0)$. Then $T_p M$ is a 2-dimensional vector space, in fact it is the 2-dimensional vector subspace of $T_p \mathbb{R}^3$ spanned by \{ $x_u(u_0, v_0), x_v(u_0, v_0)$ \},

$$T_p M = \text{span} \{ x_u(u_0, v_0), x_v(u_0, v_0) \}$$

$$= \{ Ax_u(u_0, v_0) + Bx_v(u_0, v_0) : A, B \in \mathbb{R} \}$$

**Proof.** $T_p M \subset \text{span} \{ x_u(u_0, v_0), x_v(u_0, v_0) \}$: Let $X \in T_p M$. Then there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \rightarrow M \subset \mathbb{R}^3$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Without loss of generality, by taking $\epsilon$ sufficiently small, $\sigma \subset x(U)$.

Key observation: $\sigma$ can be represented in a certain manner in terms of coordinates; we will use this representation over and over.
Let \( \hat{\sigma} = x^{-1} \circ \sigma \):

\[ \hat{\sigma} : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2, \]

and in terms of components, \( \hat{\sigma}(t) = (u(t), v(t)), \quad t \in (-\epsilon, \epsilon) \),

\[ \hat{\sigma} : \begin{cases} u = u(t) & -\epsilon < t < \epsilon, \\ v = v(t) & \end{cases} \]

\( \hat{\sigma}(0) = x^{-1}(\sigma(0)) = x^{-1}(p) = (u_0, v_0). \) Using the IFT, it can be shown that \( \hat{\sigma} \) is a smooth curve in \( \mathbb{R}^2 \), that is, \( u = u(t) \) and \( v = v(t) \) are smooth functions.

Now, \( \hat{\sigma} = x^{-1} \circ \sigma \Rightarrow \sigma = x \circ \hat{\sigma} \Rightarrow \sigma(t) = x(\hat{\sigma}(t)), \) i.e.

\[ \sigma(t) = x(u(t), v(t)), \quad t \in (-\epsilon, \epsilon). \]

**Remark.** \( \hat{\sigma} \) is the coordinate representation of \( \sigma \); \( \hat{\sigma} \) is just \( \sigma \) expressed in coordinates.

Returning to the proof, by the chain rule,

\[ \frac{d\sigma}{dt} = \frac{\partial x}{\partial u} \frac{du}{dt} + \frac{\partial x}{\partial v} \frac{dv}{dt}, \]

or, rewriting slightly,

\[ \sigma'(t) = u'(t)x_u(u(t), v(t)) + v'(t)x_v(u(t), v(t)), \]

and setting \( t = 0 \), we obtain,

\[ \sigma'(0) = u'(0)x_u(u_0, v_0) + v'(0)x_v(u_0, v_0), \]

and thus,

\[ X = Ax_u(u_0, v_0) + Bx_v(u_0, v_0), \]

where \( A = u'(0), B = v'(0) \), as was to be shown.
span \{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\} \subset T_pM: Must show that a vector of the form,

\[ A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0) \]

for any \(A, B \in \mathbb{R}\), is the velocity vector of a curve \(\sigma\) in \(M\) passing through \(p\).

**Exercise 3.6.** Show this. Hint: Let \(\sigma = \mathbf{x} \circ \hat{\sigma}\) where \(\hat{\sigma}\) is the parameterized line, \(\hat{\sigma}(t) = (At + u_0, Bt + v_0)\). Then, \(\sigma(t) = \mathbf{x}(\hat{\sigma}(t)) = \mathbf{x}(At + u_0, Bt + v_0)\), and apply the chain rule.

Tangent plane to \(M\) at \(p\):

Let \(\mathbf{x}\) be a proper patch in \(M\) containing \(p = \mathbf{x}(u_0, v_0)\). Then the tangent plane to \(M\) at \(p = \) plane through \(p\) spanned by \(\mathbf{x}_u(u_0, v_0)\) and \(\mathbf{x}_v(u_0, v_0)\) = plane through \(p\) perpendicular to \(N = \mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0)\).

Equation of tangent plane:

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \]

where \(N = (a, b, c)\) and \(p = \mathbf{x}(u_0, v_0) = (x_0, y_0, z_0)\).

**Unit normal vector field** associated to a proper patch \(\mathbf{x}: U \to M \subset \mathbb{R}^3:\)

\[ \mathbf{n} = \mathbf{n}(u, v), \mathbf{n}(u, v) \in T_{\mathbf{x}(u, v)}\mathbb{R}^3, \mathbf{n}(u, v) \perp M. \]

**Remark:** The unit normal field is unique up to sign.
**Ex.** Compute the unit normal field to the surface \( z = x^2 + y^2 \) with respect to the associated Monge patch.

We have, \( \mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3, \mathbf{x}(u, v) = (u, v, u^2 + v^2) \), \( \mathbf{x}_u = (1, 0, 2u), \mathbf{x}_v = (0, 1, 2v) \), and so,

\[
\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1).
\]

Hence,

\[
\mathbf{n} = \frac{(-2u, -2v, 1)}{|(-2u, -2v, 1)|}.
\]

\[
\mathbf{n}(u, v) = \frac{(-2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}.
\]

**Exercise 3.7** Let \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \) be a smooth function and let \( M=\text{graph} \ f \). \( M \) is a smooth surface covered by a single patch - the associated Monge patch \( \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( \mathbf{x}(u, v) = (u, v, f(u, v)) \). Show that the unit normal vector field to \( M \) wrt \( \mathbf{x} \) is given by,

\[
\mathbf{n} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}
\]

where \( f_u = \frac{\partial f}{\partial u} \) and \( f_v = \frac{\partial f}{\partial v} \).

**Some Tensor Analysis**

Consider overlapping patches,

\[
\mathbf{x} : U \to M \subset \mathbb{R}^3 \quad \mathbf{y} : V \to M \subset \mathbb{R}^3
\]

\[
\mathbf{x} = \mathbf{x}(u^1, u^2) \quad \mathbf{y} = \mathbf{y}(v^1, v^2).
\]
Let \( p \in x(U) \cap y(V) \). \( T_p M \) has the two different bases at \( p \): \( \{x_1, x_2\}, \{y_1, y_2\} \) where we are using the shorthand, \( x_1 = \frac{\partial x}{\partial u^1}, x_2 = \frac{\partial x}{\partial u^2}, y_1 = \frac{\partial y}{\partial v^1}, y_2 = \frac{\partial y}{\partial v^2} \).

Let \( X \in T_p M \). \( X \) can be expressed in two different ways,

\[
X = \sum_{i=1}^{2} X^i x_i = \sum_{k=1}^{2} \tilde{X}^k y_k
\]

Classical tensor analysis is concerned with questions like the following: How are the components \( X^i \) and \( \tilde{X}^k \) with respect to the two different bases related? We now consider this.

By the smooth overlap property, \( f = y^{-1} \circ x \) is a diffeomorphism on the overlap. We have, \( f : x^{-1}(W) \to y^{-1}(W) \), where \( W = x(U) \cap x(V) \), and \( f(u^1, u^2) = (v^1, v^2) = (f^1(u^1, u^2), f^2(u^1, u^2)) \),

\[
f : \begin{cases} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{cases}
\]

i.e., \( f \) is the change of coordinates map; \( v^1 \) and \( v^2 \) depend smoothly on \( u^1 \) and \( u^2 \). On the overlap we have, \( x = y \circ f \), and hence, \( x(u^1, u^2) = y(v^1, v^2) \), where \( v^1, v^2 \) depend on \( u^1, u^2 \) as above.

**Exercise 3.8**

(1) Use the chain rule to show,

\[
x_i = \sum_k \frac{\partial v^k}{\partial u^i} y_k
\]

(Note: This is essentially the same as the computation on p. 46, but with the role of \( x \) and \( y \) reversed from that here).

(2) Use (1) to show,

\[
\tilde{X}^k = \sum_i \frac{\partial v^k}{\partial u^i} X^i, \quad k = 1, 2.
\]

(3) Show (2) implies

\[
\begin{bmatrix} \tilde{X}^1 \\ \tilde{X}^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial v^k}{\partial u^i} \end{bmatrix}_{Df} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}
\]