Chapter 1. Calculus of Euclidean Maps

The analytic study of surfaces involves multi-variable calculus. We begin with a "brief review" of calculus in $\mathbb{R}^n$. Let

$$\begin{aligned} \mathbb{R}^n &= n-\text{dimensional Euclidean space} \\ &= \{(x^1, x^2, x^3, \ldots, x^n) : x^i \in \mathbb{R}\}\,. \end{aligned}$$

(Note the superscripts; this is standard, traditional notation in DG stemming from tensor calculus.)

<u>Standard inner product on $\mathbb{R}^n$:</u> $x = (x^1, x^2, \ldots, x^n)$, $y = (y^1, y^2, \ldots, y^n)$ then

$$\langle x, y\rangle = x \cdot y = x^1y^1 + x^2y^2 + \cdots + x^ny^n = \sum_{i=1}^{n} x^iy^i$$

<u>Norm:</u>

$$\begin{aligned} |x| = \sqrt{\langle x, x\rangle} &= \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2} \\ &= \sqrt{\sum_{i=1}^{n}(x^i)^2} \end{aligned}$$

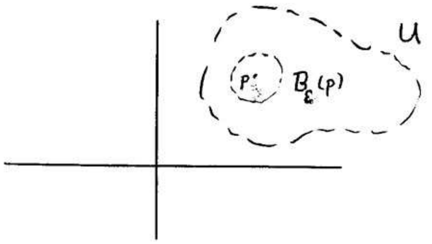
<u>Distance Function on $\mathbb{R}^n$:</u>

$$\begin{aligned} d(x, y) &= |x - y| = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + \cdots + (x^n - y^n)^2} \\ &= \sqrt{\sum_{i=1}^{n}(x^i - y^i)^2} \end{aligned}$$

<u>Open sets in $\mathbb{R}^n$:</u>

$$\begin{aligned} B_r(p) &= \text{open ball of radius } r \text{ centered at } p \\ &= \{x \in \mathbb{R}^n : d(x, p) < r\}\,. \end{aligned}$$

Def. $U \subset \mathbb{R}^n$ is *open* provided for each $p \in U$ there exists $\epsilon > 0$ such that $B_\varepsilon(p) \subset U$.



Euclidean Mappings: $F : \mathbb{R}^n \to \mathbb{R}^m$

These are the types of maps that will arise most frequently in our study, e.g.

1) $F : \mathbb{R} \to \mathbb{R}^3$: parameterized curve in space, $F(t) = (x(t), y(t), z(t))$, 1-parameter map.
2) $F : \mathbb{R}^2 \to \mathbb{R}^3$: parameterized surface in space, $F(u,v) = (x(u,v), y(u,v), z(u,v))$, 2-parameter map.
3) $F : \mathbb{R}^2 \to \mathbb{R}^2$: change of coordinates, e.g. polar coordinates,

$$F : \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta, \end{array}$$

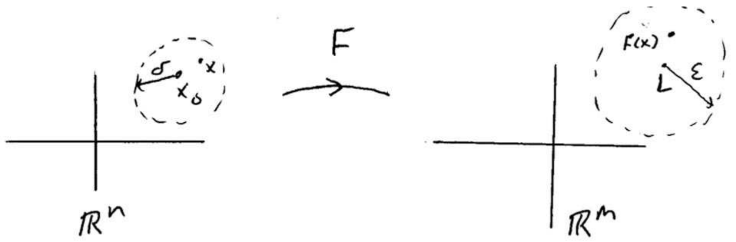
$F(r,\theta) = (r\cos\theta, r\sin\theta)$.

Limits and Continuity:

Def. Consider $F : \mathbb{R}^n \to \mathbb{R}^m$. Assume F is defined in a "deleted" neighborhood of $x_0 \in \mathbb{R}^n$. Then,

$$\lim_{x \to x_0} F(x) = L$$

means that for every $\epsilon > 0$ there exists $\delta > 0$ such that, $|F(x) - L| < \epsilon$ whenever $|x - x_0| < \delta \quad (x \neq x_0)$.



Def. $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. F is continuous at $x_0 \in U$ provided,

$$\lim_{x \to x_0} F(x) = F(x_0).$$

F is continuous on U if it is continuous at each point of U.

Fact. $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on U iff for all open sets $V \subset \mathbb{R}^m$, $F^{-1}(V)$ is open in $\mathbb{R}^n$.

Component Functions

Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ it is often useful to express F in terms of its component functions:

$$\begin{aligned} F(x^1, \dots, x^n) &= (y^1, \dots, y^m) \\ &= (f^1(x^1, \dots, x^n), ..., f^m(x^1, \dots, x^n)) \\ &= (f^1(x), \dots, f^m(x)), \quad x = (x^1, \dots, x^n). \end{aligned}$$

Component functions: $f^i : U \subset \mathbb{R}^n \to \mathbb{R}, \ i = 1, \dots, m.$

$$F : \begin{array}{rcl} y^1 &=& f^1(x^1, \dots, x^n) \\ y^2 &=& f^2(x^1, \dots, x^n) \\ &\vdots& \\ y^m &=& f^m(x^1, \dots, x^n) \end{array}$$

or,

$$F : y^i = f^i(x^1, \dots, x^n), \quad i = 1, \dots, m.$$

Ex. $F(x^1, x^2) = (2x^1x^2, x^2 - x^1)$

$$F : \begin{array}{rcl} y^1 &=& 2x^1x^2 \\ y^2 &=& x^2 - x^1 \end{array}$$

Ex. $F : \mathbb{R}^2 \to \mathbb{R}^3, \ F(u, v) = (\underbrace{uv^2}_{x}, \underbrace{u\cos v}_{y}, \underbrace{e^{u/v}}_{z})$

$$F : \begin{array}{rcl} x &=& uv^2 \\ y &=& u\cos v \\ z &=& e^{u/v} \end{array}$$

Fact: $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on U iff its component functions $f^i : U \subset \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, is continuous on U.

Differentiation of Mappings

Def. Given $f : U \subset \mathbb{R}^n \to \mathbb{R}$. f is C^k on U provided f and its partial derivatives of order k or less exist and are continuous on U. f is C^∞ on U (or *smooth* on U) provided f and its partial derivatives of all orders exist and are continuous on U.

Ex. $f : U \subset \mathbb{R}^2 \to \mathbb{R}$. $f(x, y)$ f is C^2 means that $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ exist and are continuous on U.

Ex. $f : \mathbb{R}^2 \to \mathbb{R}$. $f(x,y) = x^2 + 3xy - y^2$. f is C^∞ on $\mathbb{R}^2$.

Ex. $f(x,y) = \ln(1 - x^2 - y^2)$. f is C^∞ on $U = \{(x,y) : x^2 + y^2 < 1\}$.

Exercise 1.1. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is C^1 but not C^2.

Def. Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. F is C^k on U iff its component functions $f^1, \ldots, f^m$ are C^k on U. F is C^∞ (smooth) on U iff $f^1, \ldots, f^m$ are C^∞ (smooth) on U.

Ex. $F : \mathbb{R}^2 \to \mathbb{R}^3$, $F(x,y) = (x\cos y, x\sin y, e^{xy})$. $f^1(x,y) = x\cos y, f^2(x,y) = x\sin y, f^3(x,y) = e^{xy}$ are smooth. Therefore F is smooth.

Remark. We will usually assume the mappings we deal with are smooth - even though some results might be true with weaker differentiability assumptions.

Chain Rule for real valued functions of several variables: Given a smooth function of n variables, $w = f(x^1, \ldots, x^n)$ where $x^i = x^i(t, \ldots)$, $i = 1, \cdots, n$, depend smoothly on t. Then the composition $w = f(x^1(t,\ldots), \ldots, x^n(t,\ldots))$ depends smoothly on t and,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x^1}\frac{\partial x^1}{\partial t} + \frac{\partial w}{\partial x^2}\frac{\partial x^2}{\partial t} + \cdots + \frac{\partial w}{\partial x^n}\frac{\partial x^n}{\partial t}$$

or, using summation notation,

$$\frac{\partial w}{\partial t} = \sum_{i=1}^{n} \frac{\partial w}{\partial x^i} \cdot \frac{\partial x^i}{\partial t}\,.$$

Jacobians

Def. Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ smooth with component functions,

$$F : y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, m.$$

($\Leftrightarrow f^i : U \subset \mathbb{R}^n \to \mathbb{R}$ smooth), the *Jacobian Matrix* of F is the $m \times n$ matrix,

$$DF = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^n} \\ \vdots & & & \vdots \\ \frac{\partial y^m}{\partial x^1} & \frac{\partial y^m}{\partial x^2} & \cdots & \frac{\partial y^m}{\partial x^n} \end{bmatrix},$$

or, in short hand,

$$DF = \left[\frac{\partial y^i}{\partial x^j}\right]_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

At $p \in \mathbb{R}^n$,

$$DF(p) \;=\; \left[\frac{\partial y^i}{\partial x^j}(p)\right].$$

<u>Other notations:</u> $J(F) = DF$.

Ex. $F : \mathbb{R}^2 \to \mathbb{R}^3$, $F(x, y) = (\underbrace{x^2 + y^2}_{y^1}, \underbrace{2xy}_{y^2}, \underbrace{x\cos y}_{y^3})$. DF is 3 × 2:

$$DF = \begin{bmatrix} 2x & 2y \\ 2y & 2x \\ \cos y & -x\sin y \end{bmatrix}.$$

Ex. $f : \mathbb{R} \to \mathbb{R}$, $y = f(x)$. $DF = \left[\frac{dy}{dx}\right] \longleftrightarrow \frac{dy}{dx}$.

For mappings, the Jacobian plays the role of first derivative.

Jacobian Determinant: Consider special case $m = n$. $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$F : y^i = f^i(x^1, \dots, x^n), \quad i = 1, \dots, n.$$

Then DF is a square $n \times n$ matrix. The Jacobian determinant is then defiend as,

$$\text{Jacobian determinant} \;=\; \det DF$$

$$\frac{\partial(y^1, \dots, y^n)}{\partial(x^1, \dots, x^n)}. \;=\; \det\left[\frac{\partial y^i}{\partial x^j}\right]$$

Ex. $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x^2 - y^2, 2xy)$.

$$F : \begin{array}{rcl} u & = & x^2 - y^2 \\ v & = & 2xy \end{array}$$

$$DF = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det DF = 4(x^2+y^2)$$

<u>Chain Rule for Mappings.</u>

Re: Calc 1: $f : \mathbb{R} \to \mathbb{R},\ g : \mathbb{R} \to \mathbb{R} \Rightarrow (f \circ g)' = f' \cdot g'$.

Theorem (Chain Rule). Given smooth maps $F : V \subset \mathbb{R}^m \to \mathbb{R}^\ell,\ G : U \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $G(U) \subset V$. Then the composition $F \circ G : U \subset \mathbb{R}^n \to \mathbb{R}^\ell$ is defined and smooth, and

$$D(F \circ G)(x) = DF(G(x))DG(x)$$

or simply,

$$D(F \circ G)_{\ell \times n} = \underbrace{DF \cdot DG}_{\substack{\text{matrix multiplication} \\ \ell\times m \ \ m\times n}}$$

Proof Apply the chain rule for real valued functions of several variables. First, recall, if $A = [a_{ij}]_{\ell\times m}$ and $B = [b_{ij}]_{m\times n}$ then the product matrix $C = AB = [c_{ik}]_{\ell\times n}$ has entries given by,

$$c_{ik} = \sum_j a_{ij} b_{jk}$$

(i^{th} row of A dotted into k^{th} column of B)

Now, express F, G and $F \circ G$ in terms of component functions:

$$\begin{aligned} F : z^i &= f^i(y^1, \ldots, y^m),\ 1 \le i \le \ell \\ G : y^j &= g^j(x^1, \ldots, x^n),\ 1 \le j \le m \\ F \circ G : z^i &= f^i(g^1(x^1, \ldots, x^n), \ldots, g^m(x^1, \ldots, x^n)), 1 \le i \le \ell. \end{aligned}$$

For each $1 \leq k \leq n$: z^i depends on the y^j's and the y^j's depend on x^k. Therefore z^i depends on x^k and by the CR for real valued functions of several variables,

$$\frac{\partial z^i}{\partial x^k} = \sum_{j=1}^{m} \frac{\partial z^i}{\partial y^j} \frac{\partial y^j}{\partial x^k}$$

The term being summed is the i, kth entry of the matrix product,

$$\left[\frac{\partial z^i}{\partial y^j}\right] \cdot \left[\frac{\partial y^j}{\partial x^k}\right],$$

and hence,

$$\left[\frac{\partial z^i}{\partial x^k}\right] = \left[\frac{\partial z^i}{\partial y^j}\right] \cdot \left[\frac{\partial y^j}{\partial x^k}\right],$$

or,

$$D(F \circ G) = DF \cdot DG.$$

The Inverse Function Theorem

Analytically the Jacobian of $F : \mathbb{R}^n \to \mathbb{R}^m$ plays a role analogous to f' for functions $f : \mathbb{R} \to \mathbb{R}$. For example just as the derivative can be used to approximate f,

$$f(x + \triangle x) \approx f(x) + f'(x)\triangle x,$$

the Jacobian can be used to approximate $F : \mathbb{R}^n \to \mathbb{R}^m$,

$$F(p + \triangle p) \approx F(p) + \underbrace{DF(p)}_{m \times n} \underbrace{\triangle p}_{n \times 1}$$

(provided F is C^1 - this all can be made very precise). In the above expression we are treating points in $\mathbb{R}^n$ and $\mathbb{R}^m$ as column vectors.

Recall, given a smooth function $f : \mathbb{R} \to \mathbb{R}$, if $f'(x_0) \neq 0$ then on a small interval I about x_0, f is either increasing ($f'(x_0) > 0$) or decreasing ($f'(x_0) < 0$). In either case f has an inverse f^{-1} on I and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = [f'(f^{-1}(y))]^{-1}$$

or, more simply,

$$(f^{-1})' = \frac{1}{f'} = (f')^{-1}$$

or, in differential notation, if $y = f(x)$ then $x = f^{-1}(y)$ and,

$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}} = \left(\frac{dy}{dx}\right)^{-1}.$$

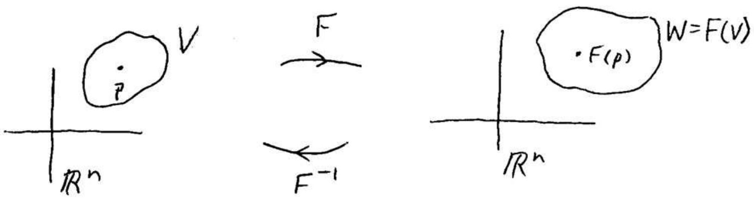
Theorem (Inverse function Theorem). *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Suppose for some $p \in U$, $DF(p)$ is nonsingular ($\Leftrightarrow \det DF(p) \neq 0$). Then there is a nbd V of p such that*

1. *$W = F(V)$ is open.*
2. *$F : V \to W$ is one-to-one and onto, and $F^{-1} : W \to V$ is smooth.*
3. *For each $q \in W$,*

$$D(F^{-1})(q) = [DF(F^{-1}(q))]^{-1}\,,$$

or simply,

$$D(F^{-1}) = (DF)^{-1}\,.$$



Exercise 1.2: Assuming (1) and (2) hold, show that (3) necessarily holds. Hint: Differentiate both sides of the equation: $F \circ F^{-1} = id$ (where $id : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, $id(x) = x$ for all $x \in \mathbb{R}^n$)

Remarks.

1. **Def.** *Let V, W be open sets in $\mathbb{R}^n$. A map $F : V \to W$ is called a diffeomorphism provided it is 1-1 and onto, and both F and F^{-1} are smooth.* Conditions (1) and (2) in the IFT say that $F : V \to W$ is a diffeomorphism.

2. Let's specialize the statement of the IFT to the case $n = 2$. Hence, consider $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (u(x, y), v(x, y))$, i.e. F has component functions

$$F : \begin{array}{lcl} u & = & u(x,y) \\ v & = & v(x,y) \end{array} \qquad (x,y) \in U \tag{$*$}$$

F is smooth $\Leftrightarrow u = u(x,y),\ v = v(x,y)$ are smooth, and

$$DF = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

Let $p = (x_0, y_0) \in U$ be such that $\det DF(x_0, y_0) \neq 0 \Leftrightarrow$

$$\det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \neq 0 \ \text{ at } \ (x_0, y_0).$$

Then, according to the IFT, there exists a neighborhood V of (x_0, y_0) such that $W = F(V)$ is an open set in the u-v plane, and $F^{-1} : W \to V$ is defined and smooth. We have $F^{-1}(u,v) = (x(u,v), y(u,v))$, i.e. F^{-1} has component functions,

$$F^{-1} : \begin{array}{c} x = x(u,v) \\ y = y(u,v) \end{array} \quad (u,v) \in W$$

i.e. the equations $(*)$ can be smoothly inverted to obtain x and y in terms of u and v. Moreover, when evaluated at the appropriate points,

$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}^{-1}$$