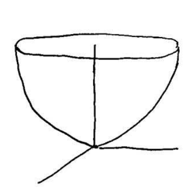
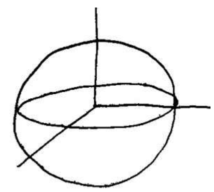
Chapter 3. Surfaces

We all understand intuitively what a surface is. In calculus we encounter surfaces in several ways.

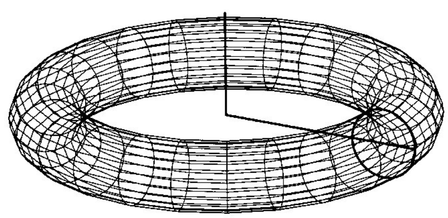
1. As graphs of functions of two variables, $z = f(x, y)$.
Ex. $z = x^2 + y^2$



2. As *level* surfaces of functions of three variables, $F(x, y, z) = c$.
Ex. $x^2 + y^2 + z^2 = 1$



3. As surfaces of revolution.
Ex. Torus: surface of a doughnut. This surface is not of type (1) or (2).

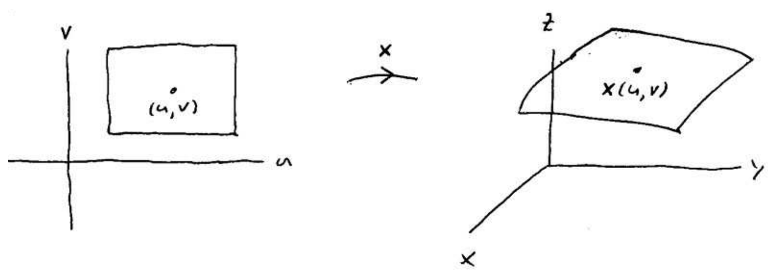


We will need to be fairly precise about what we mean by a surface. Our definition will need to cover all these cases. The key is to describe surfaces parametrically. Very roughly speaking, a surface for us is going to be a subset of $\mathbb{R}^3$ which can be broken up into overlapping pieces such that each piece is described parametrically, i.e. described by a 2-parameter map.

Hence, the starting point is the notion of *parameterized surfaces.*

Def. A smooth parameterized surface in $\mathbb{R}^3$ is a smooth map $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \to \mathbf{x}(u, v)$.

As (u, v) varies over U, $\mathbf{x}(u, v) \in \mathbb{R}^3$ traces out a "surface" in $\mathbb{R}^3$.



In terms of components, $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$,

$$\mathbf{x}: \begin{array}{l} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{array} \qquad (u,v) \in U$$

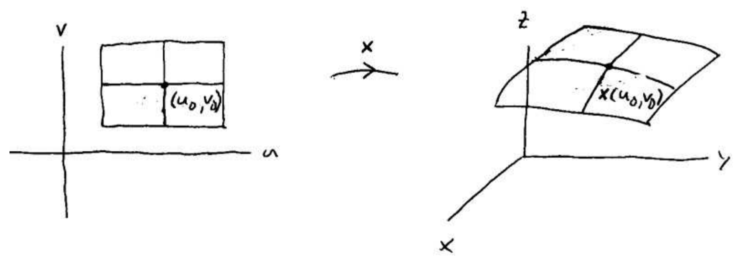
An effective way to see what gets t gets traced out is to look at the "u-curves" and "v-curves".

(1) if v is held constant, $v = v_0$ and u varies,

$$u \to \mathbf{x}(u, v_0) \qquad \text{"}u - \text{curve}''$$

(2) if u is held constant, $u = u_0$ and v varies,

$$v \to \mathbf{x}(u_0, v) \qquad \text{"}v - \text{curve}''$$



One way to examine a parameterized surface is to plot many "coordinate" curves, u=const, v=const. This is how e.g., Mathematica plots parameterized surfaces.

Ex. $\mathbf{x} : U \to \mathbb{R}^3$, $U = \{(u,v) : 0 < u < 2\pi, 0 < v < 3\}$, $\mathbf{x}(u,v) = (2\cos u, 2\sin u, v)$,

$$\mathbf{x}: \begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = v \end{array} \qquad 0 < u < 2\pi, \quad 0 < v < 3$$

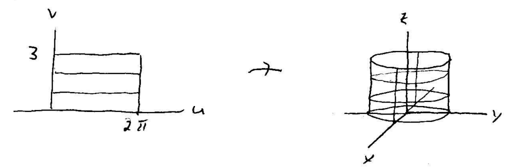
For this example it is convenient to consider closed rectangle $\overline{U} : 0 \leq u \leq 2\pi$, $0 \leq v \leq 3$. We plot some u-curves and v-curves:

$$\underline{v = 0:} \quad \begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = 0 \end{array} \qquad 0 \leq u \leq 2\pi \qquad \text{circle in } z = 0$$

$$\underline{v = 1:} \quad \begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = 1 \end{array} \qquad 0 \leq u \leq 2\pi \qquad \text{circle in } z = 1$$

etc.

$$\underline{u=0:}\quad \begin{array}{l} x=2 \\ y=0 \\ z=v \end{array} \qquad 0 \le v \le 3 \qquad \text{vertical line}$$



$$\underline{u=\pi/2:}\quad \begin{array}{l} x=0 \\ y=2 \\ z=v \end{array} \qquad 0 \le v \le 3 \qquad \text{vertical line}$$

etc.

This parameterized surface describes a cylinder. Note that the coordinate functions satisfy:

$$x^2 + y^2 = 4, \qquad 0 \le z \le 3$$

Note: On the original domain U, $\mathbf{x}$ is 1-1. We will restrict attention to parameterized surfaces $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ which are 1-1. Cylinder of radius a : $\mathbf{x}(u,v) = (a\cos u,\ a\sin u, v)$

Coordinate Vector Fields. Given a smooth surface,

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

we can differentiate wrt u and v,

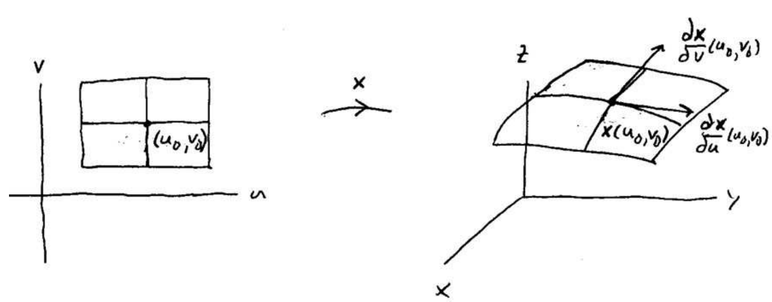
$$\frac{\partial \mathbf{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\frac{\partial \mathbf{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

These partial derivatives have natural interpretations,

$$\frac{\partial \mathbf{x}}{\partial u}(u_0,v_0) = \text{tangent vector to } u\text{-curve } u \to \mathbf{x}(u,v_0) \text{ at } \mathbf{x}(u_0,v_0)$$

$$\frac{\partial \mathbf{x}}{\partial v}(u_0,v_0) = \text{tangent vector to } v\text{-curve } v \to \mathbf{x}(u_0,v) \text{ at } \mathbf{x}(u_0,v_0)$$



Hence,

$$\frac{\partial \mathbf{x}}{\partial u} = \text{velocity vector field to } u\text{-curves}$$

$$\frac{\partial \mathbf{x}}{\partial v} = \text{velocity vector field to } v\text{-curves.}$$

Remark. The coordinate curves $u = u_0,\ v = v_0$ lie in the surface. Hence the coordinate vectors $\frac{\partial \mathbf{x}}{\partial u}(u_0, v_0), \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$ are *tangent* vectors to the surface at $\mathbf{x}(u_0, v_0)$.

Standard Picture: Grid of horizontal and vertical lines in $U \subset \mathbb{R}^2$ gives rise to a grid of curves - the coordinate curves on $\mathbf{x}(U)$. This amounts to introducing coordinates on $\mathbf{x}(U)$.

Shorthand Notation: $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u},\ \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$.

Actually, to insure that the image of a parameterized surface $\mathbf{x}$ looks like a surface (i.e. smooth 2-dimensional object), we need a *regularity* condition, akin to the regularity condition for parameterized curves ($\sigma'(t) \neq 0$).

Ex. $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3,\ \mathbf{x}(u, v) = (0, 0, 0)\ \forall (u, v)$. Image a single point! $\frac{\partial \mathbf{x}}{\partial u} = \frac{\partial \mathbf{x}}{\partial v} = \mathbf{0}$.

Ex. $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3,\ \mathbf{x}(u, v) = (\cos(u + v^2), \sin(u + v^2), 1)$

$$\mathbf{x} : \begin{array}{l} x = \cos(u + v^2) \\ y = \sin(u + v^2) \\ z = 1 \end{array}$$

Image: $x^2 + y^2 = 1, z = 1$, a circle!

Compute: $\frac{\partial \mathbf{x}}{\partial v} = -2v \frac{\partial \mathbf{x}}{\partial u}$, i.e. $\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}$ are linearly *dependent* (at every point).

To avoid this type of "degeneracy" must require that $\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}$ be linearly *independent.* There are several ways to characterize this independence.

Consider a parameterized surface, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$,

$$\mathbf{x} : \begin{array}{l} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{array}$$

$D\mathbf{x}$ = Jacobian matrix of $\mathbf{x}$, is the 2×3 matrix:

$$D\mathbf{x} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{bmatrix}$$

Recall, the

$$\begin{aligned} \text{rank of a matrix} &= \text{no. of linearly independent rows} \\ &= \text{no. of linearly independent columns} \end{aligned}$$

Prop. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface. Then the following conditions are equivalent.*

(1) $D\mathbf{x}$ *has rank 2.*

(2) $\dfrac{\partial \mathbf{x}}{\partial u}, \dfrac{\partial \mathbf{x}}{\partial v}$ *are linearly independent.*

(3) $\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v} \neq 0$.

Proof.

$$\begin{aligned} D\mathbf{x} \text{ has rank } 2 &\iff \text{columns lin. indep.} \\ &\iff \frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \text{ lin. indep.} \\ &\iff \text{one is } \textit{not} \text{ a multiple of the other} \\ &\iff \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0. \end{aligned}$$

Def. A *regular (parameterized) surface* in $\mathbb{R}^3$ is a smooth parameterized surface $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v} \neq 0$ for all $(u, v) \in U$.

A *coordinate patch* is a regular surface $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ which is one-to-one. Roughly speaking, a surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is covered by coordinate patches.

If the regularity condition is not satisfied, the image of $\mathbf{x}$ can degenerate to a point, or curve – or something that does not look like a smooth surface (surface with "folds" or "cusps"). If, however, the regularity condition is satisfied, then the image of $\mathbf{x}$ will look like a smooth surface. This is made rigorous in the following proposition.

Proposition. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface. Then for each $(u_0, v_0) \in U$ there is a neighborhood $V \subset U$ of (u_0, v_0) such that the image $\mathbf{x}(V) \subset \mathbb{R}^3$ coincides with the graph of an equation of the form,*

$$z = f(x, y) \quad \text{or} \quad y = g(x, z) \quad \text{or} \quad x = h(y, z)\,,$$

where f, g, h are smooth functions of two variables.

Proof. The proof is an application of the Inverse Function theorem. We have $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, and

$$\frac{\partial \mathbf{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad \frac{\partial \mathbf{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

Then,

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \\
&= \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \mathbf{k}\,.
\end{aligned}$$

Since, by regularity, $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0$ at (u_0, v_0), one of the components must be nonzero, say,

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0 \quad \text{at } (u_0, v_0)\,.$$

Now, consider the map $\Phi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by, $\Phi(u, v) = (x(u, v), y(u, v))$,

$$\Phi : \begin{array}{l} x = x(u, v) \\ y = y(u, v) \end{array}$$

Φ has Jacobian matrix,

$$D\Phi = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

Hence $\det D\Phi \neq 0$ at (u_0, v_0), i.e. $D\Phi$ is nonsingular at (u_0, v_0). By the IFT there exists a neighborhood V of (u_0, v_0) such that $W = \Phi(V)$ is open in $\mathbb{R}^2$ and $\Phi : V \subset \mathbb{R}^2 \to W \subset \mathbb{R}^2$ is a diffeomorphism, i.e., $\Phi^{-1} : W \subset \mathbb{R}^2 \to V \subset \mathbb{R}^2$ is smooth. In terms of components, $\Phi^{-1}(x, y) = (u(x, y), v(x, y))$,

$$\Phi^{-1} : \begin{array}{l} u = u(x, y) \\ v = v(x, y) \end{array}, (x, y) \in W$$

Now, let $f = z \circ \Phi^{-1}$, $f : W \subset \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = z(\Phi^{-1}(x, y)) = z(u(x, y), v(x, y)).$$

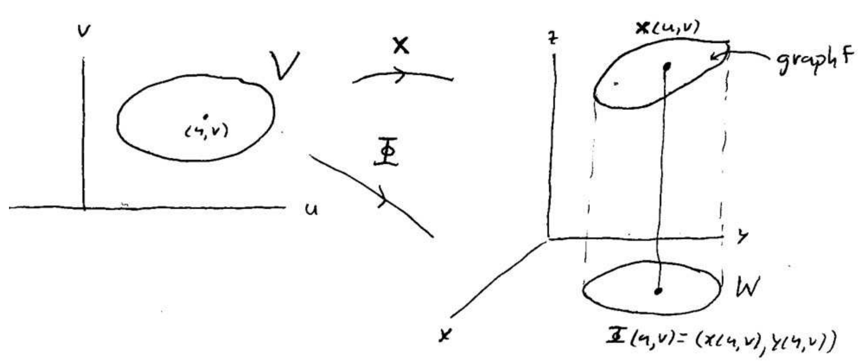
The *graph* of f is the set of points in $\mathbb{R}^3$,

$$\text{graph}\, f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in W\}$$

Claim. $\mathbf{x}(V) = \text{graph} f$:

$f = z \circ \Phi^{-1} \Rightarrow z = f \circ \Phi$, hence $z(u, v) = f(\Phi(u, v)) = f(x(u, v), y(u, v))$. Thus,

$$\begin{aligned} \mathbf{x}(u, v) &= (x(u, v), y(u, v), z(u, v)) \\ &= (x(u, v), y(u, v), f(x(u, y), y(u, v)) \in \text{graph}\, f. \end{aligned}$$



Some Parameterized Surfaces

1. Graphs of functions of two variables, $z = f(x, y)$, $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ smooth function,

$$\text{graph}\, f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in U\}.$$

Ex. $f(x, y) = x^2 + y^2$ graph f: all (x, y, z) such that $z = x^2 + y^2$
Standard parameterization: $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (u, v, f(u, v))$,

$$\mathbf{x} : \begin{array}{l} x = u \\ y = v \\ z = f(u, v) \end{array} \qquad (u, v) \in U\,.$$

$\mathbf{x}$ is a regular surface (in fact, a coordinate patch). Check regularity condition:

$$\frac{\partial \mathbf{x}}{\partial u} = \left(1, 0, \frac{\partial f}{\partial u}\right), \quad \frac{\partial \mathbf{x}}{\partial v} = \left(0, 1, \frac{\partial f}{\partial v}\right)$$

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) \neq 0\,.$$

$\mathbf{x}$ is called the *Monge patch* associated to f.

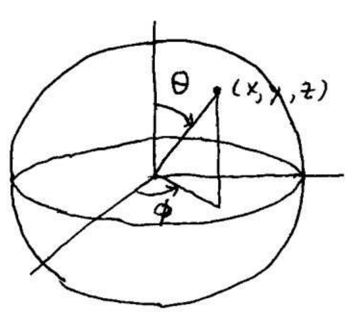
Ex. $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \sqrt{1 - x^2 - y^2}$, $U = \{(x, y) : x^2 + y^2 < 1\}$, graph $f : z = \sqrt{1 - x^2 - y^2}$, a hemisphere. Associated Monge patch:

$$\mathbf{x} : \begin{array}{l} x = u \\ y = v \\ z = \sqrt{1 - u^2 - v^2} \end{array} \quad ,$$

i.e., $\mathbf{x}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$.

2. Geographical Coordinates on a sphere.

$$S^2_R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\}$$



$\theta = \text{colatitude}, 0 \leq \theta \leq \pi$

$\phi = \text{longitude}, 0 \leq \phi \leq \pi$.

By spherical coordinates,

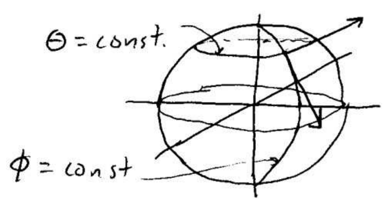
$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta\,. \end{aligned}$$

Let $U = \{(\theta,\phi) : 0 < \theta < \pi,\ \ 0 < \phi < 2\pi\}$. Define $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by,

$$\mathbf{x}(\theta,\phi) = (R\sin\theta\cos\phi,\ \ R\sin\theta\sin\phi,\ \ R\cos\theta)\,.$$

$\mathbf{x}$ is clearly a smooth parameterized surface.

Coordinate curves:



θ-curves: $\phi = \text{const}$ – longitudes (meridians)

ϕ-curves: $\theta = \text{const}$ – circles of latitude

Coordinate vector fields:

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \theta} &= (R\cos\theta\cos\phi,\ \ R\cos\theta\sin\phi, -R\sin\theta) \\ \frac{\partial \mathbf{x}}{\partial \phi} &= (-R\sin\theta\sin\phi,\ \ R\sin\theta\cos\phi,\ \ 0)\,. \end{aligned}$$

E.g., at $(\theta,\phi) = (\pi/2,\ \ \pi/2)$,

$$\frac{\partial \mathbf{x}}{\partial \theta} = (0,0,-R), \quad \frac{\partial \mathbf{x}}{\partial \phi} = (-R,0,0)$$

Exercise 3.1. Show by computation that $\left|\dfrac{\partial \mathbf{x}}{\partial \theta} \times \dfrac{\partial \mathbf{x}}{\partial \phi}\right| = R^2\sin\theta > 0, \quad 0 < \theta < \pi.$

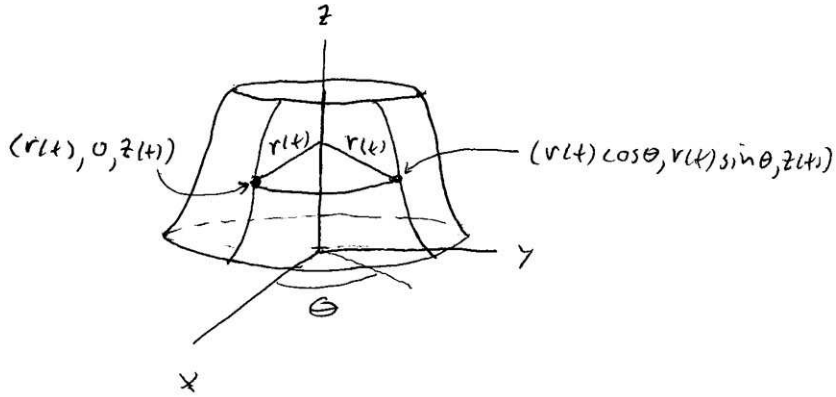
Hence, $\mathbf{x}$ is regular surface (in fact, a coordinate patch).

3. Surfaces of revolution.

Consider a regular curve σ in the x-z plane, $\sigma(t) = (r(t), 0, z(t))$, i.e,

$$\sigma : \begin{array}{l} x = r(t) \\ y = 0 \\ z = z(t) \end{array} \qquad a < t < b\,.$$

(Assume σ does not meet the z-axis.) Now rotate σ about the z-axis to generate a surface of revolution:

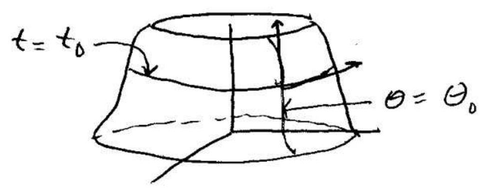


Parameterize as follows: Let $U = (a, b) \times (-\pi, \pi)$. Define $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by,

$$\mathbf{x}(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

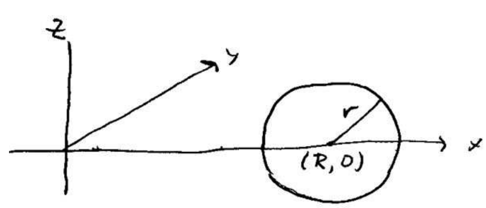
This gives a parametric description of the surface of revolution; t measures position along σ and θ measure how far σ has been rotated.

t-curves: θ=const, longitudes (meridians)
θ-curves: t = const, circles of latitude (parallels).



Exercise 3.2. Show that $\mathbf{x}$ as defined above is a regular surface (in fact a coordinate patch provided σ is 1-1).

Exercise 3.3. Rotate the circle pictured below about the z-axis to obtain a torus.



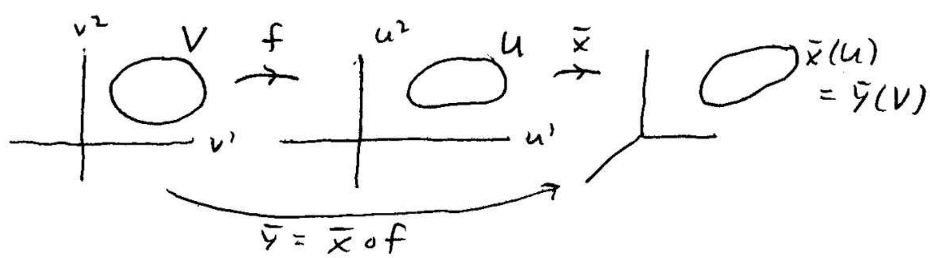
Show that the torus is parameterized by the following map: $U = (0, 2\pi) \times (-\pi, \pi)$, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathbf{x}(t, \theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t).$$

Hint: Parameterize the circle appropriately.

Reparameterizations.

Def. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface. Let $f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$ be a diffeomorphism. Then $\mathbf{y} = \mathbf{x} \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a reparameterization.



Proposition. Given a regular surface $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ and a diffeomorphism $f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$, the map $\mathbf{y} = \mathbf{x} \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a regular surface.

Proof. $\mathbf{y} = \mathbf{x} \circ f$ is smooth. Show that $\mathbf{y}$ satisfies the regularity condition. To do this we first show how the two sets of coordinate vectors $\left\{ \dfrac{\partial \mathbf{x}}{\partial u^i} \right\}$, $\left\{ \dfrac{\partial \mathbf{y}}{\partial v^i} \right\}$ are related. Some notation:

$$f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$$

$$f(v_1, v_2) = (u^1, u^2) = (f^1(v^1, v^2), f^2(v^1, v^2))$$

$$f : \begin{array}{l} u^1 = f^1(v^1, v^2) \\ u^2 = f^2(v^1, v^2) \end{array}$$

$$Df = \left[\frac{\partial u^i}{\partial v^k} \right]_{2\times 2}$$

Then,

$$\begin{aligned} \mathbf{y}(v^1, v^2) &= \mathbf{x} \circ f(v^1, v^2) = \mathbf{x}(f(v^1, v^2)) \\ &= \mathbf{x}\,\underbrace{(f^1(v^1, v^2)}_{u^1}, \underbrace{f^2(v^1, v^2))}_{u^2} \end{aligned}$$

i.e.

$$\mathbf{y} = \mathbf{x}(u^1, u^2) \quad \text{where } f : \begin{array}{l} u^1 = f^1(v^1, v^2) \\ u^2 = f^2(v^1, v^2) \end{array} .$$

Hence, by the chain rule,

$$\begin{aligned}\frac{\partial \mathbf{y}}{\partial v^k} &= \frac{\partial \mathbf{x}}{\partial u^1}\frac{\partial u^1}{\partial v^k} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{\partial u^2}{\partial v^k}, \quad k = 1,2 \\ &= \sum_{j=1}^{2} \frac{\partial \mathbf{x}}{\partial u^j}\frac{\partial u^j}{\partial v^k} \\ \frac{\partial \mathbf{y}}{\partial v^k} &= \sum_{j} \frac{\partial u^j}{\partial v^k}\frac{\partial \mathbf{x}}{\partial u^j}, \qquad\qquad k = 1,2.\end{aligned}$$

Exercise 3.4 Show that,

$$\begin{aligned}\frac{\partial \mathbf{y}}{\partial v^1} \times \frac{\partial \mathbf{y}}{\partial v^2} &= \det Df \cdot \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \\ &= \frac{\partial(u^1,u^2)}{\partial(v^1,v^2)} \cdot \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \quad (\neq 0)\end{aligned}$$

Hence, $\mathbf{y}$ is regular if $\mathbf{x}$ is.

Terminology: The reparameterization map f is called a *coordinate transformation*, and describes a change of coordinates on the surface.

Surfaces.

We now want to make the transition from the notion of a *parameterized* surface to that of a *surface*. Roughly speaking, a surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is covered by coordinate patches. For example, the sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

will, by our definition, be a surface. It can be covered by several coordinate patches – but not by a single coordinate patch.

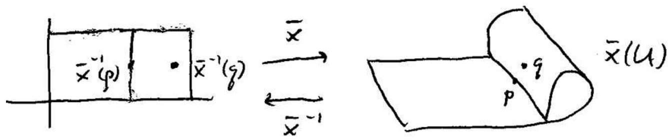
Before giving the definition, we need to refine the notion of a coordinate patch a little. Consider a *coordinate patch*, i.e. a 1-1 regular parameterized surface, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Then,

$$\mathbf{x} : U \to \mathbf{x}(U)$$

is a continuous, 1-1 and onto map. Hence we can consider the inverse,

$$\mathbf{x}^{-1} : x(U) \to U \,.$$

The inverse need not be continuous:

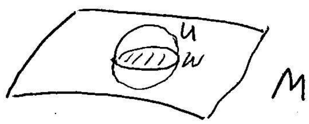


p, q close, as points in $\mathbb{R}^3$, but $\mathbf{x}^{-1}(p), \mathbf{x}^{-1}(q)$ are not close.

Def. A coordinate patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is *proper* provided the inverse $\mathbf{x}^{-1} : \mathbf{x}(U) \to U$ is continuous.

Terminology: proper patch = proper coordinate patch. Note: $\mathbf{x}$ is proper iff $\mathbf{x} : U \to \mathbf{x}(U)$ is a homeomorphism.

Subspace topology. Any subset $M \subset \mathbb{R}^3$ of $\mathbb{R}^3$ inherits a natural topology - collection of open sets:



$W \subset M$ is *open* iff $W = U \cap M$ for some open set U in $\mathbb{R}^3$.

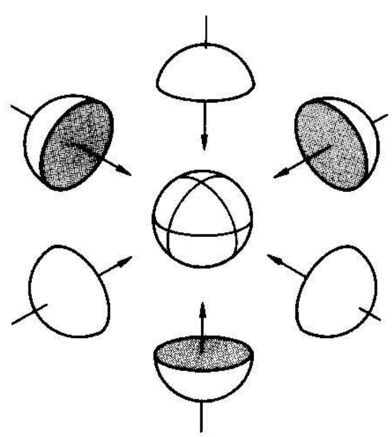
Def. A subset $M \subset \mathbb{R}^3$ is a *smooth surface* provided each point of M is contained in a proper patch, i.e. provided for each $p \in M$ there exists a proper patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, such that (i) $p \in \mathbf{x}(U)$, and (ii) $\mathbf{x}(U)$ is an open subset of M.

Equivalently, M is a smooth surface provided M is *covered* by proper patches (i.e. there is a collection of proper patches whose images are open sets in M, and the union of which equals M).

Ex: Consider the sphere,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

In this example the sphere is covered by six proper patches: $z^+, z^-, y^+, y^-, x^+, x^-$, each a parameterized hemisphere.



z^+: *upper hemisphere*: $z = \sqrt{1-x^2-y^2}$, with domain $D : x^2+y^2<1$. Associated Monge patch: $z^+ : U \to \mathbb{R}^3$, $U = \{(u,v) \in \mathbb{R}^2 : u^2+v^2<1\}$,

$$z^+ : \begin{array}{l} x = u \\ y = v \\ z = \sqrt{1-u^2-v^2}, \end{array}$$

i.e. $\quad z^+(u,v) = (u, v, \sqrt{1-u^2-v^2})$.

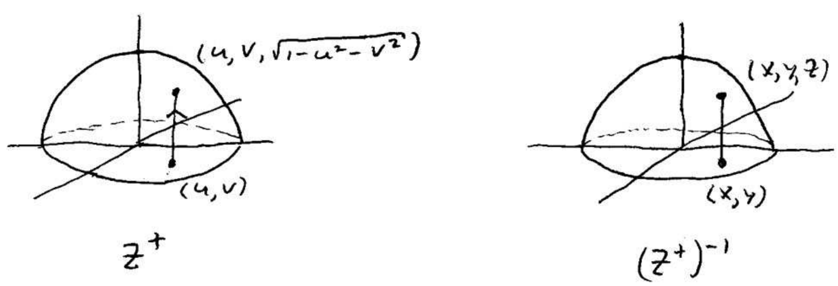
Claim. $\quad z^+$ is a proper patch in S^2.

(1) $\quad z^+$ is a coordinate patch (Monge patch)

(2) $\quad z^+(U)$ is an open subset of S^2:

$$\begin{aligned} z^+(U) &= \{(x,y,z) \in S^2 : z > 0\} \\ &= S^2 \cap \underbrace{\{z>0\}}_{\text{open in } \mathbb{R}^3}. \end{aligned}$$

(3) $\quad (z^+)^{-1} : z^+(U) \to U$ is continuous:



$(z^+)^{-1}(x,y,z) = (x,y)$ – projection onto the first two coordinates, which is continuous.

z^-: *lower hemisphere*: $z = -\sqrt{1-x^2-y^2}$; associated Monge patch: $z^- : U \to \mathbb{R}^3, z^-(u,v) = (u, v, -\sqrt{1-u^2-v^2})$.

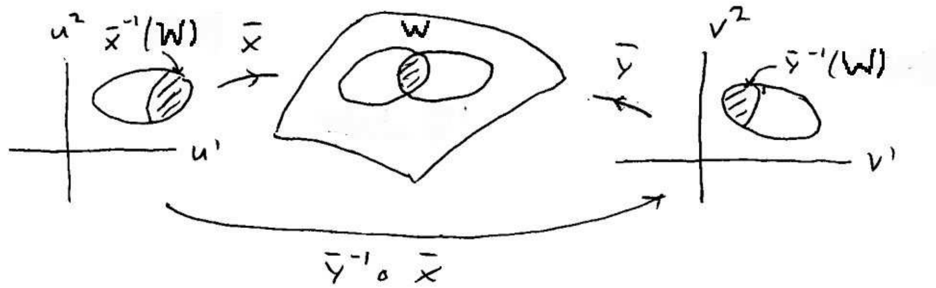
Other hemispheres.

$$\begin{array}{lll} y^+ = \text{Monge patch associated with hemisphere } & S^2 \cap \{y>0\} & (y = \sqrt{1-x^2-z^2}) \\ y^- = \ldots & S^2 \cap \{y<0\} & \\ x^+ = \ldots & S^2 \cap \{x>0\} & \\ x^- = \ldots & S^2 \cap \{x<0\}. & \end{array}$$

Proposition. (Smooth overlap property) *Let M be a surface. Let $\mathbf{x} : U \to \mathbb{R}^3$ and $\mathbf{y} : V \to \mathbb{R}^3$ be two proper patches in M which overlap, $W := x(U) \cap x(V) \neq \emptyset$ Then,*

$$\mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \subset \mathbb{R}^2 \to \mathbf{y}^{-1}(W) \subset \mathbb{R}^2$$

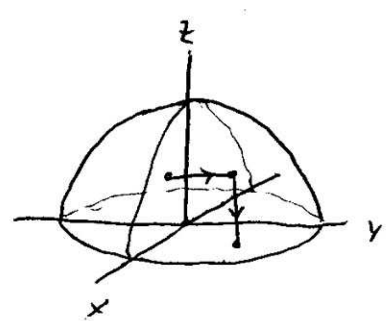
is a diffeomorphism.



Proof. Inverse function theorem! (See, e.g., DoCarmo, p. 70, Prop. 1.)

Ex. In sphere example consider the overlapping patches $z^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $z^+(u,v) = (u, v, \sqrt{1-u^2-v^2})$ and $y^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $y^+(u,v) = (u, \sqrt{1-u^2-v^2}, v)$. Observe, $y^+(U) = S^2 \cap \{y > 0\}$ and $z^+(U) = S^2 \cap \{z > 0\}$. Then,

$$W := y^+(U) \cap z^+(U) = S^2 \cap \{y > 0\} \cap \{z > 0\} \neq \emptyset\,.$$



Consider $(z^+)^{-1} \circ y^+ : (y^+)^{-1}(W) \to (z^+)^{-1}(W)$. Note, $(y^+)^{-1}(W) =$ half-disk $= U \cap \{v > 0\}$. Now,

$$\begin{aligned} y^+(u,v) &= (u, \sqrt{1-u^2-v^2}, v) \\ (z^+)^{-1}(x,y,z) &= (x,y) \end{aligned}$$

and hence

$$\begin{aligned} (z^+)^{-1} \circ y^+(u,v) &= (z^+)^{-1}(y^+(u,v)) = (z^+)^{-1}(u, \sqrt{1-u^2-v^2}, v) \\ &= (u, \sqrt{1-u^2-v^2})\,, \end{aligned}$$

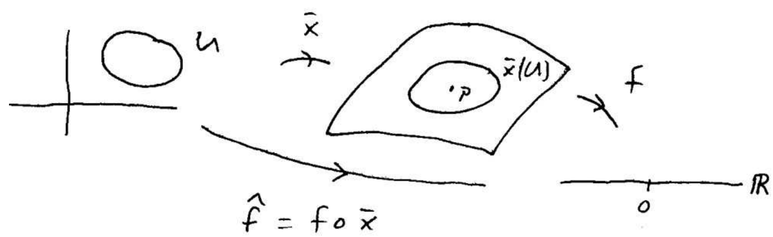
which is smooth on $U \cap \{v > 0\}$!

Remark. In above proposition, let $g = \mathbf{y}^{-1} \circ \mathbf{x}$. Then, $\mathbf{x} = \mathbf{y} \circ g$ on $\mathbf{x}^{-1}(W)$. Thus, $\mathbf{x}|_{\mathbf{x}^{-1}(W)}$ is a reparameterization of $\mathbf{y}|_{\mathbf{y}^{-1}(W)}$. Also, $\mathbf{y} = \mathbf{x} \circ g^{-1}$, so $\mathbf{y}|_{\mathbf{y}^{-1}(W)}$ is also a reparameterization of $\mathbf{x}|_{\mathbf{x}^{-1}(W)}$.

The smooth overlap property is the key ingredient used to generalize the notion of surfaces in $\mathbb{R}^3$ to *differentiable manifolds*. That this property holds for surfaces is important. For example, it is used to show that certain properties which are defined in terms of coordinate *charts* (proper charts), don't really depend on the specific coordinate charts chosen. We give an illustration.

Consider a function $f : M \to \mathbb{R}$, where M is a surface. What does it mean for f to be smooth?

Def. $f : M \to \mathbb{R}$ is *smooth* provided for each $p \in M$ there exists a proper patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ in M containing p ($p \in \mathbf{x}(U) \subset M$) such that $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth.



$\hat{f} = f$ expressed in coordinates.

Equivalently, f is smooth provided there exists a collection of charts covering M such that each coordinate expression $\hat{f}$ is smooth.

This definition of smoothness does not depend on the particular choice of proper charts covering M.

Exercise 3.5. If $\mathbf{x}$ and $\mathbf{y}$ are *any* two overlapping proper patches in M then on the overlap, $f \circ \mathbf{x}$ is smooth iff $f \circ \mathbf{y}$. (*Hint*: Smooth overlap property.)

The following proposition identifies a large and important class of surfaces.

Proposition. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Consider the level set*

$$M = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}$$

If $\nabla f = (f_x, f_y, f_z) \neq 0$ at each point of M then M is a surface.

Ex. The sphere.

$$\begin{aligned} S^2 &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}, \end{aligned}$$

where $f(x,y,z) = x^2+y^2+z^2-1$. Now,

$$\nabla f = (2x, 2y, 2z)\,,$$

and so $\nabla f \neq 0$ except at $(x,y,z) = (0,0,0) \notin S^2$. Hence, $\nabla f \neq 0$ at each point of S^2. Therefore S^2 is a surface.

Ex. Double Cone.

$$\begin{aligned} M &= \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2+y^2\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}, \end{aligned},$$

where $f(x,y,z) = x^2+y^2-z^2$. Then,

$$\nabla f = (2x, 2y, 2z) \neq 0$$

except at $(0,0,0)$. But this time $(0,0,0) \in M$. So the proposition doesn't guarantee that M is a surface, and in fact it is not. The origin is not contained in a proper patch. In general, however, away from points where the gradient vanishes we do get a surface.

Remarks

1. $Df = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \sim (f_x, f_y, f_z) = \nabla f$. I.e. the gradient of f essentially corresponds to the Jacobian of f.

2. $M = f^{-1}(0)$. For this reason, the proposition is often referred to as the *inverse image theorem.*

Sketch of proof. Uses the inverse function theorem (actually the implicit function theorem, which is a consequence of the IFT; see DoCarmo, p. 59, Prop. 2 for complete details).

We want to show M is covered by proper patches. Choose $p_0 = (x_0, y_0, z_0) \in M$; $\nabla f|_{p_o} \neq 0$. Suppose then that $\dfrac{\partial f}{\partial z}(p_0) \neq 0$.

$$M: \quad f(x,y,z) = 0 \qquad\qquad (*)$$

By the IFT, near $p_0 = (x_0, y_0, z_0)$, $(*)$ can be solved smoothly for z in terms of x and y,

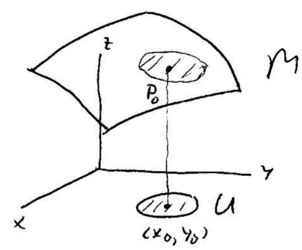
$$z = h(x, y)\,,$$

i.e., there exists a nbd U of (x_0, y_0) and a smooth function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $(x, y, h(x, y))$ satisfies $(*)$ for all (x, y) in U,

$$f(x, y, h(x, y)) = 0 \quad \forall (x, y) \in U\,.$$

Hence,

$$(x, y, h(x, y)) \in M \qquad \text{for all } (x, y) \in U.$$



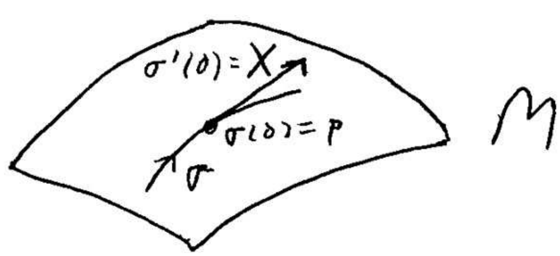
Now consider the Monge patch associated to h, $\mathbf{x} : U \to \mathbb{R}^3$,

$$\mathbf{x}(u, v) = (u, v, h(u, v))\,.$$

Then $\mathbf{x}$ is a proper patch in M which contains p_0.

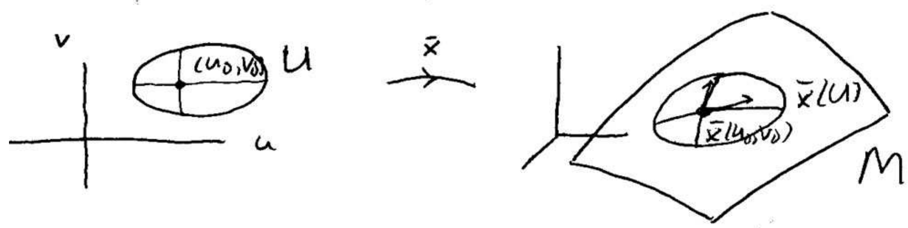
<u>Tangent Vectors to a Surface.</u>

Def. Let M be a surface, and $p \in M$. X is a tangent vector to M at p provided X is the velocity vector at p of some smooth curve σ which lies in M, i.e. provided there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3$ such that $\sigma(0) = p$ and $\sigma'(0) = X$.



Remark. This definition is independent of coordinate patches – coordinate free concept. But for computational purposes it's convenient to introduce coordinates.

Let $\mathbf{x} : U \to M \subset \mathbb{R}^3$ be a proper patch in M which contains p, $p = \mathbf{x}(u_0, v_0)$.



Observe: $\dfrac{\partial \mathbf{x}}{\partial u}(u_0, v_0)$ and $\dfrac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$ are tangent vectors to M at $p = \mathbf{x}(u_0, v_0)$, according to the definition:

$$\begin{aligned}
\mathbf{x}_u(u_0, v_0) &= \text{velocity vector to } u \to \mathbf{x}(u, v_0) \text{ at } \mathbf{x}(u_0, v_0)\text{, and,}\\
\mathbf{x}_v(u_0, v_0) &= \text{velocity vector to } v \to \mathbf{x}(u_0, v) \text{ at } \mathbf{x}(u_0, v_0)
\end{aligned}$$

Notation/Terminology.

$$\begin{aligned}
T_p\mathbb{R}^3 &:= \text{tangent space of } \mathbb{R}^3 \text{ at } p\\
&= \text{set of all vectors in } \mathbb{R}^3 \text{ based at } p.
\end{aligned}$$

$T_p\mathbb{R}^3$ is a 3-dimensional vector space. For M a surface, $p \in M$,

$$\begin{aligned}
T_pM &:= \text{tangent space of } M \text{ at } p\\
&= \text{set of all tangent vectors to } M \text{ at } p.
\end{aligned}$$

In the following proposition we show that T_pM is a 2-dimensional subspace of $T_p\mathbb{R}^3$ spanned by $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.

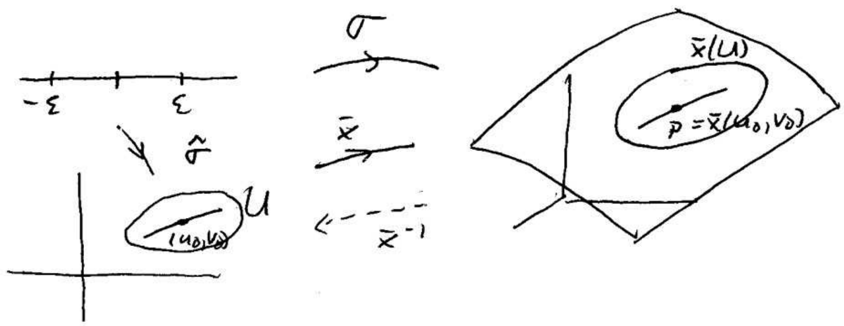
Proposition. *Let M be a surface, $p \in M$. Let $\mathbf{x} : U \to M \subset \mathbb{R}^3$ be a proper patch in M containing p, $p = \mathbf{x}(u_0, v_0)$. Then T_pM is a 2-dimensional vector space, in fact it is the 2-dimensional vector subspace of $T_p\mathbb{R}^3$ spanned by $\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\}$,*

$$\begin{aligned}
T_pM &= \operatorname{span}\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\}\\
&= \{A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0) : A, B \in \mathbb{R}\}
\end{aligned}$$

Proof. $T_pM \subset \operatorname{span}\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\}$: Let $X \in T_pM$. Then there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Without loss of generality, by taking ϵ sufficiently small, $\sigma \subset \mathbf{x}(U)$.

Key observation: σ can be represented in a certain manner in terms of coordinates; we will use this representation over and over.

Let $\hat{\sigma} = \mathbf{x}^{-1} \circ \sigma$:



$\hat{\sigma} : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$, and in terms of components, $\hat{\sigma}(t) = (u(t), v(t)), \quad t \in (-\epsilon, \epsilon)$,

$$\hat{\sigma} : \begin{array}{l} u = u(t) \\ v = v(t) \end{array} \qquad -\epsilon < t < \epsilon\,.$$

$\hat{\sigma}(0) = \mathbf{x}^{-1}(\sigma(0)) = \mathbf{x}^{-1}(p) = (u_0, v_0)$. Using the IFT, it can be shown that $\hat{\sigma}$ is a smooth curve in $\mathbb{R}^2$, that is, $u = u(t)$ and $v = v(t)$ are smooth functions.

Now, $\hat{\sigma} = \mathbf{x}^{-1} \circ \sigma \Rightarrow \quad \sigma = \mathbf{x} \circ \hat{\sigma} \Rightarrow \sigma(t) = \mathbf{x}(\hat{\sigma}(t))$, i.e.

$$\sigma(t) = \mathbf{x}(u(t), v(t))\,, \qquad t \in (-\epsilon, \epsilon)\,.$$

Remark. $\hat{\sigma}$ is the *coordinate representation* of σ; $\hat{\sigma}$ is just σ expressed in coordinates.

Returning to the proof, by the chain rule,

$$\frac{d\sigma}{dt} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt},$$

or, rewriting slightly,

$$\sigma'(t) = u'(t)\mathbf{x}_u(u(t), v(t)) + v'(t)\mathbf{x}_v(u(t), v(t)),$$

and setting $t = 0$, we obtain,

$$\sigma'(0) = u'(0)\mathbf{x}_u(u_0, v_0) + v'(0)\mathbf{x}_v(u_0, v_0)\,,$$

and thus,

$$X = A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0)\,,$$

where $A = u'(0), B = v'(0)$, as was to be shown.

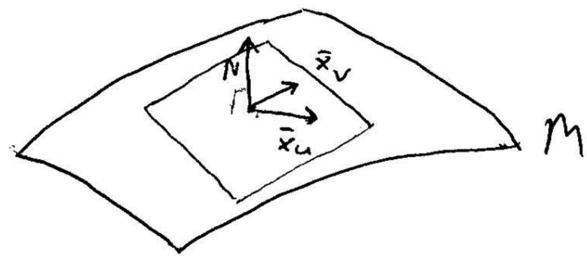
span $\{\mathbf{x}_u(u_0,v_0), \mathbf{x}_v(u_0,v_0)\} \subset T_pM$: Must show that a vector of the form,

$$A\mathbf{x}_u(u_0,v_0) + B\mathbf{x}_v(u_0,v_0)$$

for any $A, B \in \mathbb{R}$, is the velocity vector of a curve σ in M passing through p.

Exercise 3.6. Show this. Hint: Let $\sigma = \mathbf{x} \circ \hat{\sigma}$ where $\hat{\sigma}$ is the parameterized line, $\hat{\sigma}(t) = (At + u_0, Bt + v_0)$. Then, $\sigma(t) = \mathbf{x}(\hat{\sigma}(t)) = \mathbf{x}(At + u_0, Bt + v_0)$, and apply the chain rule.

Tangent plane to M at p:



Let $\mathbf{x}$ be a proper patch in M containing $p = \mathbf{x}(u_0,v_0)$. Then the

$$\begin{aligned}\text{tangent plane to } M \text{ at } p &= \text{plane through } p \text{ spanned by} \\ &\quad\ \mathbf{x}_u(u_0,v_0) \text{ and } \mathbf{x}_v(u_0,v_0) \\ &= \text{plane through } p \text{ perpendicular} \\ &\quad\ \text{to } N = \mathbf{x}_u(u_0,v_0) \times \mathbf{x}_v(u_0,v_0).\end{aligned}$$

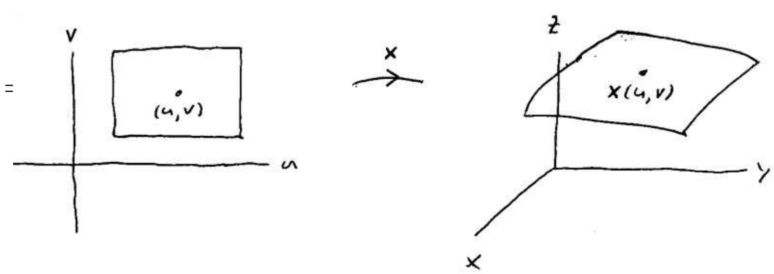
Equation of tangent plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\,,$$

where $N = (a, b, c)$ and $p = \mathbf{x}(u_0,v_0) = (x_0, y_0, z_0)$.

Unit normal vector field associated to a proper patch $\mathbf{x} : U \to M \subset \mathbb{R}^3$:

$\mathbf{n}$ =



$\mathbf{n} = \mathbf{n}(u,v)$, $\mathbf{n}(u,v) \in T_{\mathbf{x}(u,v)}\mathbb{R}^3$, $\mathbf{n}(u,v) \perp M$.

Remark: The unit normal field is *unique up to sign.*

Ex. Compute the unit normal field to the surface $z = x^2 + y^2$ with respect to the associated Monge patch.

We have, $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (u, v, u^2 + v^2)$, $\mathbf{x}_u = (1, 0, 2u), \mathbf{x}_v = (0, 1, 2v)$, and so,

$$\begin{aligned} \mathbf{x}_u \times \mathbf{x}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} \\ &= (-2u, -2v, 1)\,. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{n} &= \frac{(-2u, -2v, 1)}{|(-2u, -2v, 1)|} \\ \mathbf{n}(u, v) &= \frac{(-2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}. \end{aligned}$$

Exercise 3.7 Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let M=graph f. M is a smooth surface covered by a single patch - the associated Monge patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{x}(u, v) = (u, v, f(u, v))$. Show that the unit normal vector field to M wrt $\mathbf{x}$ is given by,

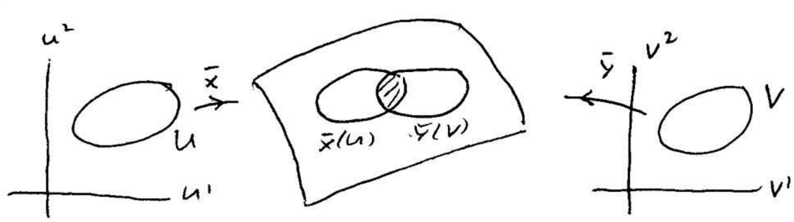
$$\mathbf{n} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

where $f_u = \dfrac{\partial f}{\partial u}$ and $f_v = \dfrac{\partial f}{\partial v}$.

<u>Some Tensor Analysis</u>

Consider overlapping patches,

$$\begin{array}{ll} \mathbf{x} : U \to M \subset \mathbb{R}^3 & \mathbf{x} = \mathbf{x}(u^1, u^2) \\ \mathbf{y} : V \to M \subset \mathbb{R}^3 & \mathbf{y} = \mathbf{y}(v^1, v^2). \end{array}$$



Let $p \in \mathbf{x}(U) \cap \mathbf{y}(V)$. T_pM has the two different bases at p: $\{\mathbf{x}_1, \mathbf{x}_2\}$, $\{\mathbf{y}_1, \mathbf{y}_2\}$ where we are using the shorthand, $\mathbf{x}_1 = \dfrac{\partial \mathbf{x}}{\partial u^1}, \mathbf{x}_2 = \dfrac{\partial \mathbf{x}}{\partial u^2}, \mathbf{y}_1 = \dfrac{\partial \mathbf{y}}{\partial v^1}, \mathbf{y}_2 = \dfrac{\partial \mathbf{u}}{\partial v^2}$.

Let $X \in T_pM$. X can be expressed in two different ways,

$$\begin{aligned} X &= \sum_{i=1}^{2} X^i \mathbf{x}_i \\ &= \sum_{k=1}^{2} \tilde{X}^k \mathbf{y}_k \end{aligned}$$

Classical tensor analysis is concerned with questions like the following: How are the components X^i and $\tilde{X}^k$ with respect to the two different bases related? We now consider this.

By the smooth overlap property, $f = \mathbf{y}^{-1} \circ \mathbf{x}$ is a diffeomorphism on the overlap. We have, $f : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$, where $W = \mathbf{x}(U) \cap \mathbf{x}(V)$, and $f(u^1, u^2) = (v^1, v^2) = (f^1(u^1, u^2), f^2(u^1, u^2))$,

$$f : \begin{array}{l} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{array} ,$$

i.e., f is the change of coordinates map; v^1 and v^2 depend smoothly on u^1 and u^2. On the overlap we have, $\mathbf{x} = \mathbf{y} \circ f$, and hence, $\mathbf{x}(u^1, u^2) = \mathbf{y}(v^1, v^2)$, where v^1, v^2 depend on u^1, u^2 as above.

Exercise 3.8

(1) Use the chain rule to show,

$$\mathbf{x}_i = \sum_k \frac{\partial v^k}{\partial u^i} \mathbf{y}_k$$

(Note: This is essentially the same as the computation on p. 46, but with the role of $\mathbf{x}$ and $\mathbf{y}$ reversed from that here).

(2) Use (1) to show,

$$\tilde{X}^k = \sum_i \frac{\partial v^k}{\partial u^i} X^i, \quad k = 1, 2.$$

(3) Show (2) implies

$$\begin{bmatrix} \tilde{X}^1 \\ \tilde{X}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial v^k}{\partial u^i} \end{bmatrix}}_{Df} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}$$