Introduction to Differential Geometry

Contents

1 Calculus of Euclidean Maps 2
2 Parameterized Curves in $\mathbb{R}^3$ 11
3 Surfaces 35
4 The First Fundamental Form (Induced Metric) 58
5 The Second Fundamental Form 74
6 Geodesics and the Gauss-Bonnet Theorem 109
1 Calculus of Euclidean Maps

The analytic study of surfaces involves multi-variable calculus. We begin with a “brief review” of calculus in $\mathbb{R}^n$. Let

$$\mathbb{R}^n = n - \text{dimensional Euclidean space}$$

$$= \{(x^1, x^2, x^3, \ldots, x^n) : x^i \in \mathbb{R}\}.$$

(Note the superscripts; this is standard, traditional notation in DG stemming from tensor calculus.)

Standard inner product on $\mathbb{R}^n$: $x = (x^1, x^2, \ldots, x^n)$, $y = (y^1, y^2, \ldots, y^n)$ then

$$\langle x, y \rangle = x \cdot y = x^1 y^1 + x^2 y^2 + \cdots + x^n y^n = \sum_{i=1}^{n} x^i y^i$$

Norm:

$$|x| = \sqrt{\langle x, x \rangle} = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (x^i)^2}$$

Distance Function on $\mathbb{R}^n$:

$$d(x, y) = |x - y| = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + \cdots + (x^n - y^n)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$$

Open sets in $\mathbb{R}^n$:

$$B_r(p) = \text{open ball of radius } r \text{ centered at } p$$

$$= \{x \in \mathbb{R}^n : d(x, p) < r\}.$$

Def. $U \subset \mathbb{R}^n$ is open provided for each $p \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset U$. 

\[ U \]

\[ B_r(p) \]

\[ \mathbb{R}^n \]
Euclidean Mappings: $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$

These are the types of maps that will arise most frequently in our study, e.g.

1) $F : \mathbb{R} \rightarrow \mathbb{R}^3$: parameterized curve in space, $F(t) = (x(t), y(t), z(t))$, 1-parameter map.

2) $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$: parameterized surface in space, $F(u, v) = (x(u, v), y(u, v), z(u, v))$, 2-parameter map.

3) $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$: change of coordinates, e.g. polar coordinates, $F : x = r \cos \theta$
   $y = r \sin \theta$,
   $F(r, \theta) = (r \cos \theta, r \sin \theta)$.

Limits and Continuity:

**Def.** Consider $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Assume $F$ is defined in a “deleted” neighborhood of $x_0 \in \mathbb{R}^n$. Then,

$$\lim_{x \to x_0} F(x) = L$$

means that for every $\epsilon > 0$ there exists $\delta > 0$ such that, $|F(x) - L| < \epsilon$ whenever $|x - x_0| < \delta \ (x \neq x_0)$.

**Def.** $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. $F$ is continuous at $x_0 \in U$ provided,

$$\lim_{x \to x_0} F(x) = F(x_0).$$

$F$ is continuous on $U$ if it is continuous at each point of $U$.

**Fact.** $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous on $U$ if for all open sets $V \subset \mathbb{R}^m$, $F^{-1}(V)$ is open in $\mathbb{R}^n$.

**Component Functions**
Given $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ it is often useful to express $F$ in terms of its component functions:

$$F(x^1, \ldots, x^n) = (y^1, \ldots, y^m) = (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n)) = (f^1(x), \ldots, f^m(x)), \quad x = (x^1, \ldots, x^n).$$

Component functions: $f^i : U \subset \mathbb{R}^n \rightarrow \mathbb{R}, \ i = 1, \ldots, m.$

$$F : y^1 = f^1(x^1, \ldots, x^n) \quad y^2 = f^2(x^1, \ldots, x^n) \quad \vdots \quad y^m = f^m(x^1, \ldots, x^n)$$

or,

$$F : y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, m.$$

**Ex.** $F(x^1, x^2) = (2x^1x^2, x^2 - x^1)$

$$F : \quad y^1 = 2x^1x^2 \quad y^2 = x^2 - x^1$$

**Ex.** $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \ F(u, v) = (u^2, u \cos v, e^{u/v})$

$$x = uv^2 \quad F : \quad y = u \cos v \quad z = e^{u/v}$$

**Fact:** $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous on $U$ iff its component functions $f^i : U \subset \mathbb{R}^n \rightarrow \mathbb{R}, \ i = 1, \ldots, m,$ is continuous on $U.$

**Differentiation of Mappings**

**Def.** Given $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}.$ $f$ is $C^k$ on $U$ provided $f$ and its partial derivatives of order $k$ or less exist and are continuous on $U.$ $f$ is $C^\infty$ on $U$ (or smooth on $U$) provided $f$ and its partial derivatives of all orders exist and are continuous on $U.$

**Ex.** $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}.$ $f(x, y)$ is $C^2$ means that $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ exist and are continuous on $U.$
Ex. \( f : \mathbb{R}^2 \to \mathbb{R}. \) \( f(x, y) = x^2 + 3xy - y^2. \) \( f \) is \( C^\infty \) on \( \mathbb{R}^2. \)

Ex. \( f(x, y) = \ln(1 - x^2 - y^2). \) \( f \) is \( C^\infty \) on \( U = \{(x, y) : x^2 + y^2 < 1\}. \)

Exercise 1.1. Construct a function \( f : \mathbb{R} \to \mathbb{R} \) which is \( C^1 \) but not \( C^2. \)

Def. Given \( F : U \subset \mathbb{R}^n \to \mathbb{R}^m. \) \( F \) is \( C^k \) on \( U \) iff its component functions \( f^1, \ldots, f^m \) are \( C^k \) on \( U. \) \( F \) is \( C^\infty \) (smooth) on \( U \) iff \( f^1, \ldots, f^m \) are \( C^\infty \) (smooth) on \( U. \)

Ex. \( F : \mathbb{R}^2 \to \mathbb{R}^3, \) \( F(x, y) = (x \cos y, x \sin y, e^{xy}). \) \( f^1(x, y) = x \cos y, f^2(x, y) = x \sin y, f^3(x, y) = e^{xy} \) are smooth. Therefore \( F \) is smooth.

Remark. We will usually assume the mappings we deal with are smooth - even though some results might be true with weaker differentiability assumptions.

Chain Rule for real valued functions of several variables: Given a smooth function of \( n \) variables, \( w = f(x^1, \ldots, x^n) \) where \( x^i = x^i(t, \ldots), i = 1, \ldots, n, \) depend smoothly on \( t. \) Then the composition \( w = f(x^1(t, \ldots), \ldots, x^n(t, \ldots)) \) depends smoothly on \( t \) and,

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x^1} \frac{\partial x^1}{\partial t} + \frac{\partial w}{\partial x^2} \frac{\partial x^2}{\partial t} + \cdots + \frac{\partial w}{\partial x^n} \frac{\partial x^n}{\partial t}
\]

or, using summation notation,

\[
\frac{\partial w}{\partial t} = \sum_{i=1}^{n} \frac{\partial w}{\partial x^i} \frac{\partial x^i}{\partial t}.
\]

Jacobians

Def. Given \( F : U \subset \mathbb{R}^n \to \mathbb{R}^m \) smooth with component functions,

\( F : y^i = f^i(x^1, \ldots, x^n), \ i = 1, \ldots, m. \)\n
(\( \Leftrightarrow f^i : U \subset \mathbb{R}^n \to \mathbb{R} \) smooth), the Jacobian Matrix of \( F \) is the \( m \times n \) matrix,

\[
DF = \begin{bmatrix}
\frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^n} \\
\frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial y^m}{\partial x^1} & \frac{\partial y^m}{\partial x^2} & \cdots & \frac{\partial y^m}{\partial x^n}
\end{bmatrix}
\]
or, in short hand,

\[
DF = \left[ \frac{\partial y^i}{\partial x^j} \right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}.
\]

At \( p \in \mathbb{R}^n \),

\[
DF(p) = \left[ \frac{\partial y^i}{\partial x^j}(p) \right].
\]

Other notations: \( J(F) = DF \).

**Ex.** \( F : \mathbb{R}^2 \to \mathbb{R}^3 \), \( F(x, y) = (x^2 + y^2, 2xy, x \cos y) \). \( DF \) is \( 3 \times 2 \):

\[
DF = \begin{bmatrix}
2x & 2y \\
2y & 2x \\
\cos y & -x \sin y
\end{bmatrix}.
\]

**Ex.** \( f : \mathbb{R} \to \mathbb{R} \), \( y = f(x) \). \( DF = \left[ \frac{dy}{dx} \right] \left\left\left\left. \longrightarrow \frac{dy}{dx} \right) \right. \).

For mappings, the Jacobian plays the role of first derivative.

**Jacobian Determinant:** Consider special case \( m = n \). \( F : U \subset \mathbb{R}^n \to \mathbb{R}^n \),

\[
F : y^i = f^i(x^1, \ldots , x^n), \quad i = 1, \ldots , n.
\]

Then \( DF \) is a square \( n \times n \) matrix. The Jacobian determinant is then defined as,

\[
\text{Jacobian determinant} = \det DF = \frac{\partial (y^1, \ldots , y^n)}{\partial (x^1, \ldots , x^n)} = \det \left[ \frac{\partial y^i}{\partial x^j} \right].
\]

**Ex.** \( F : \mathbb{R}^2 \to \mathbb{R}^2 \), \( F(x, y) = (x^2 - y^2, 2xy) \).

\[
F : \begin{array}{c}
u = x^2 - y^2 \\
v = 2xy
\end{array}
\]
\[
DF = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} = \begin{bmatrix}
2x & -2y \\
2y & 2x
\end{bmatrix}
\]

\[
\frac{\partial(u, v)}{\partial(x, y)} = \det DF = 4(x^2 + y^2)
\]

Chain Rule for Mappings.

Re: Calc 1: \(f : \mathbb{R} \rightarrow \mathbb{R}, \ g : \mathbb{R} \rightarrow \mathbb{R} \Rightarrow (f \circ g)' = f' \cdot g'.\)

**Theorem** (Chain Rule). Given smooth maps \(F : V \subset \mathbb{R}^m \rightarrow \mathbb{R}^\ell, \ G : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m\) such that \(G(U) \subset V\). Then the composition \(F \circ G : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^\ell\) is defined and smooth, and

\[
D(F \circ G)(x) = DF(G(x))DG(x)
\]

or simply,

\[
D(F \circ G)_{\ell \times n} = \underbrace{DF \cdot DG}_{\text{matrix multiplication}}_{\ell \times m \times m \times n}
\]

**Proof** Apply the chain rule for real valued functions of several variables. First, recall, if \(A = [a_{ij}]_{\ell \times m}\) and \(B = [b_{ij}]_{m \times n}\) then the product matrix \(C = AB = [c_{ik}]_{\ell \times n}\) has entries given by,

\[
c_{ik} = \sum_j a_{ij}b_{jk}
\]

\((i^{th} \text{ row of } A \text{ dotted into } k^{th} \text{ column of } B)\)

Now, express \(F, G\) and \(F \circ G\) in terms of component functions:

\[
F : z^i = f^i(y^1, \ldots, y^m), \ 1 \leq i \leq \ell
\]

\[
G : y^j = g^j(x^1, \ldots, x^n), \ 1 \leq j \leq m
\]

\[
F \circ G : z^i = f^i(g^1(x^1, \ldots, x^n), \ldots, g^m(x^1, \ldots, x^n)), \ 1 \leq i \leq \ell.
\]

For each \(1 \leq k \leq n : z^i\) depends on the \(y^j\)'s and the \(y^j\)'s depend on \(x^k\). Therefore \(z^i\) depends on \(x^k\) and by the CR for real valued functions of several variables,
\[
\frac{\partial z^i}{\partial x^k} = \sum_{j=1}^{m} \frac{\partial z^i}{\partial y^j} \frac{\partial y^j}{\partial x^k}
\]

The term being summed is the \(i,k\)th entry of the matrix product,

\[
\begin{bmatrix}
\frac{\partial z^i}{\partial y^j}
\end{bmatrix} \cdot \begin{bmatrix}
\frac{\partial y^j}{\partial x^k}
\end{bmatrix},
\]

and hence,

\[
\begin{bmatrix}
\frac{\partial z^i}{\partial x^k}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial z^i}{\partial y^j}
\end{bmatrix} \cdot \begin{bmatrix}
\frac{\partial y^j}{\partial x^k}
\end{bmatrix},
\]

or,

\[D(F \circ G) = DF \cdot DG.\]

**The Inverse Function Theorem**

Analytically the Jacobian of \(F : \mathbb{R}^n \rightarrow \mathbb{R}^m\) plays a role analogous to \(f'\) for functions \(f : \mathbb{R} \rightarrow \mathbb{R}\). For example just as the derivative can be used to approximate \(f\),

\[f(x + \Delta x) \approx f(x) + f'(x)\Delta x,\]

the Jacobian can be used to approximate \(F : \mathbb{R}^n \rightarrow \mathbb{R}^m\),

\[F(p + \Delta p) \approx F(p) + DF(p) \Delta p\]

(provided \(F\) is \(C^1\) - this all can be made very precise). In the above expression we are treating points in \(\mathbb{R}^n\) and \(\mathbb{R}^m\) as column vectors.

Recall, given a smooth function \(f : \mathbb{R} \rightarrow \mathbb{R}\), if \(f'(x_0) \neq 0\) then on a small interval \(I\) about \(x_0\), \(f\) is either increasing \((f'(x_0) > 0)\) or decreasing \((f'(x_0) < 0)\). In either case \(f\) has an inverse \(f^{-1}\) on \(I\) and

\[(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \left[f'(f^{-1}(y))\right]^{-1}\]

or, more simply,

\[(f^{-1})' = \frac{1}{f'} = (f')^{-1}\]

or, in differential notation, if \(y = f(x)\) then \(x = f^{-1}(y)\) and,

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \left(\frac{dy}{dx}\right)^{-1}.
\]
Theorem (Inverse function Theorem). Let \( F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a smooth map. Suppose for some \( p \in U \), \( DF(p) \) is nonsingular (\( \Leftrightarrow \det DF(p) \neq 0 \)). Then there is a nbd \( V \) of \( p \) such that

1. \( W = F(V) \) is open.
2. \( F : V \rightarrow W \) is one-to-one and onto, and \( F^{-1} : W \rightarrow V \) is smooth.
3. For each \( q \in W \),

\[
D(F^{-1})(q) = (DF(F^{-1}(q)))^{-1},
\]

or simply,

\[
D(F^{-1}) = (DF)^{-1}.
\]

Exercise 1.2: Assuming (1) and (2) hold, show that (3) necessarily holds. Hint: Differentiate both sides of the equation: \( F \circ F^{-1} = id \) (where \( id : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the identity map, \( id(x) = x \) for all \( x \in \mathbb{R}^n \))

Remarks.

1. Def. Let \( V, W \) be open sets in \( \mathbb{R}^n \). A map \( F : V \rightarrow W \) is called a diffeomorphism provided it is 1-1 and onto, and both \( F \) and \( F^{-1} \) are smooth. Conditions (1) and (2) in the IFT say that \( F : V \rightarrow W \) is a diffeomorphism.

2. Let’s specialize the statement of the IFT to the case \( n = 2 \). Hence, consider \( F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), \( F(x, y) = (u(x, y), v(x, y)) \), i.e. \( F \) has component functions

\[
F : \begin{array}{c}
u \\
v
\end{array} = \begin{array}{c}u(x, y) \\
v(x, y)
\end{array} \quad (x, y) \in U \quad (\ast)
\]
If smooth $\iff u = u(x, y), \ v = v(x, y)$ are smooth, and

$$DF = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Let $p = (x_0, y_0) \in U$ be such that $\det DF(x_0, y_0) \neq 0 \iff$

$$\det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \neq 0 \text{ at } (x_0, y_0).$$

Then, according to the IFT, there exists a neighborhood $V$ of $(x_0, y_0)$ such that $W = F(V)$ is an open set in the $u$-$v$ plane, and $F^{-1} : W \to V$ is defined and smooth. We have $F^{-1}(u, v) = (x(u, v), y(u, v))$, i.e. $F^{-1}$ has component functions,

$$F^{-1} : \begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases} (u, v) \in W$$

i.e. the equations ($\ast$) can be smoothly inverted to obtain $x$ and $y$ in terms of $u$ and $v$. Moreover, when evaluated at the appropriate points,

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}^{-1}$$
2 Parameterized Curves in $\mathbb{R}^3$

**Def.** A smooth curve in $\mathbb{R}^3$ is a smooth map $\sigma : (a, b) \to \mathbb{R}^3$.

For each $t \in (a, b)$, $\sigma(t) \in \mathbb{R}^3$. As $t$ increases from $a$ to $b$, $\sigma(t)$ traces out a curve in $\mathbb{R}^3$. In terms of components,

$$\sigma(t) = (x(t), y(t), z(t)), \quad (1)$$

or

$$x = x(t)$$
$$\sigma : y = y(t) \quad a < t < b,$$
$$z = z(t)$$

**velocity at time $t$:** $\frac{d\sigma}{dt}(t) = \sigma'(t) = (x'(t), y'(t), z'(t))$

**speed at time $t$:** $\left| \frac{d\sigma}{dt}(t) \right| = |\sigma'(t)|$

**Ex.** $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (r \cos t, r \sin t, 0)$ - the standard parameterization of the unit circle,

$$x = r \cos t$$
$$\sigma : y = r \sin t$$
$$z = 0$$

$$\sigma'(t) = (-r \sin t, r \cos t, 0)$$
$$|\sigma'(t)| = r \quad \text{(constant speed)}$$
**Ex.** \( \sigma : \mathbb{R} \rightarrow \mathbb{R}^3, \sigma(t) = (r \cos t, r \sin t, ht), r, h > 0 \) constants (helix).

\[
\sigma'(t) = (-r \sin t, r \cos t, h) \\
|\sigma'(t)| = \sqrt{r^2 + h^2} \text{ (constant)}
\]

**Def** A *regular* curve in \( \mathbb{R}^3 \) is a smooth curve \( \sigma : (a, b) \rightarrow \mathbb{R}^3 \) such that \( \sigma'(t) \neq 0 \) for all \( t \in (a, b) \).

That is, a regular curve is a smooth curve with everywhere nonzero velocity.

**Ex.** Examples above are regular.

**Ex.** \( \sigma : \mathbb{R} \rightarrow \mathbb{R}^3, \sigma(t) = (t^3, t^2, 0) \). \( \sigma \) is smooth, but not regular:

\[
\sigma'(t) = (3t^2, 2t, 0), \quad \sigma'(0) = (0, 0, 0)
\]

**Graph:**

\[
\begin{align*}
x &= t^3 \\
y &= t^2 \\
z &= 0
\end{align*}
\]

\[
\Rightarrow \quad y = t^2 = (x^{1/3})^2 \\
y = x^{2/3}
\]

There is a cusp, not because the curve isn’t smooth, but because the velocity = 0 at the origin. A regular curve has a well-defined smoothly turning tangent, and hence its graph will appear smooth.

**The Geometric Action of the Jacobian** (exercise)

Given smooth map \( F : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, p \in U \). Let \( X \) be any vector based at the point \( p \). To \( X \) at \( p \) we associate a vector \( Y \) at \( F(p) \) as follows.

Let \( \sigma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3 \) be any smooth curve such that,

\[
\sigma(0) = p \quad \text{and} \quad \frac{d\sigma}{dt}(0) = X,
\]
i.e. $\sigma$ is a curve which passes through $p$ at $t = 0$ with velocity $X$. (E.g. one can take $\sigma(t) = p + tX$.) Now, look at the image of $\sigma$ under $F$, i.e. consider $\beta = F \circ \sigma$, $\beta : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$, $\beta(t) = F \circ \sigma(t) = F(\sigma(t))$. We have, $\beta(0) = F(\sigma(0)) = F(p)$, i.e., $\beta$ passes through $F(p)$ at $t = 0$. Finally, let

$$Y = \frac{d\beta}{dt}(0).$$

i.e. $Y$ is the velocity vector of $\beta$ at $t = 0$.

Exercise 2.1. Show that

$$Y = DF(p)X.$$

Note: In the above, $X$ and $Y$ are represented as column vectors, and the $rhs$ of the equation involves matrix multiplication. Hint: Use the chain rule.

Thus, roughly speaking, the geometric effect of the Jacobian is to “send velocity vectors to velocity vectors”. The same result holds for mappings $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ (i.e. it is not necessary to restrict to dimension three).

Reparameterizations

Given a regular curve $\sigma : (a, b) \rightarrow \mathbb{R}^3$. Traversing the same path at a different speed (and perhaps in the opposite direction) amounts to what is called a reparameterization.

Def. Let $\sigma : (a, b) \rightarrow \mathbb{R}^3$ be a regular curve. Let $h : (c, d) \subset \mathbb{R} \rightarrow (a, b) \subset \mathbb{R}$ be a diffeomorphism (i.e. $h$ is 1-1, onto such that $h$ and $h^{-1}$ are smooth). Then $\tilde{\sigma} = \sigma \circ h : (c, d) \rightarrow \mathbb{R}^3$ is a regular curve, called a reparameterization of $\sigma$.

$$\tilde{\sigma}(u) = \sigma \circ h(u) = \sigma(h(u))$$

I.e., start with curve $\sigma = \sigma(t)$, make a change of parameter $t = h(u)$, obtain reparameterized curve $\tilde{\sigma} = \sigma(h(u))$; $t =$ original parameter, $u =$ new parameter.
Remarks.

1. \( \sigma \) and \( \tilde{\sigma} \) describe the same path in space, just traversed at different speeds (and perhaps in opposite directions).

2. Compare velocities:

\[
\tilde{\sigma} = \sigma(h(u)) \quad \text{i.e.,} \\
\tilde{\sigma} = \sigma(t), \text{ where } t = h(u).
\]

By the chain rule,

\[
\frac{d\tilde{\sigma}}{du} = \frac{d\sigma}{dt} \cdot \frac{dt}{du} = \frac{d\sigma}{dt} \cdot h'
\]

\( h' > 0 \): orientation preserving reparameterization.
\( h' < 0 \): orientation reversing reparameterization.

**Ex.** \( \sigma : (0, 2\pi) \to \mathbb{R}^3, \sigma(t) = (\cos t, \sin t, 0) \). Reparameterization function:

\( h : (0, \pi) \to (0, 2\pi), h(t) = \pi t \),

\[
h : t = h(u) = 2u, \quad u \in (0, \pi),
\]

Reparameterized curve:

\[
\tilde{\sigma}(u) = \sigma(t) = \sigma(2u) \\
\tilde{\sigma}(u) = (\cos 2u, \sin 2u, 0)
\]

\( \tilde{\sigma} \) describes the same circle, but traversed twice as fast,

speed of \( \sigma = \left| \frac{d\sigma}{dt} \right| = 1 \), speed of \( \tilde{\sigma} = \left| \frac{d\tilde{\sigma}}{du} \right| = 2 \).

**Remark** Regular curves always admit a very important reparameterization: they can always be parameterized in terms of arc length.

Length Formula: Consider a smooth curve defined on a closed interval, \( \sigma : [a, b] \to \mathbb{R}^3 \).
\( \sigma \) is a smooth curve segment. Its \textit{length} is defined by,

\[
\text{length of } \sigma = \int_a^b |\sigma'(t)| \, dt.
\]

I.e., to get the length, integrate speed \textit{wrt} time.

\textbf{Ex.} \( \sigma(t) = (r \cos t, \ r \sin t, 0) \quad 0 \leq t \leq 2\pi. \)

Length of \( \sigma = \int_0^{2\pi} |\sigma'(t)| \, dt = \int_0^{2\pi} r \, dt = 2\pi r. \)

\textbf{Fact.} The length formula is independent of parameterization, i.e., if \( \tilde{\sigma} : [c, d] \to \mathbb{R}^3 \) is a reparameterization of \( \sigma : [a, b] \to \mathbb{R}^3 \) then length of \( \tilde{\sigma} = \text{length of } \sigma. \)

\textbf{Exercise 2.2} Prove this fact.

\textbf{Arc Length Parameter:}

Along a regular curve \( \sigma : (a, b) \to \mathbb{R}^3 \) there is a distinguished parameter called \textit{arc length} parameter. Fix \( t_0 \in (a, b). \) Define the following function (\textit{arc length function}).

\[
s = s(t), \quad t \in (a, b), \quad s(t) = \int_{t_0}^t |\sigma'(t)| \, dt.
\]

Thus,

if \( t > t_0, \quad s(t) = \text{length of } \sigma \text{ from } t_0 \text{ to } t \)

if \( t < t_0, \quad s(t) = -\text{length of } \sigma \text{ from } t_0 \text{ to } t. \)

\( s = s(t) \) is smooth and by the Fundamental Theorem of calculus,

\[
s'(t) = |\sigma'(t)| > 0 \quad \text{for all } t \in (a, b)
\]

Hence \( s = s(t) \) is strictly increasing, and so has a smooth inverse - can solve smoothly for \( t \) in terms of \( s, \ t = t(s) \) (reparameterization function). Then,

\[
\tilde{\sigma}(s) = \sigma(t(s))
\]

is the arc length reparameterization of \( \sigma. \)

\textbf{Fact.} A regular curve admits a reparameterization in terms of arc length.
Exercise 2.2. Reparameterize the circle \( \sigma(t) = (r \cos t, r \sin t, 0) \), \(-\infty < t < \infty\), in terms of arc length parameter.

Obtain the arc length function \( s = s(t) \),

\[
\begin{align*}
  s &= \int_0^t |\sigma'(t)| \, dt = \int_0^t r \, dt \\
  s &= rt \implies t = \frac{s}{r} \quad \text{(reparam. function)}
\end{align*}
\]

Hence,

\[
\tilde{\sigma}(s) = \sigma(t(s)) = \sigma\left(\frac{s}{r}\right) = (r \cos \left(\frac{s}{r}\right), r \sin \left(\frac{s}{r}\right), 0).
\]

Remarks

1. Often one relaxes the notation and writes \( \sigma(s) \) for \( \tilde{\sigma}(s) \) (i.e. one drops the tilde).

2. Let \( \sigma = \sigma(t), \ t \in (a, b) \) be a unit speed curve, \( |\sigma'(t)| = 1 \) for all \( t \in (a, b) \). Then,

\[
\begin{align*}
  s &= \int_{t_0}^t |\sigma'(t)| \, dt = \int_{t_0}^t 1 \, dt \\
  s &= t - t_0.
\end{align*}
\]

I.e. up to a trivial translation of parameter, \( s = t \). Hence unit speed curves are already parameterized \textit{wrt} arc length (as measured from some point). Conversely, if \( \sigma = \sigma(s) \) is a regular curve parameterized \textit{wrt} arc length \( s \) then \( \sigma \) is unit speed, i.e. \( |\sigma'(s)| = 1 \) for all \( s \) (why?). Hence the phrases “unit speed curve” and “curve parameterized \textit{wrt} arc length” are used interchangeably.

Exercise 2.3. Reparameterize the helix, \( \sigma : \mathbb{R} \to \mathbb{R}^3, \ \sigma(t) = (r \cos t, r \sin t, ht) \) in terms of arc length.

Vector fields along a curve.

We will frequently use the notion of a vector field along a curve \( \sigma \).

Definition. Given a smooth curve \( \sigma : (a, b) \to \mathbb{R}^3 \) a vector field along \( \sigma \) is a vector-valued map \( X : (a, b) \to \mathbb{R}^3 \) which assigns to each \( t \in (a, b) \) a vector \( X(t) \) at the point \( \sigma(t) \).
Ex. Velocity vector field along $\sigma : (a, b) \to \mathbb{R}^3$.

$$\sigma' : (a, b) \to \mathbb{R}^3, \ t \to \sigma'(t);$$
if $\sigma(t) = (x(t), y(t), z(t))$, $\sigma'(t) = (x'(t), y'(t), z'(t))$.

Ex. Unit tangent vector field along $\sigma$.

$$T(t) = \frac{\sigma'(t)}{|\sigma'(t)|}.$$  
$|T(t)| = 1$ for all $t$. (Note $\sigma$ must be regular for $T$ to be defined).

Ex. Find unit tangent vector field along $\sigma(t) = (r \cos t, r \sin t, ht)$.

$$\sigma'(t) = (-r \sin t, r \cos t, h)$$

$$|\sigma'(t)| = \sqrt{r^2 + h^2}$$

$$T(t) = \frac{1}{\sqrt{r^2 + h^2}}(-r \sin t, r \cos t, h)$$

Note. If $s \to \sigma(s)$ is parameterized wrt arc length then $|\sigma'(s)| = 1$ (unit speed) and so,

$$T(s) = \sigma'(s).$$

Differentiation. Analytically vector fields along a curve are just maps,

$$X : (a, b) \subset \mathbb{R} \to \mathbb{R}^3.$$  
Can differentiate by expressing $X = X(t)$ in terms of components,
\[ X(t) = (X'(t), X^2(t), X^3(t)), \]
\[ \frac{dX}{dt} = \begin{pmatrix} \frac{dX^1}{dt} & \frac{dX^2}{dt} & \frac{dX^3}{dt} \end{pmatrix}. \]

**Ex.** Consider the unit tangent field to the helix,
\[ T(t) = \frac{1}{\sqrt{r^2 + h^2}}(-r \sin t, r \cos t, h) \]
\[ T'(t) = \frac{1}{\sqrt{r^2 + h^2}}(-r \cos t, -r \sin t, 0). \]

**Exercise 2.4.** Let \( X = X(t) \) and \( Y = Y(t) \) be two smooth vector fields along \( \sigma : (a, b) \to \mathbb{R}^3 \). Prove the following product rules,
\[ (1) \quad \frac{d}{dt} \langle X, Y \rangle = \langle \frac{dX}{dt}, Y \rangle + \langle X, \frac{dY}{dt} \rangle \]
\[ (2) \quad \frac{d}{dt} X \times Y = \frac{dX}{dt} \times Y + X \times \frac{dY}{dt} \]

Hint: Express in terms of components.

**Curvature**

Curvature of a curve is a measure of how much a curve bends at a given point:

This is quantified by measuring the rate at which the unit tangent turns \textit{wrt distance} along the curve. Given regular curve, \( t \to \sigma(t) \), reparameterize in terms of arc length, \( s \to \sigma(s) \), and consider the unit tangent vector field,
\[ T = T(s) \quad (T(s) = \sigma'(s)). \]

Now differentiate \( T = T(s) \) \textit{wrt} arc length,
\[ \frac{dT}{ds} = \text{curvature vector} \]
The direction of $\frac{dT}{ds}$ tells us which way the curve is bending. Its magnitude tells us how much the curve is bending,

$$\left| \frac{dT}{ds} \right| = \text{curvature}$$

**Def.** Let $s \to \sigma(s)$ be a unit speed curve. The curvature $\kappa = \kappa(s)$ of $\sigma$ is defined as follows,

$$\kappa(s) = |T'(s)| \text{ (or } |\sigma''(s)|),$$

where $' = \frac{d}{ds}$.

**Ex.** Compute the curvature of a circle of radius $r$.

*Standard parameterization:* $\sigma(t) = (r \cos t, r \sin t, 0)$.

*Arc length parameterization:* $\sigma(s) = \left( r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right), 0 \right)$.

$$T(s) = \sigma'(s) = \left( -\sin \left( \frac{s}{r} \right), \cos \left( \frac{s}{r} \right), 0 \right)$$

$$T'(s) = \left( -\frac{1}{r} \cos \left( \frac{s}{r} \right), -\frac{1}{r} \sin \left( \frac{s}{r} \right), 0 \right)$$

$$= -\frac{1}{r} \left( \cos \left( \frac{s}{r} \right), \sin \left( \frac{s}{r} \right), 0 \right)$$

$$\kappa(s) = |T'(s)| = \frac{1}{r}$$

(Does this answer agree with intuition?)

**Exercise 2.5.** Let $s \to \sigma(s)$ be a unit speed plane curve,

$$\sigma(s) = (x(s), y(s), 0).$$

For each $s$ let,

$$\phi(s) = \text{angle between positive } x\text{-axis and } T(s).$$

Show: $\kappa(s) = |\phi'(s)|$ (i.e. $\kappa = \left| \frac{d\phi}{ds} \right|$).

Hint: Observe, $T(s) = \cos \phi(s)i + \sin \phi(s)j$ (why?).
Conceptually, the definition of curvature is the right one. But for computational purposes it’s not so good. For one thing, it would be useful to have a formula for computing curvature which does not require that the curve be parameterized with respect to arc length. Using the chain rule, such a formula is easy to obtain.

Given a regular curve \( t \to \sigma(t) \), it can be reparameterized \textit{wrt} arc length \( s \to \sigma(s) \). Let \( T = T(s) \) be the unit tangent field to \( \sigma \).

\[
T = T(s), \quad s = s(t),
\]

So by the chain rule,

\[
\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} \left| \frac{d\sigma}{dt} \right| = \left| \frac{dT}{ds} \right| \left| \frac{d\sigma}{dt} \right|
\]

and hence,

\[
\kappa = \frac{dT}{ds} \left| \frac{d\sigma}{dt} \right|, \quad \kappa = \frac{d}{dt}.
\]

**Exercise 2.6.** Use the above formula to compute the curvature of the helix \( \sigma(t) = (r \cos t, r \sin t, ht) \).

**Frenet-Equations**

Let \( s \to \sigma(s) \), \( s \in (a, b) \) be a regular unit speed curve such that \( \kappa(s) \neq 0 \) for all \( s \in (a, b) \). (We will refer to such a curve as \textit{strongly} regular). Along \( \sigma \) we are going to introduce the vector fields,

\[
T = T(s) \quad - \text{unit tangent vector field}
\]
\[
N = N(s) \quad - \text{principal normal vector field}
\]
\[
B = B(s) \quad - \text{binormal vector field}
\]

\( \{T, N, B\} \) is called a Frenet frame.
At each point of $\sigma$ 
$\{T, N, B\}$ forms an
orthonormal basis, i.e.
$T, N, B$ are mutually
perpendicular unit vectors.

To begin the construction of the Frenet frame, we have the unit tangent vector field,

$$T(s) = \sigma'(s), \quad ' = \frac{d}{ds}$$

Consider the derivative $T' = T'(s)$.

**Claim.** $T' \perp T$ along $\sigma$.

**Proof.** It suffices to show $\langle T', T \rangle = 0$ for all $s \in (a, b)$. Along $\sigma$,

$$\langle T, T \rangle = |T|^2 = 1.$$ 

Differentiating both sides,

$$\frac{d}{ds} \langle T, T \rangle = \frac{d}{ds} 1 = 0$$

$$\langle \frac{dT}{ds}, T \rangle + \langle T, \frac{dT}{ds} \rangle = 0$$

$$2 \langle \frac{dT}{ds}, T \rangle = 0$$

$$\langle T', T \rangle = 0.$$ 

**Def.** Let $s \to \sigma(s)$ be a strongly regular unit speed curve. The *principal normal* vector field along $\sigma$ is defined by

$$N(s) = \frac{T'(s)}{|T'(s)|} = \frac{T'(s)}{\kappa(s)} \quad (\kappa(s) \neq 0)$$

The *binormal* vector field along $\sigma$ is defined by

$$B(s) = T(s) \times N(s).$$

Note, the definition of $N = N(s)$ implies the equation

$$T' = \kappa N$$
**Claim.** For each $s$, $\{T(s), N(s), B(s)\}$ is an orthonormal basis for vectors in space based at $\sigma(s)$.

**Mutually perpendicular:**

\[
\langle T, N \rangle = \langle T, \frac{T'}{\kappa} \rangle = \frac{1}{\kappa} \langle T, T' \rangle = 0.
\]

\[
B = T \times N \Rightarrow \langle B, T \rangle = \langle B, N \rangle = 0.
\]

**Unit length:** $|T| = 1$, and

\[
|N| = \left| \frac{T'}{|T'} \right| = \frac{|T'|}{|T'|} = 1,
\]

\[
|B|^2 = |T \times N|^2
\]

\[
= |T|^2 |N|^2 - \langle T, N \rangle^2 = 1.
\]

**Remark on o.n. bases.**

\[
X = \text{vector at } \sigma(s).
\]

$X$ can be expressed as a linear combination of $T(s), N(s), B(s)$,

\[
X = aT + bN + cB
\]

The constants $a, b, c$ are determined as follows,

\[
\langle X, T \rangle = \langle aT + bN + cB, T \rangle
\]

\[
= a \langle T, T \rangle + b \langle N, T \rangle + c \langle B, T \rangle
\]

\[
= a
\]

Hence, $a = \langle X, T \rangle$, and similarly, $b = \langle X, N \rangle$, $c = \langle X, B \rangle$. Hence $X$ can be expressed as,

\[
X = \langle X, T \rangle T + \langle X, N \rangle N + \langle X, B \rangle B.
\]
**Torsion:** Torsion is a measure of “twisting”. Curvature is associated with $T'$; torsion is associated with $B'$:

\[
\begin{align*}
B &= T \times N \\
B' &= T' \times N + T \times N' \\
&= \kappa N \times N + T \times N'
\end{align*}
\]

Therefore $B' = T \times N'$ which implies $B' \perp T$, i.e.

\[
\langle B', T \rangle = 0
\]

Also, since $B = B(s)$ is a unit vector along $\sigma$, $\langle B, B \rangle = 1$ which implies by differentiation,

\[
\langle B', B \rangle = 0
\]

It follows that $B'$ is a multiple of $N$,

\[
\begin{align*}
B' &= \langle B', T \rangle T + \langle B', N \rangle N + \langle B', B \rangle B \\
B' &= \langle B', N \rangle N.
\end{align*}
\]

Hence, we may write,

\[
B' = -\tau N
\]

where $\tau = \text{torsion} := -\langle B', N \rangle$.

**Remarks**

1. $\tau$ is a function of $s$, $\tau = \tau(s)$.
2. $\tau$ is signed i.e. can be positive or negative.
3. $|\tau(s)| = |B'(s)|$, i.e., $\tau = \pm |B'|$, and hence $\tau$ measures how $B$ wiggles.

Given a strongly regular unit speed curve $\sigma$, the collection of quantities $T, N, B, \kappa, \tau$ is sometimes referred to as the Frenet apparatus.

**Ex.** Compute $T, N, B, \kappa, \tau$ for the unit speed circle.

\[
\begin{align*}
\sigma(s) &= \left( r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right), 0 \right) \\
T &= \sigma' = \left( -\sin \left( \frac{s}{r} \right), \cos \left( \frac{s}{r} \right), 0 \right) \\
T' &= -\frac{1}{r} \left( \cos \left( \frac{s}{r} \right), \sin \left( \frac{s}{r} \right), 0 \right)
\end{align*}
\]
\[ \kappa = |T'| = \frac{1}{r} \]

\[ N = \frac{T'}{k} = -\left(\cos \left(\frac{s}{r}\right), \sin \left(\frac{s}{r}\right), 0\right) \]

\[ B = T \times N \]

\[ = \left| \begin{array}{ccc} i & j & k \\ -s & c & 0 \\ -c & -s & 0 \end{array} \right| = k = (0, 0, 1), \]

(where \( c = \cos \left(\frac{s}{r}\right) \) and \( s = \sin \left(\frac{s}{r}\right) \)). Finally, since \( B' = 0 \), \( \tau = 0 \), i.e. the torsion vanishes.

**Conjecture.** Let \( s \to \sigma(s) \) be a strongly regular unit speed curve. Then, \( \sigma \) is a *plane curve* iff its torsion vanishes, \( \tau \equiv 0 \).

**Exercise 2.7.** Consider the helix, 
\[ \sigma(t) = (r \cos t, r \sin t, ht). \]
Show that, when parameterized *wrt* arc length, we obtain, 
\[ \sigma(s) = (r \cos \omega s, r \sin \omega s, h \omega s), \quad (*) \]
where \( \omega = \frac{1}{\sqrt{r^2 + h^2}} \).

**Ex.** Compute \( T, N, B, \kappa, \tau \) for the unit speed helix \((*)\).

\[ T = \sigma' = (-r \omega \sin \omega s, r \omega \cos \omega s, h \omega) \]

\[ T' = -\omega^2 r (\cos \omega s, \sin \omega s, 0) \]

\[ \kappa = |T'| = \omega^2 r = \frac{r}{r^2 + h^2} = \text{const.} \]

\[ N = \frac{T'}{\kappa} = (- \cos \omega s, - \sin \omega s, 0) \]
\[ B = T \times N = \begin{vmatrix} i & j & k \\
 -r \omega \sin \omega s & r \omega \cos \omega s & h \omega \\
 -\cos \omega s & -\sin \omega s & 0 \end{vmatrix} \]

\[ B = (h \omega \sin \omega s, -h \omega \cos \omega s, r \omega) \]

\[ B' = (h \omega^2 \cos \omega s, h \omega^2 \sin \omega s, 0) \]

\[ = h \omega^2 (\cos \omega s, \sin \omega s, 0) \]

\[ B' = -h \omega^2 N \]

\[ B' = -\tau N \Rightarrow \tau = h \omega^2 = \frac{h}{r^2 + h^2}. \]

Remarks.

\[ \Pi(s) = osculating \ plane \ of \ \sigma \ at \ \sigma(s) \]
\[ = plane \ passing \ through \ \sigma(s) \ spanned \ by \ N(s) \ and \ T(s) \]
\[ (or \ equivalently, \ perpendicular \ to \ B(s)). \]

(1) \( s \rightarrow \Pi(s) \) is the family of osculating planes along \( \sigma \). The Frenet equation \( B' = -\tau N \) shows that the torsion \( \tau \) measures how the osculating plane is twisting along \( \sigma \).

(2) \( \Pi(s_0) \) passes through \( \sigma(s_0) \) and is spanned by \( \sigma'(s_0) \) and \( \sigma''(s_0) \). Hence, in a sense that can be made precise, \( s \rightarrow \sigma(s) \) lies in \( \Pi(s_0) \) “to second order in \( s \)”. If \( \tau(s_0) \neq 0 \) then \( \sigma'''(s_0) \) is not tangent to \( \Pi(s_0) \). Hence the torsion \( \tau \) gives a measure of the extent to which \( \sigma \) twists out of a given fixed osculating plane.
Theorem. (Frenet Formulas) Let \( s \to \sigma(s) \) be a strongly regular unit speed curve. Then the Frenet frame, \( T, N, B \) satisfies,

\[
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N
\end{align*}
\]

Proof. We have already established the first and third formulas. To establish the second, observe \( B = T \times N \Rightarrow N = B \times T \). Hence,

\[
\begin{align*}
N' &= (B \times T)' = B' \times T + B \times T' \\
&= -\tau N \times T + \kappa B \times N \\
&= -\tau (-B) + \kappa (-T) \\
&= -\kappa T + \tau B.
\end{align*}
\]

We can express Frenet formulas as a matrix equation,

\[
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}' =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\]

\( A \) is skew symmetric: \( A^t = -A \). \( A = [a_{ij}] \), then \( a_{ji} = -a_{ij} \).

The Frenet equations can be used to derive various properties of space curves.

Proposition. Let \( s \to \sigma(s), s \in (a,b), \) be a strongly regular unit speed curve. Then, \( \sigma \) is a plane curve iff its torsion vanishes, \( \tau \equiv 0 \).

Proof. Recall, the plane II which passes through the point \( x_0 \in \mathbb{R}^3 \) and is perpendicular to the unit vector \( n \) consists of all points \( x \in \mathbb{R}^3 \) which satisfy the equation,

\[
\langle n, x - x_0 \rangle = 0
\]

\( \Rightarrow \): Assume \( s \to \sigma(s) \) lies in the plane II. Then, for all \( s \),

\[
\langle n, \sigma(s) - x_0 \rangle = 0
\]
Since $n$ is constant, differentiating twice gives,
\[
\frac{d}{ds} \langle n, \sigma(s) - x_0 \rangle = \langle n, \sigma' \rangle = \langle n, T \rangle = 0,
\]
\[
\frac{d}{ds} \langle n, T \rangle = \langle n, T' \rangle = \kappa \langle n, N \rangle = 0.
\]
Since $n$ is a unit vector perpendicular to $T$ and $N$, $n = \pm B$, so $B = \pm n$. I.e., $B = B(s)$ is constant which implies $B' = 0$. Therefore $\tau \equiv 0$.

$\Leftarrow$: Now assume $\tau \equiv 0$. $B' = -\tau N \Rightarrow B' = 0$, i.e. $B(s)$ is constant,
\[
B(s) = B = \text{constant vector}.
\]
We show $s \to \sigma(s)$ lies in the plane, $\langle B, x - \sigma(s_0) \rangle = 0$, passing through $\sigma(s_0)$, $s_0 \in (a,b)$, and perpendicular to $B$, i.e., will show,
\[
\langle B, \sigma(s) - \sigma(s_0) \rangle = 0. \quad (*)
\]
for all $s \in (a,b)$. Consider the function, $f(s) = \langle B, \sigma(s) - \sigma(s_0) \rangle$. Differentiating,
\[
f'(s) = \frac{d}{ds} \langle B, \sigma(s) - \sigma(s_0) \rangle
\]
\[
= \langle B', \sigma(s) - \sigma(s_0) \rangle + \langle B, \sigma'(s) \rangle
\]
\[
= 0 + \langle B, T \rangle = 0.
\]
Hence, $f(s) = c = \text{const.}$ Since $f(s_0) = \langle B, \sigma(s_0) - \sigma(s_0) \rangle = 0$, $c = 0$ and thus $f(s) \equiv 0$. Therefore $(*)$ holds, i.e., $s \to \sigma(s)$ lies in the plane $\langle B, x - \sigma(s_0) \rangle = 0$.

**Sphere Curves.** A sphere curve is a curve in $\mathbb{R}^3$ which lies on a sphere,
\[
|x - x_0|^2 = r^2, \quad \text{(sphere of radius } r \text{ centered at } x_0)
\]
\[
\langle x - x_0, x - x_0 \rangle = r^2
\]

Thus, $s \to \sigma(s)$ is a sphere curve iff there exists $x_0 \in \mathbb{R}^3$, $r > 0$ such that
\[
\langle \sigma(s) - x_0, \sigma(s) - x_0 \rangle = r^2, \quad \text{for all } s. \quad (*)
\]
If $s \to \sigma(s)$ lies on a sphere of radius $r$, it is reasonable to conjecture that $\sigma$ has curvature $\kappa \geq \frac{1}{r}$ (why?). We prove this.

27
Proposition. Let \( s \rightarrow \sigma(s), s \in (a, b) \), be a unit speed curve which lies on a sphere of radius \( r \). Then its curvature function \( \kappa = \kappa(s) \) satisfies, \( \kappa \geq \frac{1}{r} \).

Proof Differentiating (*) gives,

\[
2\langle \sigma', \sigma - x_0 \rangle = 0
\]
i.e.,

\[
\langle T, \sigma - x_0 \rangle = 0.
\]

Differentiating again gives:

\[
\begin{align*}
\langle T', \sigma - x_0 \rangle + \langle T, \sigma' \rangle &= 0 \\
\langle T', \sigma - x_0 \rangle + \langle T, T \rangle &= 0 \\
\langle T', \sigma - x_0 \rangle &= -1 \quad (\Rightarrow T' \neq 0) \\
\kappa \langle N, \sigma - x_0 \rangle &= -1
\end{align*}
\]

But,

\[
|\langle N, \sigma - x_0 \rangle| = |N||\sigma - x_0||\cos \theta|
\]

\[
= r|\cos \theta|
\]
and so,

\[
\kappa = |\kappa| = \frac{1}{|\langle N, \sigma - x_0 \rangle|} = \frac{1}{r|\cos \theta|} \geq \frac{1}{r}
\]

Exercise 2.8. Prove that any unit speed sphere curve \( s \rightarrow \sigma(s) \) having constant curvature is a circle (or part of a circle). (Suggestion: Show that the torsion vanishes (why is this sufficient?). To show this differentiate (*) a few times.)

Lancrets Theorem.

Consider the unit speed circular helix \( \sigma(s) = (r \cos \omega s, r \sin \omega s, h \omega s), \omega = 1/\sqrt{r^2 + h^2} \). This curve makes a constant angle \( \text{wrt} \) the \( z \)-axis: \( T = (-r \omega \sin \omega s, r \cos \omega s, h \omega) \),

\[
\cos \theta = \frac{\langle T, k \rangle}{|T||k|} = h \omega = \text{const}.
\]

Def. A unit speed curve \( s \rightarrow \sigma(s) \) is called a generalized helix if its unit tangent \( T \) makes a constant angle with a fixed unit direction vector \( u \) \( (\Leftrightarrow \langle T, u \rangle = \cos \theta = \text{const}) \).

Theorem. (Lancret) Let \( s \rightarrow \sigma(s), s \in (a, b) \) be a strongly regular unit speed curve such that \( \tau(s) \neq 0 \) for all \( s \in (a, b) \). Then \( \sigma \) is a generalized helix iff \( \kappa/\tau = \text{constant} \).
Non-unit Speed Curves.

Given a regular curve \( t \to \sigma(t) \), it can be reparameterized in terms of arc length \( s \to \tilde{\sigma}(s), \tilde{\sigma}(s) = \sigma(t(s)) \). The quantities \( T, N, B, \kappa, T \) can then be computed (provided \( \kappa \neq 0 \)). But it is convenient to have formulas for these quantities which do not involve reparameterizing in terms of arc length.

**Proposition.** Let \( t \to \sigma(t) \) be a strongly regular curve in \( \mathbb{R}^3 \). Then

1. \[ T = \frac{\dot{\sigma}}{|\dot{\sigma}|}, \quad \cdot = \frac{d}{dt} \]
2. \[ B = \frac{\dot{\sigma} \times \ddot{\sigma}}{|\dot{\sigma} \times \dot{\sigma}|} \]
3. \[ N = B \times T \]
4. \[ \kappa = \frac{|\dot{\sigma} \times \ddot{\sigma}|}{|\dot{\sigma}|^3} \]
5. \[ \tau = \frac{\langle \dot{\sigma} \times \ddot{\sigma}, \ddot{\sigma} \rangle}{|\dot{\sigma} \times \ddot{\sigma}|^2} \]

**Proof.** We derive some of these. See e.g. Millman and Parker, Section 2-6, for further details. Interpreting physically, \( t=\text{time}, \dot{\sigma}=\text{velocity}, \ddot{\sigma}=\text{acceleration} \). The unit tangent may be expressed as,

\[ T = \frac{\dot{\sigma}}{|\dot{\sigma}|} = \frac{\dot{\sigma}}{v} \]

where \( v = |\dot{\sigma}| = \text{speed} \). Hence,

\[
\begin{align*}
\dot{\sigma} &= vT \\
\ddot{\sigma} &= \frac{d}{dt}vT + v\frac{dT}{dt} \\
&= \frac{dv}{dt}T + v\frac{dT}{dt} \cdot \frac{ds}{dt} \\
&= \frac{dv}{dt}T + v(\kappa N)v \\
\dddot{\sigma} &= \ddot{v}T + v^2\kappa N
\end{align*}
\]

**Side Comment:** This is the well-known expression for acceleration in terms of its tangential and normal components.
\[ \dot{v} = \text{tangential component of acceleration (} \dot{s} \text{)} \]

\[ v^2 \kappa = \text{normal component of acceleration} \]

\[ = \text{centripetal acceleration (for a circle, } v^2 \kappa = \frac{v^2}{r} \text{).} \]

\( \dot{\sigma}, \ddot{\sigma} \) lie in osculating plane; if \( \tau \neq 0 \), \( \ddot{\sigma} \) does not.

Continuing the derivation,

\[
\dot{\sigma} \times \ddot{\sigma} = vT \times (\dot{v}T + v^2 \kappa N) = v\dot{v} T \times T + v^3 \kappa T \times N
\]

\[
\dot{\sigma} \times \ddot{\sigma} = v^3 \kappa B \]

\[
|\dot{\sigma} \times \ddot{\sigma}| = v^3 \kappa |B| = v^3 \kappa
\]

Hence,

\[
\kappa = \frac{|\dot{\sigma} \times \ddot{\sigma}|}{v^3} = \frac{|\dot{\sigma} \times \ddot{\sigma}|}{|\dot{\sigma}|^3}
\]

Also,

\[
B = \text{const} \cdot \dot{\sigma} \times \ddot{\sigma} = \frac{\dot{\sigma} \times \ddot{\sigma}}{|\dot{\sigma} \times \ddot{\sigma}|}.
\]

**Exercise 2.9.** Derive the expression for \( \tau \). Hint: Compute \( \ddot{\sigma} \) and use Frenet formulas.

**Exercise 2.10.** Suppose \( \sigma \) is a regular curve in the \( x-y \) plane, \( \sigma(t) = (x(t), y(t), 0) \), i.e.,

\[
\sigma : \begin{align*}
x &= x(t) \\
y &= y(t)
\end{align*}
\]

(a) Show that the curvature of \( \sigma \) is given by,

\[
\kappa = \frac{|\dot{x}y - \dot{y}x|}{|x^2 + y^2|^{3/2}}
\]

(b) Use this formula to compute the curvature \( \kappa = \kappa(t) \) of the ellipse,

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]
Fundamental Theorem of Space Curves

This theorem says basically that any strongly regular unit speed curve is completely determined by its curvature and torsion (up to a Euclidean motion).

**Theorem.** Let $\overline{\kappa} = \overline{\kappa}(s)$ and $\overline{\tau} = \overline{\tau}(s)$ be smooth functions on an interval $(a, b)$ such that $\overline{\kappa}(s) > 0$ for all $s \in (a, b)$. Then there exists a strongly regular unit speed curve $s \to \sigma(s)$, $s \in (a, b)$ whose curvature and torsion functions are $\overline{\kappa}$ and $\overline{\tau}$, respectively. Moreover, $\sigma$ is essentially unique, i.e. any other such curve $\tilde{\sigma}$ can be obtained from $\sigma$ by a Euclidean motion (translation and/or rotation).

**Remarks**

1. The FTSC shows that curvature and torsion are the essential quantities for describing space curves.

2. The FTSC also illustrates a very important issue in differential geometry. The problem of establishing the existence of some geometric object having certain geometric properties often reduces to a problem concerning the existence of a solution to some differential equation, or system of differential equations.

**Proof:** Fix $s_0 \in (a, b)$, and in space fix $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and a positively oriented orthonormal frame of vectors at $P_0$, $\{T_0, N_0, B_0\}$.

We show that there exists a unique unit speed curve $\sigma : (a, b) \to \mathbb{R}^3$ having curvature $\overline{\kappa}$ and torsion $\overline{\tau}$ such that $\sigma(s_0) = P_0$ and $\sigma$ has Frenet frame $\{T_0, N_0, B_0\}$ at $\sigma(s_0)$.

The proof is based on the Frenet formulas:

\[
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N
\end{align*}
\]

or, in matrix form,

\[
\frac{d}{ds} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.
\]

The idea is to mimic these equations using the given functions $\overline{\kappa}, \overline{\tau}$. Consider the following system of O.D.E.’s in the (as yet unknown) vector-valued functions $e_1 = e_1(s), e_2 = e_2(s), e_3 = e_3(s)$,
\[
\frac{de_1}{ds} = \kappa e_2 \\
\frac{de_2}{ds} = -\kappa e_1 + \tau e_3 \\
\frac{de_3}{ds} = -\tau e_2
\] (\*)

We express this system of ODE's in a notation convenient for the proof:

\[
\frac{d}{ds} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},
\]

Set,

\[
\Omega = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}
\]

i.e. \( \Omega^1 = 0, \Omega^2 = \kappa, \Omega^3 = 0 \), etc. Note that \( \Omega \) is skew symmetric, \( \Omega^t = -\Omega \iff \Omega^j = -\Omega^i \), \( 1 \leq i, j \leq 3 \). Thus we may write,

\[
\frac{d}{ds} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = [\Omega^i]^j \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},
\]

or,

\[
\frac{d}{ds} e_i = \sum_{j=1}^{3} \Omega^i j e_j, \quad 1 \leq i \leq 3
\]

IC :
\[
e_1(s_0) = T_0 \\
e_2(s_0) = N_0 \\
e_3(s_0) = B_0
\]

Now, basic existence and unique result for systems of linear ODE’s guarantees that this system has a unique solution:

\( s \to e_1(s), s \to e_2(s), s \to e_3(s), s \in (a, b) \)

We show that \( e_1 = T, e_2 = N, e_3 = B, \kappa = \kappa \) and \( \tau = \tau \) for some unit speed curve \( s \to \sigma(s) \).

Claim \( \{e_1(s), e_2(s), e_3(s)\} \) is an orthonormal frame for all \( s \in (a, b) \), i.e.,
\[ \langle e_i(s), e_j(s) \rangle = \delta_{ij} \quad \forall \ s \in (a, b) \]

where \( \delta_{ij} \) is the “Kronecker delta” symbol:

\[
\delta_{ij} = \begin{cases} 
0 & i \neq j \\
1 & i = j.
\end{cases}
\]

Proof of the claim: We make use of the “Einstein summation convention”:

\[
\frac{d}{ds} e_i = \sum_{j=1}^{3} \Omega_i^j e_j = \Omega_i^j e_j
\]

Let \( g_{ij} = \langle e_i, e_j \rangle \), \( g_{ij}(s) = g_{ij}(s), 1 \leq i, j \leq 3 \). Note,

\[
g_{ij}(s_0) = \langle e_i(s_0), e_j(s_0) \rangle = \delta_{ij}
\]

The \( g_{ij} \)'s satisfy a system of linear ODE's,

\[
\frac{d}{ds} g_{ij} = \frac{d}{ds} \langle e_i, e_j \rangle
\]

\[
= \langle e'_i, e_j \rangle + \langle e_i, e'_j \rangle
\]

\[
= \langle \Omega_k^i e_k, e_j \rangle + \langle e_i, \Omega_j^\ell e_\ell \rangle
\]

\[
= \Omega_i^k \langle e_k, e_j \rangle + \Omega_j^\ell \langle e_i, e_\ell \rangle
\]

Hence,

\[
\frac{d}{ds} g_{ij} = \Omega_i^k g_{kj} + \Omega_j^\ell g_{i\ell}
\]

IC : \( g_{ij}(s_0) = \delta_{ij} \)

Observe, \( g_{ij} = \delta_{ij} \) is a solution to this system,

\[
LHS = \frac{d}{ds} \delta_{ij} = \frac{d}{ds} \text{const} = 0.
\]

\[
RHS = \Omega_i^k \delta_{kj} + \Omega_j^\ell \delta_{i\ell}
\]

\[
= \Omega_i^j + \Omega_j^i
\]

\[
= 0 \ (\text{skew symmetry!}).
\]
But ODE theory guarantees a unique solution to this system. Therefore $g_{ij} = \delta_{ij}$ is the solution, and hence the claim follows.

How to define $\sigma$: Well, if $s \to \sigma(s)$ is a unit speed curve then

$$\sigma'(s) = T(s) \Rightarrow \sigma(s) = \sigma(s_0) + \int_{s_0}^{s} T(s)ds.$$ 

Hence, we define $s \to \sigma(s), s \in (a,b)$ by,

$$\sigma(s) = P_0 + \int_{s_0}^{s} e_1(s)ds$$ 

Claim $\sigma$ is unit speed, $\kappa = \pi$, $\tau = \tau$, $T = e_1$, $N = e_2$, $B = e_3$.

We have,

$$\sigma' = \frac{d}{ds}(P_0 + \int_{s_0}^{s} e_1(s)ds) = e_1$$

$$|\sigma'| = |e_1| = 1, \text{ therefore } \sigma \text{ is unit speed},$$

$$T \equiv \sigma' = e_1$$

$$\kappa = |T'| = |e_1'| = |\pi e_2| = \pi$$

$$N = \frac{T'}{\kappa} = \frac{e_1'}{\pi} = \frac{\pi e_2}{\pi} = e_2$$

$$B \equiv T \times N = e_1 \times e_2 = e_3$$

$$B' = e_3' = -\pi e_2 = -\pi N \Rightarrow$$

$$\tau \equiv \tau.$$
3 Surfaces

We all understand intuitively what a surface is. In calculus we encounter surfaces in several ways.

1. As graphs of functions of two variables, \( z = f(x, y) \).
   Ex. \( z = x^2 + y^2 \)

2. As level surfaces of functions of three variables, \( F(x, y, z) = c \).
   Ex. \( x^2 + y^2 + z^2 = 1 \)

3. As surfaces of revolution.
   Ex. Torus: surface of a doughnut.

We will need to be fairly precise about what we mean by a surface. Our definition will need to cover all these cases. The key is to describe surfaces parametrically. Very roughly speaking, a surface for us is going to be a subset of \( \mathbb{R}^3 \) which can be broken up into overlapping pieces such that each piece is described parametrically, i.e. described by a 2-parameter map.

Hence, the starting point is the notion of parameterized surfaces.

Def. A smooth parameterized surface in \( \mathbb{R}^3 \) is a smooth map \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, (u, v) \rightarrow \mathbf{x}(u, v) \).

As \( (u, v) \) varies over \( U \), \( \mathbf{x}(u, v) \in \mathbb{R}^3 \) traces out a “surface” in \( \mathbb{R}^3 \).
In terms of components, \( \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \),

\[
\begin{align*}
  x &= x(u, v) \\
  y &= y(u, v) \\
  z &= z(u, v)
\end{align*}
\]

\( (u, v) \in U \)

An effective way to see what gets traced out is to look at the “u-curves” and “v-curves”.

(1) if \( v \) is held constant, \( v = v_0 \) and \( u \) varies,
\[
u \rightarrow \mathbf{x}(u, v_0) \quad "u - curve"\]

(2) if \( u \) is held constant, \( u = u_0 \) and \( v \) varies,
\[
v \rightarrow \mathbf{x}(u_0, v) \quad "v - curve"\]

One way to examine a parameterized surface is to plot many “coordinate” curves, \( u=\text{const}, \ v=\text{const} \). This is how e.g., Mathematica plots parameterized surfaces.

Ex. \( \mathbf{x} : U \rightarrow \mathbb{R}^3, U = \{ (u, v) : 0 < u < 2\pi, 0 < v < 3 \} \), \( \mathbf{x}(u, v) = (2 \cos u, 2 \sin u, v) \),

\[
\begin{align*}
  x &= 2 \cos u \\
  y &= 2 \sin u \\
  z &= v
\end{align*}
\]

\( 0 < u < 2\pi, \ 0 < v < 3 \)

For this example it is convenient to consider closed rectangle \( U : 0 \leq u \leq 2\pi, \ 0 \leq v \leq 3 \). We plot some u-curves and v-curves:

\[
\begin{align*}
  v = 0 : & \quad x = 2 \cos u \\
            & \quad y = 2 \sin u \quad 0 \leq u \leq 2\pi \quad \text{circle in } z = 0 \\
            & \quad z = 0 \\
  v = 1 : & \quad x = 2 \cos u \\
            & \quad y = 2 \sin u \quad 0 \leq u \leq 2\pi \quad \text{circle in } z = 1 \\
            & \quad z = 1 \\
\end{align*}
\]

etc.
This parameterized surface describes a cylinder. Note that the coordinate functions satisfy:

\[ x^2 + y^2 = 4, \quad 0 \leq z \leq 3 \]

**Note:** On the original domain \( U \), \( \mathbf{x} \) is 1-1. We will restrict attention to parameterized surfaces \( \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) which are 1-1. Cylinder of radius \( a \) : \( \mathbf{x}(u,v) = (a \cos u, a \sin u, v) \)

**Coordinate Vector Fields.** Given a smooth surface,

\[ \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \]

we can differentiate wrt \( u \) and \( v \),

\[
\frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{pmatrix}, \\
\frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}.
\]

These partial derivatives have natural interpretations,

\[
\frac{\partial \mathbf{x}}{\partial u}(u_0,v_0) = \text{tangent vector to } u\text{-curve} \\
\quad u \to \mathbf{x}(u,v_0) \text{ at } \mathbf{x}(u_0,v_0)
\]

\[
\frac{\partial \mathbf{x}}{\partial v}(u_0,v_0) = \text{tangent vector to } v\text{-curve} \\
\quad v \to \mathbf{x}(u_0,v) \text{ at } \mathbf{x}(u_0,v_0)
\]
Hence,
\[ \frac{\partial \mathbf{x}}{\partial u} \text{ = velocity vector field to } u\text{-curves} \]
\[ \frac{\partial \mathbf{x}}{\partial v} \text{ = velocity vector field to } v\text{-curves.} \]

**Remark.** The coordinate curves \( u = u_0, \ v = v_0 \) lie in the surface. Hence the coordinate vectors \( \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0), \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0) \) are tangent vectors to the surface at \( \mathbf{x}(u_0, v_0) \).

**Standard Picture:** Grid of horizontal and vertical lines in \( U \subset \mathbb{R}^2 \) gives rise to a grid of curves - the coordinate curves on \( \mathbf{x}(U) \). This amounts to introducing coordinates on \( \mathbf{x}(U) \).

**Shorthand Notation:** \( \mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}, \ \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v} \).

Actually, to insure that the image of a parameterized surface \( \mathbf{x} \) looks like a surface (i.e. smooth 2-dimensional object), we need a *regularity* condition, akin to the regularity condition for parameterized curves (\( \sigma'(t) \neq 0 \)).

**Ex.** \( \mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3, \ \mathbf{x}(u, v) = (0, 0, 0) \ \forall (u, v) \). Image a single point! \( \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial \mathbf{x}}{\partial v} = \mathbf{0} \).

**Ex.** \( \mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3, \ \mathbf{x}(u, v) = (\cos(u + v^2), \sin(u + v^2), 1) \)

\[ x = \cos(u + v^2) \]
\[ \mathbf{x} : \ y = \sin(u + v^2) \]
\[ z = 1 \]

Image: \( x^2 + y^2 = 1, z = 1 \), a circle!

Compute: \( \frac{\partial \mathbf{x}}{\partial v} = -2v \frac{\partial \mathbf{x}}{\partial u} \), i.e. \( \frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \) are linearly dependent (at every point).

To avoid this type of “degeneracy” must require that \( \frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \) be linearly independent. There are several ways to characterize this independence.

Consider a parameterized surface, \( \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3, \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \),

\[ x = x(u, v) \]
\[ \mathbf{x} : \ y = y(u, v) \]
\[ z = z(u, v) \]
\[ D\mathbf{x} = \text{Jacobian matrix of } \mathbf{x}, \text{ is the } 3 \times 2 \text{ matrix:} \]
\[
D\mathbf{x} = \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{bmatrix}
\]

Recall, the

rank of a matrix = no. of linearly independent rows
= no. of linearly independent columns

**Prop.** Let \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a smooth parameterized surface. Then the following conditions are equivalent.

1. \( D\mathbf{x} \) has rank 2.
2. \( \frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \) are linearly independent.
3. \( \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0. \)

**Proof.**

\( D\mathbf{x} \) has rank 2 \( \iff \) columns lin. indep.

\( \iff \frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \) lin. indep.

\( \iff \) one is not a multiple of the other

\( \iff \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0. \)

**Def.** A regular (parameterized) surface in \( \mathbb{R}^3 \) is a smooth parameterized surface \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) such that \( \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0 \) for all \((u, v) \in U.\)
A coordinate patch is a regular surface \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) which is one-to-one. Roughly speaking, a surface in \( \mathbb{R}^3 \) is a subset of \( \mathbb{R}^3 \) which is covered by coordinate patches.

If the regularity condition is not satisfied, the image of \( \mathbf{x} \) can degenerate to a point, or curve – or something that does not look like a smooth surface (surface with “folds” or “cusps”). If, however, the regularity condition is satisfied, then the image of \( \mathbf{x} \) will look like a smooth surface. This is made rigorous in the following proposition.

**Proposition.** Let \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a regular surface. Then for each \( (u_0, v_0) \in U \) there is a neighborhood \( V \subset U \) of \( (u_0, v_0) \) such that the image \( \mathbf{x}(V) \subset \mathbb{R}^3 \) coincides with the graph of an equation of the form,

\[
z = f(x, y) \quad \text{or} \quad y = g(x, z) \quad \text{or} \quad x = h(y, z),
\]

where \( f, g, h \) are smooth functions of two variables.

**Proof.** The proof is an application of the Inverse Function theorem. We have \( \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \), and

\[
\frac{\partial \mathbf{x}}{\partial u} = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right), \quad \frac{\partial \mathbf{x}}{\partial v} = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).
\]

Then,

\[
\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
x_u & y_u & z_u \\
x_v & y_v & z_v
\end{vmatrix} = \begin{vmatrix}
y_u & z_u \\
y_v & z_v
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
x_u & z_u \\
x_v & z_v
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
x_u & y_u \\
x_v & y_v
\end{vmatrix} \mathbf{k}.
\]

Since, by regularity, \( \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0 \) at \( (u_0, v_0) \), one of the components must be nonzero, say,

\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}
\end{vmatrix} \neq 0 \quad \text{at} \quad (u_0, v_0).
\]

Now, consider the map \( \Phi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by, \( \Phi(u, v) = (x(u, v), y(u, v)) \),

\[
\Phi : \begin{align*}
x &= x(u, v) \\
y &= y(u, v)
\end{align*}
\]
Φ has Jacobian matrix,

\[
D\Phi = \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{bmatrix}
\]

Hence \( \det D\Phi \neq 0 \) at \((u_0,v_0)\), i.e. \( D\Phi \) is nonsingular at \((u_0,v_0)\). By the IFT there exists a neighborhood \( V \) of \((u_0,v_0)\) such that \( W = \Phi(V) \) is open in \( \mathbb{R}^2 \) and \( \Phi : V \subset \mathbb{R}^2 \to W \subset \mathbb{R}^2 \) is a diffeomorphism, i.e., \( \Phi^{-1} : W \subset \mathbb{R}^2 \to V \subset \mathbb{R}^2 \) is smooth. In terms of components, \( \Phi^{-1}(x,y) = (u(x,y), v(x,y)) \),

\[
\Phi^{-1} : \begin{align*}
u &= u(x,y) \\
v &= v(x,y)
\end{align*}, (x,y) \in W
\]

Now, let \( f = z \circ \Phi^{-1}, f : W \subset \mathbb{R}^2 \to \mathbb{R} \),

\[
f(x,y) = z(\Phi^{-1}(x,y)) = z(u(x,y), v(x,y)).
\]

The graph of \( f \) is the set of points in \( \mathbb{R}^3 \),

\[
\text{graph } f = \{ (x,y,z) \in \mathbb{R}^3 : z = f(x,y), (x,y) \in W \}
\]

Claim. \( x(V) = \text{graph } f \):

\[f = z \circ \Phi^{-1} \Rightarrow z = f \circ \Phi, \text{ hence } z(u,v) = f(\Phi(u,v)) = f(x(u,v), y(u,v)). \]

Thus,

\[
\begin{align*}
x(u,v) &= (x(u,v), y(u,v), z(u,v)) \\
&= (x(u,v), y(u,v), f(x(u,v), y(u,v))) \in \text{graph } f.
\end{align*}
\]
Some Parameterized Surfaces

1. Graphs of functions of two variables, \( z = f(x, y) \), \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \) smooth function,

\[
\text{graph } f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in U \}.
\]

**Ex.** \( f(x, y) = x^2 + y^2 \) graph \( f \): all \((x, y, z)\) such that \( z = x^2 + y^2 \)

Standard parameterization: \( \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \), \( \mathbf{x}(u, v) = (u, v, f(u, v)) \),

\[
\begin{align*}
  x &= u \\
  y &= v \\
  z &= f(u, v)
\end{align*}
\]

\( \mathbf{x} \) is a regular surface (in fact, a coordinate patch). Check regularity condition:

\[
\frac{\partial \mathbf{x}}{\partial u} = \left( 1, 0, \frac{\partial f}{\partial u} \right), \quad \frac{\partial \mathbf{x}}{\partial v} = \left( 0, 1, \frac{\partial f}{\partial v} \right)
\]

\[
\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \left( -\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \right) \neq 0.
\]

\( \mathbf{x} \) is called the Monge patch associated to \( f \).

**Ex.** \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \), \( f(x, y) = \sqrt{1 - x^2 - y^2} \), \( U = \{(x, y) : x^2 + y^2 < 1\} \),

graph \( f : z = \sqrt{1 - x^2 - y^2} \), a hemisphere. Associated Monge patch:

\[
\begin{align*}
  x &= u \\
  y &= v \\
  z &= \sqrt{1 - u^2 - v^2}
\end{align*}
\]

i.e., \( \mathbf{x}(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \), \( \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \).

2. Geographical Coordinates on a sphere.

\[
S^2_R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2 \}
\]

\( \theta = \text{colatitude}, 0 \leq \theta \leq \pi \)

\( \phi = \text{longitude}, 0 \leq \phi \leq \pi \).
By spherical coordinates,
\[ x = R \sin \theta \cos \phi \]
\[ y = R \sin \theta \sin \phi \]
\[ z = R \cos \theta . \]

Let \( U = \{ (\theta, \phi) : 0 < \theta < \pi, \ 0 < \phi < 2\pi \} \). Define \( x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) by,
\[ x(\theta, \phi) = (R \sin \theta \cos \phi, \ R \sin \theta \sin \phi, \ R \cos \theta) . \]

\( x \) is clearly a smooth parameterized surface.

Coordinate curves:
\[ \theta \text{-curves: } \phi = \text{const} - \text{longitudes (meridians)} \]
\[ \phi \text{-curves: } \theta = \text{const} - \text{circles of latitude} \]

Coordinate vector fields:
\[
\frac{\partial x}{\partial \theta} = \left( R \cos \theta \cos \phi, \ R \cos \theta \sin \phi, -R \sin \theta \right) \\
\frac{\partial x}{\partial \phi} = \left( -R \sin \theta \sin \phi, \ R \sin \theta \cos \phi, \ 0 \right).
\]

E.g., at \((\theta, \phi) = (\pi/2, \ \pi/2)\),
\[
\frac{\partial x}{\partial \theta} = (0, 0, -R), \quad \frac{\partial x}{\partial \phi} = (-R, 0, 0)
\]

**Exercise 3.1.** Show by computation that \( \left| \frac{\partial x}{\partial \theta} \times \frac{\partial x}{\partial \phi} \right| = R^2 \sin \theta > 0, \ 0 < \theta < \pi \).

Hence, \( x \) is regular surface (in fact, a coordinate patch).


Consider a regular curve \( \sigma \) in the \( x-z \) plane, \( \sigma(t) = (r(t), 0, z(t)) \), i.e,
\[
\sigma : \ y = 0 \quad a < t < b .
\]

(Assume \( \sigma \) does not meet the \( z \)-axis.) Now rotate \( \sigma \) about the \( z \)-axis to generate a surface of revolution:
Parameterize as follows: Let $U = (a, b) \times (-\pi, \pi)$. Define $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by,

$$\mathbf{x}(t, \theta) = (r(t) \cos \theta, r(t) \sin \theta, z(t)).$$

This gives a parametric description of the surface of revolution; $t$ measures position along $\sigma$ and $\theta$ measure how far $\sigma$ has been rotated.

$t$-curves: $\theta = \text{const}$, longitudes (meridians)

$\theta$-curves: $t = \text{const}$, circles of latitude (parallels).

**Exercise 3.2.** Show that $\mathbf{x}$ as defined above is a regular surface (in fact a coordinate patch provided $\sigma$ is 1-1).

**Exercise 3.3.** Rotate the circle pictured below about the $z$-axis to obtain a torus.

Show that the torus is parameterized by the following map: $U = (0, 2\pi) \times (-\pi, \pi)$, $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$,

$$\mathbf{x}(t, \theta) = ((R + r \cos t) \cos \theta, (R + r \cos t) \sin \theta, r \sin t).$$

Hint: Parameterize the circle appropriately.
Reparameterizations.

**Def.** Let \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) be a regular surface. Let \( f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \) be a diffeomorphism. Then \( y = x \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3 \) is called a reparameterization.

![Diagram](image)

**Proposition.** Given a regular surface \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) and a diffeomorphism \( f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \), the map \( y = x \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3 \) is a regular surface.

**Proof.** \( y = x \circ f \) is smooth. Show that \( y \) satisfies the regularity condition. To do this we first show how the two sets of coordinate vectors \( \left\{ \frac{\partial x}{\partial u^i} \right\}, \left\{ \frac{\partial y}{\partial v^i} \right\} \) are related. Some notation:

\[
\begin{align*}
f &: V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \\
f(v_1, v_2) &= (u^1, u^2) = (f^1(v_1, v_2), f^2(v_1, v_2)) \\
f &: u^1 = f^1(v_1, v_2) \\
&u^2 = f^2(v_1, v_2) \\
Df &= \begin{bmatrix} \frac{\partial u^i}{\partial v^k} \end{bmatrix}_{2 \times 2}
\end{align*}
\]

Then,

\[
y(v^1, v^2) = x \circ f(v^1, v^2) = x(f(v^1, v^2)) = x\left(\begin{array}{c} f^1(v^1, v^2) \\ f^2(v^1, v^2) \end{array}\right)_{u^1}^{u^2}
\]

i.e.

\[
y = x(u^1, u^2) \quad \text{where} \quad f : \begin{align*}
u^1 &= f^1(v^1, v^2) \\
u^2 &= f^2(v^1, v^2)
\end{align*}
\]
Hence, by the chain rule,

\[
\frac{\partial y}{\partial v^k} = \frac{\partial x}{\partial u^1} \frac{\partial u^1}{\partial v^k} + \frac{\partial x}{\partial u^2} \frac{\partial u^2}{\partial v^k}, \quad k = 1, 2
\]

\[
= \sum_{j=1}^{2} \frac{\partial x}{\partial w^j} \frac{\partial w^j}{\partial v^k}
\]

\[
\frac{\partial y}{\partial v^k} = \sum_{j} \frac{\partial w^j}{\partial v^k} \frac{\partial x}{\partial w^j}, \quad k = 1, 2.
\]

Exercise 3.4 Show that,

\[
\frac{\partial y}{\partial v^1} \times \frac{\partial y}{\partial v^2} = \det Df \cdot \frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2}
\]

\[
= \frac{\partial(u^1, u^2)}{\partial(v^1, v^2)} \cdot \frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2} \quad (\neq 0)
\]

Hence, \( y \) is regular if \( x \) is.

**Terminology.** The reparameterization map \( f \) is called a **coordinate transformation**, and describes a change of coordinates on the surface.

**Surfaces.**

We now want to make the transition from the notion of a parameterized surface to that of a surface. Roughly speaking, a surface in \( \mathbb{R}^3 \) is a subset of \( \mathbb{R}^3 \) which is covered by coordinate patches. For example, the sphere

\[
S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}
\]

will, by our definition, be a surface. It can be covered by several coordinate patches – but not by a single coordinate patch.

Before giving the definition, we need to refine the notion of a coordinate patch a little. Consider a **coordinate patch**, i.e. a 1-1 regular parameterized surface, \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \). Then,

\[
x : U \to x(U)
\]

is a continuous, 1-1 and onto map. Hence we can consider the inverse,

\[
x^{-1} : x(U) \to U.
\]

The inverse need not be continuous.
$p, q$ close, as points in $\mathbb{R}^3$, but $x^{-1}(p), x^{-1}(q)$ are not close.

**Def.** A coordinate patch $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is *proper* provided the inverse $x^{-1} : x(U) \rightarrow U$ is continuous.

**Terminology:** proper patch = proper coordinate patch. Note: $x$ is proper iff $x : U \rightarrow x(U)$ is a homeomorphism.

**Subspace topology.** Any subset $M \subset \mathbb{R}^3$ of $\mathbb{R}^3$ inherits a natural topology - collection of open sets:

\[ W \subset M \text{ is open iff } W = U \cap M \]

for some open set $U$ in $\mathbb{R}^3$.

**Def.** A subset $M \subset \mathbb{R}^3$ is a *smooth surface* provided each point of $M$ is contained in a proper patch, i.e. provided for each $p \in M$ there exists a proper patch $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, such that (i) $p \in x(U)$, and (ii) $x(U)$ is an open subset of $M$.

Equivalently, $M$ is a smooth surface provided $M$ is *covered* by proper patches (i.e. there is a collection of proper patches whose images are open sets in $M$, and the union of which equals $M$).

**Ex:** Consider the sphere,

\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \]

In this example the sphere is covered by six proper patches: $z^+, z^-, y^+, y^-, x^+, x^-$, each a parameterized hemisphere.
\( z^+ \): upper hemisphere: \( z = \sqrt{1 - x^2 - y^2} \), with domain \( D : x^2 + y^2 < 1 \). Associated Monge patch: \( z^+ : U \to \mathbb{R}^3, U = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} \),
\[
\begin{align*}
x &= u \\
y &= v \\
z &= \sqrt{1 - u^2 - v^2},
\end{align*}
\]
i.e. \( z^+(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \).

**Claim.** \( z^+ \) is a proper patch in \( S^2 \).

1. \( z^+ \) is a coordinate patch (Monge patch)
2. \( z^+(U) \) is an open subset of \( S^2 \):
\[
z^+(U) = \{(x, y, z) \in S^2 : z > 0\}
= S^2 \cap \{z > 0\}.
\]
3. \((z^+)^{-1} : z^+(U) \to U \) is continuous:

\[
(z^+)^{-1}(x, y, z) = (x, y) \quad \text{projection onto the first two coordinates, which is continuous.}
\]

\( z^- \): lower hemisphere: \( z = -\sqrt{1 - x^2 - y^2} \); associated Monge patch: \( z^- : U \to \mathbb{R}^3, z^-(u, v) = (u, v, -\sqrt{1 - u^2 - v^2}) \).

Other hemispheres.

\[
\begin{align*}
y^+ &= \text{Monge patch associated with hemisphere } S^2 \cap \{y > 0\} \quad (y = \sqrt{1 - x^2 - z^2}) \\
y^- &= \ldots \quad S^2 \cap \{y < 0\} \\
x^+ &= \ldots \quad S^2 \cap \{x > 0\} \\
x^- &= \ldots \quad S^2 \cap \{x < 0\}.
\end{align*}
\]
**Proposition.** (Smooth overlap property) Let \( M \) be a surface. Let \( x : U \to \mathbb{R}^3 \) and \( y : V \to \mathbb{R}^3 \) be two proper patches in \( M \) which overlap, \( W := x(U) \cap y(V) \neq \emptyset \). Then,

\[
y^{-1} \circ x : x^{-1}(W) \subset \mathbb{R}^2 \to y^{-1}(W) \subset \mathbb{R}^2
\]

is a diffeomorphism.

**Proof.** Inverse function theorem! (See, e.g., DoCarmo, p. 70, Prop. 1.)

**Ex.** In sphere example consider the overlapping patches \( z^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \), \( z^+(u,v) = (u,v,\sqrt{1-u^2-v^2}) \) and \( y^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \), \( y^+(u,v) = (u,\sqrt{1-u^2-v^2},v) \). Observe, \( y^+(U) = S^2 \cap \{y > 0\} \) and \( z^+(U) = S^2 \cap \{z > 0\} \). Then,

\[
W := y^+(U) \cap z^+(U) = S^2 \cap \{y > 0\} \cap \{z > 0\} \neq \emptyset.
\]

Consider \((z^+)^{-1} \circ y^+ : (y^+)^{-1}(W) \to (z^+)^{-1}(W)\). Note, \((y^+)^{-1}(W) = \text{half-disk} = U \cap \{v > 0\}\). Now,

\[
y^+(u,v) = (u,\sqrt{1-u^2-v^2},v) \quad \quad (z^+)^{-1}(x,y,z) = (x,y)
\]

and hence

\[
(z^+)^{-1} \circ y^+(u,v) = (z^+)^{-1}(y^+(u,v)) = (z^+)^{-1}(u,\sqrt{1-u^2-v^2},v) = (u,\sqrt{1-u^2-v^2}),
\]

49
which is smooth on $U \cap \{v > 0\}$!

**Remark.** In above proposition, let $g = y^{-1} \circ x$. Then, $x = y \circ g$ on $x^{-1}(W)$. Thus, $x|_{x^{-1}(W)}$ is a reparameterization of $y|_{y^{-1}(W)}$. Also, $y = x \circ g^{-1}$, so $y|_{y^{-1}(W)}$ is also a reparameterization of $x|_{x^{-1}(W)}$.

The smooth overlap property is the key ingredient used to generalize the notion of surfaces in $\mathbb{R}^3$ to differentiable manifolds. That this property holds for surfaces is important. For example, it is used to show that certain properties which are defined in terms of coordinate charts (proper charts), don’t really depend on the specific coordinate charts chosen. We give an illustration.

Consider a function $f : M \to \mathbb{R}$, where $M$ is a surface. What does it mean for $f$ to be smooth?

**Def.** $f : M \to \mathbb{R}$ is smooth provided for each $p \in M$ there exists a proper patch $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ in $M$ containing $p$ ($p \in x(U) \subset M$) such that $f \circ x : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth.

\[ \hat{f} = f \circ x \text{ expressed in coordinates.} \]

Equivalently, $f$ is smooth provided there exists a collection of charts covering $M$ such that each coordinate expression $\hat{f}$ is smooth.

This definition of smoothness does not depend on the particular choice of proper charts covering $M$.

**Exercise 3.5.** If $x$ and $y$ are any two overlapping proper patches in $M$ then on the overlap, $f \circ x$ is smooth iff $f \circ y$. (Hint: Smooth overlap property.)

The following proposition identifies a large and important class of surfaces.

**Proposition.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Consider the level set

\[ M = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\} \]

If $\nabla f = (f_x, f_y, f_z) \neq 0$ at each point of $M$ then $M$ is a surface.
**Ex.** The sphere.

\[ S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \} = \{ (x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0 \}, \]

where \( f(x, y, z) = x^2 + y^2 + z^2 - 1 \). Now,

\[ \nabla f = (2x, 2y, 2z), \]

and so \( \nabla f \neq 0 \) except at \((x, y, z) = (0, 0, 0) \notin S^2 \). Hence, \( \nabla f \neq 0 \) at each point of \( S^2 \). Therefore \( S^2 \) is a surface.

**Ex.** Double Cone.

\[ M = \{ (x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \} = \{ (x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0 \}, \]

where \( f(x, y, z) = x^2 + y^2 - z^2 \). Then,

\[ \nabla f = (2x, 2y, 2z) \neq 0 \]

except at \((0, 0, 0) \). But this time \((0, 0, 0) \in M \). So the proposition doesn’t guarantee that \( M \) is a surface, and in fact it is not. The origin is not contained in a proper patch. In general, however, away from points where the gradient vanishes we do get a surface.

**Remarks**

1. \( Df = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \sim (f_x, f_y, f_z) = \nabla f. \) I.e. the gradient of \( f \) essentially corresponds to the Jacobian of \( f \).

2. \( M = f^{-1}(0) \). For this reason, the proposition is often referred to as the *inverse image theorem*.

*Sketch of proof.* Uses the inverse function theorem (actually the implicit function theorem, which is a consequence of the IFT; see DoCarmo, p. 59, Prop. 2 for complete details).

We want to show \( M \) is covered by proper patches. Choose \( p_0 = (x_0, y_0, z_0) \in M \); \( \nabla f|_{p_0} \neq 0 \). Suppose then that \( \frac{\partial f}{\partial z}(p_0) \neq 0 \).

\[ M : \quad f(x, y, z) = 0 \quad (\ast) \]
By the IFT, near \( p_0 = (x_0, y_0, z_0) \), (*) can be solved smoothly for \( z \) in terms of \( x \) and \( y \),

\[
z = h(x, y),
\]
i.e., there exists a nbd \( U \) of \((x_0, y_0)\) and a smooth function \( h : U \subset \mathbb{R}^2 \to \mathbb{R} \) such that \((x, y, h(x, y))\) satisfies (*) for all \((x, y)\) in \( U \),

\[
f(x, y, h(x, y)) = 0 \quad \forall (x, y) \in U.
\]

Hence,

\[
(x, y, h(x, y)) \in M \quad \text{for all } (x, y) \in U.
\]

Now consider the Monge patch associated to \( h \), \( x : U \to \mathbb{R}^3 \),

\[
x(u, v) = (u, v, h(u, v)).
\]

Then \( x \) is a proper patch in \( M \) which contains \( p_0 \).

**Tangent Vectors to a Surface.**

**Def.** Let \( M \) be a surface, and \( p \in M \). \( X \) is a tangent vector to \( M \) at \( p \) provided \( X \) is the velocity vector at \( p \) of some smooth curve \( \sigma \) which lies in \( M \), i.e. provided there exists a smooth curve \( \sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3 \) such that \( \sigma(0) = p \) and \( \sigma'(0) = X \).

**Remark.** This definition is independent of coordinate patches – coordinate free concept. But for computational purposes it's convenient to introduce coordinates.
Let \( x : U \to M \subset \mathbb{R}^3 \) be a proper patch in \( M \) which contains \( p, \ p = x(u_0, v_0) \).

Observe: \( \frac{\partial x}{\partial u}(u_0, v_0) \) and \( \frac{\partial x}{\partial v}(u_0, v_0) \) are tangent vectors to \( M \) at \( p = x(u_0, v_0) \), according to the definition:

\[
\begin{align*}
x_u(u_0, v_0) &= \text{velocity vector to } u \to x(u, v_0) \text{ at } x(u_0, v_0), \text{ and}, \\
x_v(u_0, v_0) &= \text{velocity vector to } v \to x(u_0, v) \text{ at } x(u_0, v_0)
\end{align*}
\]

Notation/Terminology.

\( T_p\mathbb{R}^3 := \) tangent space of \( \mathbb{R}^3 \) at \( p \)

\[= \text{set of all vectors in } \mathbb{R}^3 \text{ based at } p.\]

\( T_p\mathbb{R}^3 \) is a 3-dimensional vector space. For \( M \) a surface, \( p \in M \),

\( T_pM := \) tangent space of \( M \) at \( p \)

\[= \text{set of all tangent vectors to } M \text{ at } p.\]

In the following proposition we show that \( T_pM \) is a 2-dimensional subspace of \( T_p\mathbb{R}^3 \) spanned by \( x_u(u_0, v_0) \) and \( x_v(u_0, v_0) \).

**Proposition.** Let \( M \) be a surface, \( p \in M \). Let \( x : U \to M \subset \mathbb{R}^3 \) be a proper patch in \( M \) containing \( p, \ p = x(u_0, v_0) \). Then \( T_pM \) is a 2-dimensional vector space, in fact it is the 2-dimensional vector subspace of \( T_p\mathbb{R}^3 \) spanned by \( \{ x_u(u_0, v_0), x_v(u_0, v_0) \} \),

\[
T_pM = \text{span}\{ x_u(u_0, v_0), x_v(u_0, v_0) \} = \{ Ax_u(u_0, v_0) + Bx_v(u_0, v_0) : A, B \in \mathbb{R} \}
\]

**Proof.** \( T_pM \subset \text{span}\{ x_u(u_0, v_0), x_v(u_0, v_0) \} \): Let \( X \in T_pM \). Then there exists a smooth curve \( \sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3 \) such that \( \sigma(0) = p \) and \( \sigma'(0) = X \). Without loss of generality, by taking \( \epsilon \) sufficiently small, \( \sigma \subset x(U) \).

Key observation: \( \sigma \) can be represented in a certain manner in terms of coordinates; we will use this representation over and over.
Let $\hat{\sigma} = x^{-1} \circ \sigma$:

$\hat{\sigma} : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$, and in terms of components, $\hat{\sigma}(t) = (u(t), v(t)), \ t \in (-\epsilon, \epsilon)$,

$\hat{\sigma} : \begin{cases} 
    u = u(t) \\
    v = v(t)
\end{cases} \ - \epsilon < t < \epsilon.$

$\hat{\sigma}(0) = x^{-1}(\sigma(0)) = x^{-1}(p) = (u_0, v_0)$. Using the IFT, it can be shown that $\hat{\sigma}$ is a smooth curve in $\mathbb{R}^2$, that is, $u = u(t)$ and $v = v(t)$ are smooth functions.

Now, $\hat{\sigma} = x^{-1} \circ \sigma \Rightarrow \sigma = x \circ \hat{\sigma} \Rightarrow \sigma(t) = x(\hat{\sigma}(t))$, i.e. $\sigma(t) = x(u(t), v(t)), \ t \in (-\epsilon, \epsilon)$.

**Remark.** $\hat{\sigma}$ is the coordinate representation of $\sigma$; $\hat{\sigma}$ is just $\sigma$ expressed in coordinates.

Returning to the proof, by the chain rule,

$$\frac{d\sigma}{dt} = \frac{\partial x}{\partial u} \frac{du}{dt} + \frac{\partial x}{\partial v} \frac{dv}{dt},$$

or, rewriting slightly,

$$\sigma'(t) = u'(t)x_u(u(t), v(t)) + v'(t)x_v(u(t), v(t)),$$

and setting $t = 0$, we obtain,

$$\sigma'(0) = u'(0)x_u(u_0, v_0) + v'(0)x_v(u_0, v_0),$$

and thus,

$$X = Ax_u(u_0, v_0) + Bx_v(u_0, v_0),$$

where $A = u'(0), B = v'(0)$, as was to be shown.
span \( \{ \mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0) \} \subset T_p M \): Must show that a vector of the form,
\[ A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0) \]
for any \( A, B \in \mathbb{R} \), is the velocity vector of a curve \( \sigma \) in \( M \) passing through \( p \).

**Exercise 3.6.** Show this. Hint: Let \( \sigma = \mathbf{x} \circ \hat{\sigma} \) where \( \hat{\sigma} \) is the parameterized line, \( \hat{\sigma}(t) = (At + u_0, Bt + v_0) \). Then, \( \sigma(t) = \mathbf{x}(\hat{\sigma}(t)) = \mathbf{x}(At + u_0, Bt + v_0) \), and apply the chain rule.

**Tangent plane to** \( M \) **at** \( p \):

![Tangent plane diagram]

Let \( \mathbf{x} \) be a proper patch in \( M \) containing \( p = \mathbf{x}(u_0, v_0) \). Then the tangent plane to \( M \) at \( p = \) plane through \( p \) spanned by \( \mathbf{x}_u(u_0, v_0) \) and \( \mathbf{x}_v(u_0, v_0) \) = plane through \( p \) perpendicular to \( N = \mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) \).

**Equation of tangent plane:**
\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 , \]
where \( N = (a, b, c) \) and \( p = \mathbf{x}(u_0, v_0) = (x_0, y_0, z_0) \).

**Unit normal vector field** associated to a proper patch \( \mathbf{x} : U \to M \subset \mathbb{R}^3 \):

\[ \mathbf{n} = \mathbf{n}(u, v), \mathbf{n}(u, v) \in T_{\mathbf{x}(u, v)} \mathbb{R}^3, \mathbf{n}(u, v) \perp M . \]

**Remark:** The unit normal field is unique up to sign.
Ex. Compute the unit normal field to the surface $z = x^2 + y^2$ with respect to the associated Monge patch.

We have, $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (u, v, u^2 + v^2)$, $\mathbf{x}_u = (1, 0, 2u)$, $\mathbf{x}_v = (0, 1, 2v)$, and so,

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1).$$

Hence,

$$\mathbf{n} = \frac{(-2u, -2v, 1)}{|(-2u, -2v, 1)|},$$

$$\mathbf{n}(u, v) = \frac{(-2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}.$$

Exercise 3.7 Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let $M=\text{graph } f$. $M$ is a smooth surface covered by a single patch - the associated Monge patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{x}(u, v) = (u, v, f(u, v))$. Show that the unit normal vector field to $M$ wrt $\mathbf{x}$ is given by,

$$\mathbf{n} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

where $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$.

Some Tensor Analysis

Consider overlapping patches,

$$\mathbf{x} : U \to M \subset \mathbb{R}^3 \quad \mathbf{x} = \mathbf{x}(u^1, u^2)$$

$$\mathbf{y} : V \to M \subset \mathbb{R}^3 \quad \mathbf{y} = \mathbf{y}(v^1, v^2).$$
Let \( p \in x(U) \cap y(V). \) \( T_p M \) has the two different bases at \( p: \{x_1, x_2\}, \{y_1, y_2\} \) where we are using the shorthand, \( x_1 = \frac{\partial x}{\partial u_1}, x_2 = \frac{\partial x}{\partial u_2}, y_1 = \frac{\partial y}{\partial v_1}, y_2 = \frac{\partial y}{\partial v_2}. \)

Let \( X \in T_p M. \) \( X \) can be expressed in two different ways,

\[
X = \sum_{i=1}^{2} X^i x_i = \sum_{k=1}^{2} \tilde{X}^k y_k
\]

Classical tensor analysis is concerned with questions like the following: How are the components \( X^i \) and \( \tilde{X}^k \) with respect to the two different bases related? We now consider this.

By the smooth overlap property, \( f = y^{-1} \circ x \) is a diffeomorphism on the overlap. We have, \( f : x^{-1}(W) \rightarrow y^{-1}(W), \) where \( W = x(U) \cap x(V), \) and \( f(u^1, u^2) = (v^1, v^2) = (f^1(u^1, u^2), f^2(u^1, u^2)), \)

\[
f: v^1 = f^1(u^1, u^2), \quad v^2 = f^2(u^1, u^2),
\]

i.e., \( f \) is the change of coordinates map; \( v^1 \) and \( v^2 \) depend smoothly on \( u^1 \) and \( u^2. \) On the overlap we have, \( x = y \circ f, \) and hence, \( x(u^1, u^2) = y(v^1, v^2), \) where \( v^1, v^2 \) depend on \( u^1, u^2 \) as above.

**Exercise 3.8**

(1) Use the chain rule to show,

\[
x_i = \sum_{k} \frac{\partial v^k}{\partial u^i} y_k
\]

(Note: This is essentially the same as the computation on p. 46, but with the role of \( x \) and \( y \) reversed from that here).

(2) Use (1) to show,

\[
\tilde{X}^k = \sum_{i} \frac{\partial v^k}{\partial u^i} X^i, \quad k = 1, 2.
\]

(3) Show (2) implies

\[
\begin{bmatrix}
\tilde{X}^1 \\
\tilde{X}^2
\end{bmatrix} = \underbrace{\left[ \frac{\partial v^k}{\partial u^i} \right]}_{Df} \begin{bmatrix}
X^1 \\
X^2
\end{bmatrix}
\]

57
4 The First Fundamental Form (Induced Metric)

We begin with some definitions from linear algebra.

**Def.** Let $V$ be a vector space (over $\mathbb{R}$). A *bilinear form* on $V$ is a map of the form $B : V \times V \to \mathbb{R}$ which is *bilinear*, i.e. linear in each “slot”,

\[
B(aX + bY, Z) = aB(X, Z) + bB(Y, Z), \\
B(X, cY + dZ) = cB(X, Y) + dB(X, Z).
\]

A bilinear form $B$ is *symmetric* provided $B(X, Y) = B(Y, X)$ for all $X, Y \in V$.

**Def.** Let $V$ be a vector space. An *inner product* on $V$ is a bilinear form $\langle \ , \rangle : V \times V \to \mathbb{R}$ which is symmetric and positive definite.

1. *bilinear*: linear in each slot,
2. *symmetric*: $\langle X, Y \rangle = \langle Y, X \rangle$ for all $X, Y$.
3. *positive definite*: $\langle X, X \rangle \geq 0 \ \forall X$, and $= 0$ iff $X = 0$.

**Ex.** $\langle \ , \rangle : T_p\mathbb{R}^3 \times T_p\mathbb{R}^3 \to \mathbb{R}$,

\[
\langle X, Y \rangle = X \cdot Y \quad \text{(usual Euclidean dot product).}
\]

**Exercise 4.1** Verify carefully that the Euclidean dot product is indeed an inner product.

**Def.** Let $M$ be a surface. A *metric* on $M$ is an assignment, to each point $p \in M$, of an inner product $\langle \ , \rangle : T_pM \times T_pM \to \mathbb{R}$.

Because our surfaces sit in Euclidean space, they inherit in a natural way, a metric called the *induced metric* or *first fundamental form*.

**Def.** Let $M$ be a surface. The *induced metric* (or *first fundamental form*) of $M$ is the assignment to each $p \in M$ of the inner product,

\[
\langle \ , \rangle : T_pM \times T_pM \to \mathbb{R}, \\
\langle X, Y \rangle = X \cdot Y \quad \text{(ordinary scalar product of $X$ and $Y$ viewed as vectors in $\mathbb{R}^3$ at $p$)}
\]
I.e., the induced metric is just the Euclidean dot product, restricted to the tangent spaces of $M$.

We will only consider surfaces in the induced metric. Just as the Euclidean dot product contains all geometric information about $\mathbb{R}^3$, the induced metric contains all geometric information about $M$, as we shall see.

The Metric in a Coordinate Patch.

Let $x : U \subset \mathbb{R}^2 \rightarrow M \subset \mathbb{R}^3$ be a proper patch in $M$. Let $p \in x(U)$ be any point in $x(U)$, $p = x(u^1, u^2)$, and let $X, Y \in T_pM$. Then,

\[
X = X^1 \frac{\partial x}{\partial u^1} + X^2 \frac{\partial x}{\partial u^2} = X^1 x_1 + X^2 x_2,
\]

and similarly,

\[
Y = \sum_j Y^j x_j.
\]

Then,

\[
\langle X, Y \rangle = \langle \sum_i X^i x_i, \sum_j Y^j x_j \rangle = \sum_{i,j} X^i Y^j \langle x_i, x_j \rangle.
\]

The metric components are the functions $g_{ij} : U \rightarrow \mathbb{R}$, $1 \leq i, j \leq 2$, defined by

\[
g_{ij} = \langle x_i, x_j \rangle, \quad g_{ij} = g_{ij}(u^1, u^2).
\]

Thus, in coordinates,

\[
\langle X, Y \rangle = \sum_{i,j=1}^2 g_{ij} X^i Y^j.
\]
Note that the metric in $\mathbf{x}(U)$ is completely determined by the $g_{ij}$'s. The metric components may be displayed by a $2 \times 2$ matrix,

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

Note: $g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{x}_j, \mathbf{x}_i \rangle = g_{ji}$. Hence, the matrix of metric components is symmetric; and there are only three distinct components,

$$g_{11} = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle, \quad g_{12} = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = g_{21}, \quad g_{22} = \langle \mathbf{x}_2, \mathbf{x}_2 \rangle$$

Notation:

1. Gauss: $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$.

2. $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. Then one writes:

   $$g_{uu} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad g_{uv} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad g_{vv} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

**Ex.** Consider the parameterization of $S^2_r$ in terms of geographic coordinates,

$$\mathbf{x}(\theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta),$$

$0 < \theta < \pi, \quad 0 < \phi < 2\pi$. We compute the metric components in these coordinates. We have,

$$\mathbf{x}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r(\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),$$

$$\mathbf{x}_\phi = r(-\sin \theta \sin \phi, \sin \theta \cos \phi, 0),$$

$$g_{\theta\theta} = \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle$$

$$= r^2[\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta]$$

$$= r^2(\cos^2 \theta + \sin^2 \theta) = r^2,$$

$$g_{\theta\phi} = \langle \mathbf{x}_\theta, \mathbf{x}_\phi \rangle$$

$$= r^2[-\cos \theta \cos \phi \sin \theta \sin \phi + \cos \theta \sin \phi \sin \theta \cos \phi]$$

$$= 0 \quad \text{(geometric significance?)},$$

$$g_{\phi\phi} = r^2[\sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi]$$

$$= r^2 \sin^2 \theta.$$
Thus,

\[
[g_{ij}] = \begin{bmatrix}
g_{\theta\theta} & g_{\theta\phi} \\
g_{\theta\phi} & g_{\phi\phi}
\end{bmatrix} = \begin{bmatrix}
r^2 & 0 \\
0 & r^2\sin^2\theta
\end{bmatrix}
\]

Length and Angle Measurement in \( M \).

Let \( \sigma: [a, b] \to M \subset \mathbb{R}^3 \) be a smooth curve in a surface \( M \). Viewed as a curve in \( \mathbb{R}^3 \), \( \sigma(t) = (x(t), y(t), z(t)) \), we can compute its length by the formula,

\[
\text{Length of } \sigma = \int_{a}^{b} \left| \frac{d\sigma}{dt} \right| dt = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt
\]

But this formula does not make sense to creatures living in the surface: \( x, y, z \) are Euclidean space coordinates. Creatures living in the surface must use surface coordinates – i.e., we must express \( \sigma \) in terms of surface coordinates.

Let \( x: U \to M \subset \mathbb{R}^3 \) be a proper patch in \( M \) and suppose \( \sigma \) is contained in this patch, \( \sigma \subset x(U) \):

We express \( \sigma \) in terms of coordinates: \( \hat{\sigma} = x^{-1} \circ \sigma : [a, b] \to U \subset \mathbb{R}^2 \), \( \hat{\sigma}(t) = (u'(t), u^2(t)) \), i.e,

\[
\hat{\sigma} : \begin{align*}
   u^1 &= u^1(t) \\
   u^2 &= u^2(t)
\end{align*}, \quad a \leq t \leq b.
\]

Then, \( \sigma = x \circ \hat{\sigma} \), i.e., \( \sigma(t) = x(\hat{\sigma}(t)) \), hence,

\[
\sigma(t) = x(u^1(t), u^2(t)).
\]

By the chain rule,

\[
\frac{d\sigma}{dt} = \frac{\partial x}{\partial u^1} \frac{du^1}{dt} + \frac{\partial x}{\partial u^2} \frac{du^2}{dt} = \frac{du^1}{dt} x_1 + \frac{du^2}{dt} x_2,
\]
or,
\[
\frac{d\sigma}{dt} = \sum_i \frac{du^i}{dt} x_i.
\]

This shows that \(\frac{du^i}{dt}, i = 1, 2\), are the components of the velocity vector with respect to the basis \(\{x_1, x_2\}\).

Computing the dot product,
\[
\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \rangle = \left(\sum_i \frac{du^i}{dt} x_i; \sum_j \frac{du^j}{dt} x_j\right)
\]
\[
= \sum_{i,j} \frac{du^i}{dt} \frac{du^j}{dt} \langle x_i, x_j \rangle
\]
\[
= \sum_{i,j=1}^2 g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}.
\]

Hence, for the speed in surface coordinates, we have,
\[
\left| \frac{d\sigma}{dt} \right| = \sqrt{\sum_{i,j=1}^2 g_{ij} \left( \frac{du^i}{dt} \right)^2 \left( \frac{du^j}{dt} \right)^2}.
\]

For length, we then have,
\[
\text{Length of } \sigma = \int_a^b \sqrt{\sum_{i,j} g_{ij} \left( \frac{du^i}{dt} \right) \left( \frac{du^j}{dt} \right)} dt
\]
\[
= \int_a^b \sqrt{g_{11} \left( \frac{du^1}{dt} \right)^2 + 2g_{12} \frac{du^1}{dt} \frac{du^2}{dt} + g_{22} \left( \frac{du^2}{dt} \right)^2} dt.
\]

Arc length. Let \(s\) denote arc length along \(\sigma\), \(s\) can be computed in terms of \(t\) as follows. \(s = s(t), a \leq t \leq b\),

\[
s(t) = \text{length of } \sigma \text{ from time } a \text{ to time } t
\]
\[
= \int_a^t \sqrt{\sum_{i,j} g_{ij} \left( \frac{du^i}{dt} \right) \left( \frac{du^j}{dt} \right)} dt.
\]
Arc length element:

\[ \frac{ds}{dt} = \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}} \]

In terms of differentials,

\[ ds = \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}} \, dt \]

\[ ds^2 = \left( \sum_{i,j} g_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \right) \, dt^2 \]

\[ = \sum_{i,j} g_{ij} \left( \frac{du^i}{dt} \right) \left( \frac{du^j}{dt} \right) \]

\[ ds^2 = \sum_{i,j=1}^2 g_{ij} du^i du^j . \]

Heuristics: element of arc length

\[ ds^2 = \sum_{i,j} g_{ij} du^i du^j \]

Traditionally, one displays the metric (or, metric components \( g_{ij} \)) by writing out the arc length element.

Notations:

\[ ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1 du^2 + g_{22}(du^2)^2 \]

\[ ds^2 = g_{uu}du^2 + 2g_{uv}dudv + g_{vv}dv^2 \]

\((u^1 = u, u^2 = v)\)

\[ ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad \text{(Gauss)}. \]
Remark. These expressions for arc length element of a surface $M$ generalize the expression for the arc length element in the Euclidean $u-v$ plane we encounter in calculus,

$$ds^2 = du^2 + dv^2$$

(i.e. $g_{uu} = 1$, $g_{uv} = 0$, $g_{vv} = 1$).

Ex. Write out the arc length element for the sphere $S^2$ parameterized in terms of geographic coordinates,

$$\mathbf{x}(\theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta).$$

We previously computed the $g_{ij}$’s,

$$[g_{ij}] = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix}$$

i.e., $g_{\theta\theta} = r^2$, $g_{\theta\phi} = g_{\phi\theta} = 0$, $g_{\phi\phi} = r^2 \sin^2 \theta$. So,

$$ds^2 = g_{\theta\theta} d\theta^2 + 2g_{\theta\phi} d\theta d\phi + g_{\phi\phi} d\phi^2$$

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$ 

But this expression is familiar from calculus as the arc length element which can be derived from heuristic geometric considerations.

$$ds^2 = dl_1^2 + dl_2^2.$$ 

$$dl_1 = rd\theta, \quad dl_2 = r \sin \theta d\phi$$

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$ 

Exercise 4.2. Consider the parameterization of the $x$-$y$ plane in terms of polar coordinates,

$$x = r \cos \theta$$

$$\mathbf{x}: \quad y = r \sin \theta \quad , 0 < r < \infty, \quad 0 < \theta < 2\pi,$$

$$z = 0$$

i.e., $\mathbf{x}(r, \theta) = (r \cos \theta, r \sin \theta, 0)$, $0 < r < \infty$, $0 < \theta < 2\pi$. Compute the $g_{ij}$’s with respect to these coordinates. Show that the arc length element in this case is:

$$ds^2 = dr^2 + r^2 d\theta^2.$$
Angle Measurement.

\[ X = \sum_i X^i x_i, \]
\[ Y = \sum_j Y^j x_j, \]

\[ \cos \theta = \frac{\langle X, Y \rangle}{|X||Y|} \]
\[ = \frac{\sum g_{ij} X^i Y^j}{\sqrt{\sum g_{ij} X^i X^j} \sqrt{\sum g_{ij} Y^i Y^j}}. \]

**Ex.** Determine the angle between the coordinate vectors \( x_1 = \frac{\partial x}{\partial u^1} \) and \( x_2 = \frac{\partial x}{\partial u^2} \) in terms of the \( g_{ij} \)'s.

\[ \cos \theta = \frac{\langle x_1, x_2 \rangle}{|x_1| |x_2|} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}} \]

\(|x_1| = \sqrt{\langle x_1, x_1 \rangle} = \sqrt{g_{11}}, \text{ etc.}\)

**The Metric is intrinsic:**

This discussion is somewhat heuristic. We claim that the \( g_{ij} \)'s are *intrinsic*, i.e. in principle they can be determined by measurements made in the surface.

Let \( x : U \to M \) be a proper patch in \( M \); \( x = x(u^1, u^2) = x(u, v) \) (i.e., \( u^1 = u \), \( u^2 = v \)). Consider the coordinate curve \( u \longrightarrow x(u, v_0) \) passing through \( x(u_0, v_0) \).
Let \( s = s(u) \) be the arc length function along \( \sigma \), i.e.,

\[
\begin{align*}
    s(u) &= \text{length of } \sigma \text{ from } u_0 \text{ to } u \\
    &= \int_{u_0}^{u} \left| \frac{\partial \mathbf{x}}{\partial u} \right| \, du \\
    &= \int_{u_0}^{u} \sqrt{g_{uu}} \, du \quad \left( \left| \frac{\partial \mathbf{x}}{\partial u} \right| = \sqrt{g_{uu}} \right).
\end{align*}
\]

By making length measurements in the surface the function \( s = s(u) \) is known. Then by calculus, the derivative,

\[
\frac{ds}{du} = \sqrt{g_{uu}}.
\]

is known. Therefore \( g_{11} = g_{uu} \), and similarly \( g_{22} = g_{vv} \), can in principal be determined by measurements made in the surface.

The metric component \( g_{12} \) can then be determined by angle measurement,

\[
g_{12} = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = |\mathbf{x}_1||\mathbf{x}_2|\cos \theta
\]

\[
= \sqrt{g_{11}}\sqrt{g_{22}} \cdot \cos(\text{angle between } \mathbf{x}_1, \mathbf{x}_2).
\]

Hence \( g_{12} \) is also measurable. Thus all metric components can be determined by measurements made in the surface, i.e.

*the metric components and all quantities determined from them are intrinsic.*

**Surface Area.**

Let \( M \) be a surface, and let \( \mathbf{x} : U \to M \) be a proper patch in \( M \). Consider a bounded region \( \mathcal{R} \) contained in \( \mathbf{x}(U) \); we have \( \mathcal{R} = \mathbf{x}(W) \) for some bounded region \( W \) in \( U \):

We want to obtain (i.e. heuristically motivate) a formula for the area of \( \mathcal{R} = \mathbf{x}(W) \). Restrict attention to \( \mathcal{R} = \mathbf{x}(W) \); partition \( W \) into small rectangles:
Let $\Delta S = \text{area of the small patch corresponding to the coordinate rectangle}$. Then,

$$\Delta S \approx \text{area of the parallelogram spanned by } \vec{PQ} \text{ and } \vec{PR},$$

$$\Delta S \approx | \vec{PQ} \times \vec{PR} | .$$

But,

$$\vec{PQ} = \mathbf{x}(u + \Delta u, v) - \mathbf{x}(u, v) \approx \frac{\partial \mathbf{x}}{\partial u} \Delta u ,$$

$$\vec{PR} = \mathbf{x}(u, v + \Delta v) - \mathbf{x}(u, v) \approx \frac{\partial \mathbf{x}}{\partial v} \Delta v ,$$

and thus,

$$\Delta S \approx \left| \frac{\partial \mathbf{x}}{\partial u} \Delta u \times \frac{\partial \mathbf{x}}{\partial v} \Delta v \right|$$

$$\approx \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \Delta u \Delta v .$$

The smaller the increments $\Delta u$ and $\Delta v$, the better the approximation.

$$dS = \text{the area element of the surface corresponding to the coordinate increments } du, dv ,$$

$$dS = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du \, dv .$$

To obtain the total area of $\mathcal{R}$, we must sum up all these area elements - but the summing up process is integration:

$$\text{Area of } \mathcal{R} = \iint dS ,$$

$$\text{Area of } \mathcal{R} = \iint_{W} \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du \, dv ,$$

67
where $\mathcal{R} = \mathbf{x}(W)$.

This is a perfectly reasonable formula for computing surface area - but not for 2-dimensional creatures living in the surface. It involves the cross product which is an $\mathbb{R}^3$ concept. We now show how this area formula can be expressed in an intrinsic way (i.e. involving the $g_{ij}$’s).

Using generic notation, $u^1 = u$, $u^2 = v$, $\mathbf{x} = \mathbf{x}(u^1, u^2)$ we write,

$$\text{Area of } \mathcal{R} = \iint_W \left| \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \right| du^1 du^2$$

$$= \iint_W |\mathbf{x}_1 \times \mathbf{x}_2| du^1 du^2$$

Now introduce the notation,

$$g = \det[g_{ij}], \quad g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle .$$

**Lemma.** $g = |\mathbf{x}_1 \times \mathbf{x}_2|^2$

**Proof.** Recall the vector identity,

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 .$$

Hence,

$$|\mathbf{x}_1 \times \mathbf{x}_2|^2 = |\mathbf{x}_1|^2 |\mathbf{x}_2|^2 - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2$$

$$= \langle \mathbf{x}_1, \mathbf{x}_1 \rangle \langle \mathbf{x}_2, \mathbf{x}_2 \rangle - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2$$

$$= g_{11}g_{22} - g_{12}^2 = g,$$

$$g = \det[g_{ij}] = \det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = g_{11}g_{22} - g_{12}^2 ,$$

where we have used $g_{21} = g_{12}$. Thus, the surface area formula may be expressed as,

$$\text{Area of } \mathcal{R} = \iint_W \sqrt{g} du^1 du^2 \quad (\mathcal{R} = \mathbf{x}(W))$$

$$= \iint_W dS ,$$

where,

$$dS = \sqrt{g} du^1 du^2 .$$

68
Ex. Compute the area of the sphere of radius $r$.

$$S^2_r : x^2 + y^2 + z^2 = r^2.$$ 

Parameterize with respect to geographical coordinates, $\mathbf{x} : U \rightarrow S^2_r$,

$$\mathbf{x}(\theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta),$$

$U : 0 < \theta < \pi, 0 < \phi < 2\pi$.

We have,

$$\text{Area of } S^2_r = \int \int_U dS, \quad \text{where } dS = \sqrt{g} d\theta d\phi.$$ 

Now,

$$g = \det[g_{ij}] = \det \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix}$$

$$= \det \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix}$$

$$g = r^4 \sin^2 \theta$$

Thus,

$$dS = \sqrt{r^4 \sin^2 \theta} d\theta d\phi = r^2 \sin \theta d\theta d\phi$$

Remark: This expression for the surface area element of a sphere is familiar from calculus or physics where it is usually derived by heuristic considerations:
\[ dS = d\ell_1 d\ell_2, \]
\[ d\ell_1 = r d\theta, \quad d\ell_2 = r \sin \theta d\phi \]
\[ dS = (r d\theta)(r \sin \theta d\phi) \]
\[ = r^2 \sin \theta d\theta d\phi. \]

Continuing the computation of the surface area of \( S^2_r \),
\[
\text{Area of } S^2_r = \int \int_U r^2 \sin \theta d\theta d\phi = \int \int_U r^2 \sin \theta d\theta d\phi \\
= \int_0^{2\pi} \int_0^\pi r^2 \sin \theta d\theta d\phi = \int_0^{2\pi} r^2 [- \cos \theta]_0^\pi d\phi \\
= \int_0^{2\pi} 2r^2 d\phi = 2r^2 \phi |_0^{2\pi} = 4\pi r^2.
\]

The surface area formula involves a choice of coordinates, i.e. a choice of proper patch. It is important to recognize that the formula is independent of this choice.

**Proposition.** The area formula is independent of the choice of coordinate patch.

Let \( x : U \to M, \ y : V \to M \) be proper patches, and suppose \( R \) is contained in \( x(U) \cap y(V) \):

Set,
\[ g_{ij} = \langle x_i, x_j \rangle, \quad g = \det[g_{ij}], \]
\[ \tilde{g}_{ij} = \langle y_i, y_j \rangle, \quad \tilde{g} = \det[\tilde{g}_{ij}]. \]
Then the claim is that,

$$\int\int_{x^{-1}(\mathcal{R})} \sqrt{g} \, du^1 \, du^2 = \int\int_{y^{-1}(\mathcal{R})} \sqrt{\tilde{g}} \, dv^1 \, dv^2$$

**Proof.** The proof is an application of the change of variable formula for double integrals.

Let $f : U \subset \mathbb{R}^2 \rightarrow V \subset \mathbb{R}^2$ be a diffeomorphism, where $U, V$ are bounded regions in $\mathbb{R}^2$.

$$f : \begin{align*}
v^1 &= f^1(u^1, u^2) \\
v^2 &= f^2(u^1, u^2)
\end{align*}$$

Then, the change of variable formula for double integrals is as follows,

$$\int\int_{V} h(v^1, v^2) \, dv^1 \, dv^2 = \int\int_{U} h \circ f(u^1, u^2) \cdot \det Df \, du^1 \, du^2$$

$$= \int\int_{U} h(f^1(u^1, u^2), f^2(u^1, u^2)) \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| \, du^1 \, du^2$$

or, in briefer notation,

$$\int\int_{V} h \, dv^1 \, dv^2 = \int\int_{U} h \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| \, du^1 \, du^2.$$

In the case at hand, $f = y^{-1} \circ x : x^{-1}(\mathcal{R}) \rightarrow y^{-1}(\mathcal{R})$, and $h = \sqrt{g}$. So, by the change of variable formula,

$$\int\int_{y^{-1}(\mathcal{R})} \sqrt{\tilde{g}} \, dv^1 \, dv^2 = \int\int_{x^{-1}(\mathcal{R})} \sqrt{g} \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| \, du^1 \, du^2$$

Thus, to complete the proof, it suffices to establish the following lemma.

**Lemma.** $g = \det[g_{ij}], \quad \tilde{g} = \det[\tilde{g}_{ij}]$ are related by,

$$\sqrt{\tilde{g}} = \sqrt{g} \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right|.$$
Proof of the lemma: From Exercise 3.8 it follows that,
\[
\frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} = \frac{\partial (v^1, v^2)}{\partial (u^1, u^2)} \frac{\partial \mathbf{y}}{\partial v^1} \times \frac{\partial \mathbf{y}}{\partial v^2}
\]
or,
\[
\mathbf{x}_1 \times \mathbf{x}_2 = \frac{\partial (v^1, v^2)}{\partial (u^1, u^2)} \mathbf{y}_1 \times \mathbf{y}_2
\]
Hence,
\[
g = \det [g_{ij}] = |\mathbf{x}_1 \times \mathbf{x}_2|^2
\]
\[
= \left( \frac{\partial (v^1, v^2)}{\partial (u^1, u^2)} \right)^2 |\mathbf{y}_1 \times \mathbf{y}_2|^2
\]
\[
= \left( \frac{\partial (v^1, v^2)}{\partial (u^1, u^2)} \right)^2 \tilde{g}.
\]
Taking square roots yields the result.

Exercise 4.3 Consider the torus of large radius $R$ and small radius $r$ described in Exercise 3.3. Use the intrinsic surface area formula and the parameterization given in Exercise 3.3 to compute the surface area of the torus. Answer: $4\pi^2 Rr$.

Exercise 4.4 Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function of two variables. Let $M$ be the graph of $f|_W = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in W\}$, where $W$ is a bounded subset of $U$. Derive the following standard formula from calculus for the surface area of $M$,
\[
\text{Area of } M = \int \int_W \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} \, dx \, dy,
\]
by considering the Monge patch associated to $f$.

More Tensor Analysis

Let $\mathbf{x} : U \to M$, $\mathbf{y} : V \to M$ be overlapping patches in a surface $M$, $W := \mathbf{x}(U) \cap \mathbf{y}(V) \neq \emptyset$. Let $f = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$,
\[
f : \begin{cases} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{cases}
\]
be the smooth overlap map, cf., p. 15 of Chapter 3. Introduce the metric components with respect to each patch,

\[ g_{ij} = \langle x_i, x_j \rangle, \quad \tilde{g}_{ij} = \langle y_i, y_j \rangle. \]

How are these metric components related on the overlap?

**Exercise 4.5** Show that,

\[ g_{ij} = \sum_{a,b=1}^{2} \tilde{g}_{ab} \frac{\partial v^a}{\partial u^i} \frac{\partial v^b}{\partial u^j}, \quad i, j = 1, 2. \]

These equations can be expressed as a single matrix equation,

\[
\begin{bmatrix} g_{ij} \end{bmatrix} = \begin{bmatrix} \frac{\partial v^a}{\partial u^i} \end{bmatrix}^t \begin{bmatrix} \tilde{g}_{ab} \end{bmatrix} \begin{bmatrix} \frac{\partial v^b}{\partial u^j} \end{bmatrix}.
\]

Taking determinants we obtain,

\[ g = \det[g_{ij}] = \det[*]^t[*][*] \]
\[ = \det[*]^t \det[*] \det[*] \]
\[ = \det \left[ \frac{\partial v^a}{\partial u^i} \right] \det[\tilde{g}_{ij}] \det \left[ \frac{\partial v^b}{\partial u^j} \right] \]
\[ = \tilde{g} (\det Df)^2 \]
\[ g = \tilde{g} \left[ \frac{\partial (v^1, v^2)}{\partial (u^1, u^2)} \right]^2, \]

our second derivation of this formula.

**Remark:** Interchanging the roles of \( x \) and \( y \) above we obtain,

\[ \tilde{g}_{ab} = \sum_{i,j} g_{ij} \frac{\partial u^i}{\partial v^a} \frac{\partial u^j}{\partial v^b}, \]

which involves the Jacobian of \( f^{-1} \). Compare this “transformation law” for the metric components to the transformation law for vector components considered in Exercise 3.8. Vector fields are “contravariant” tensors. The metric \( \langle \ , \ \rangle \) is a “covariant” tensor.
5 The Second Fundamental Form

Directional Derivatives in $\mathbb{R}^3$.

Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function defined on an open subset of $\mathbb{R}^3$. Fix $p \in U$ and $X \in T_p\mathbb{R}^3$. The directional derivative of $f$ at $p$ in the direction $X$, denoted $D_X f$ is defined as follows. Let $\sigma : \mathbb{R} \to \mathbb{R}^3$ be the parameterized straight line, $\sigma(t) = p + tX$. Note $\sigma(0) = p$ and $\sigma'(0) = X$. Then,

$$D_X f = \frac{d}{dt} f \circ \sigma(t)|_{t=0}$$

$$= \frac{d}{dt} f(p + tX)|_{t=0}$$

$$= \left( \lim_{t \to 0} \frac{f(p + tX) - f(p)}{t} \right).$$

Fact: The directional derivative is given by the following formula,

$$D_X f = X \cdot \nabla f(p)$$

$$= (X^1, X^2, X^3) \cdot \left( \frac{\partial f}{\partial x^1}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right)$$

$$= X^1 \frac{\partial f}{\partial x^1}(p) + X^2 \frac{\partial f}{\partial x^2}(p) + X^3 \frac{\partial f}{\partial x^3}(p)$$

$$= \sum_{i=1}^{3} X^i \frac{\partial f}{\partial x^i}(p).$$

Proof Chain rule!

Vector Fields on $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a rule which assigns to each point of $\mathbb{R}^3$ a vector at the point,

$$x \in \mathbb{R}^3 \to Y(x) \in T_x \mathbb{R}^3$$
Analytically, a vector field is described by a mapping of the form,

\[ Y: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \]

\[ Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \in T_x \mathbb{R}^3. \]

Components of \( Y \): \( Y^i: U \rightarrow \mathbb{R}, i = 1, 2, 3. \)

Ex. \( Y: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \) \( Y(x, y, z) = (y + z, z + x, x + y). \) E.g., \( Y(1, 2, 3) = (5, 4, 3), \) etc. \( Y^1 = y + z, \) \( Y^2 = z + x, \) and \( Y^3 = x + y. \)

The directional derivative of a vector field is defined in a manner similar to the directional derivative of a function: Fix \( p \in U, \ X \in T_p \mathbb{R}^3. \) Let \( \sigma: \mathbb{R} \rightarrow \mathbb{R}^3 \) be the parameterized line \( \sigma(t) = p + tX \ (\sigma(0) = p, \sigma'(0) = X). \) Then \( t \rightarrow Y \circ \sigma(t) \) is a vector field along \( \sigma \) in the sense of the definition in Chapter 2. Then, the directional derivative of \( Y \) in the direction \( X \) at \( p, \) is defined as,

\[ D_X Y = \frac{d}{dt} Y \circ \sigma(t)|_{t=0} \]

I.e., to compute \( D_X Y, \) restrict \( Y \) to \( \sigma \) to obtain a vector valued function of \( t, \) and differentiate with respect to \( t. \)

In terms of components, \( Y = (Y^1, Y^2, Y^3), \)

\[ D_X Y = \frac{d}{dt} (Y^1 \circ \sigma(t), Y^2 \circ \sigma(t), Y^3 \circ \sigma(t))|_{t=0} \]

\[ = \left( \frac{d}{dt} Y^1 \circ \sigma(t)|_{t=0}, \frac{d}{dt} Y^2 \circ \sigma(t)|_{t=0}, \frac{d}{dt} Y^3 \circ \sigma(t)|_{t=0} \right) \]

\[ = (D_X Y^1, D_X Y^2, D_X Y^3). \]
**Directional derivatives on surfaces.**

Let $M$ be a surface, and let $f : M \to \mathbb{R}$ be a smooth function on $M$. Recall, this means that $\hat{f} = f \circ \mathbf{x}$ is smooth for all proper patches $\mathbf{x} : U \to M$ in $M$.

**Def.** For $p \in M$, $X \in T_p M$, the directional derivative of $f$ at $p$ in the direction $X$, denoted $\nabla_X f$, is defined as follows. Let $\sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3$ be any smooth curve in $M$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then,

$$\nabla_X f = \frac{d}{dt} f \circ \sigma(t) |_{t=0}$$

I.e., to compute $\nabla_X f$, restrict $f$ to $\sigma$ and differentiate with respect to parameter $t$.

**Proposition.** The directional derivative is well-defined, i.e. independent of the particular choice of $\sigma$.

**Proof.** Let $\mathbf{x} : U \to M$ be a proper patch containing $p$. Express $\sigma$ in terms of coordinates in the usual manner,

$$\sigma(t) = \mathbf{x}(u^1(t), u^2(t)).$$

By the chain rule,

$$\frac{d\sigma}{dt} = \sum du_i \frac{dx_i}{dt} \quad \left( x_i = \frac{\partial \mathbf{x}}{\partial u_i} \right)$$

$X \in T_p M \Rightarrow X = \sum X^i x_i$. The initial condition, $\frac{d\sigma}{dt}(0) = X$ then implies

$$\frac{du^i}{dt}(0) = X^i, \quad i = 1, 2.$$ 

Now,

$$f \circ \sigma(t) = f(\sigma(t)) = f(\mathbf{x}(u^1(t), u^2(t)))$$

$$= f(\mathbf{x}(u^1(t), u^2(t)))$$

$$= \hat{f}(u^1(t), u^2(t)).$$

Hence, by the chain rule,

$$\frac{d}{dt} f \circ \sigma(t) = \frac{\partial \hat{f}}{\partial u^1} \frac{du^1}{dt} + \frac{\partial \hat{f}}{\partial u^2} \frac{du^2}{dt}$$

$$= \sum_i \frac{\partial \hat{f}}{\partial u^i} \frac{du^i}{dt} = \sum_i \frac{du^i}{dt} \frac{\partial \hat{f}}{\partial u^i}. $$

76
Therefore,
\[ \nabla_{X} f = \frac{d}{dt} f \circ \sigma(t)|_{t=0} \]
\[ = \sum_{i} \frac{du^i}{dt} (0) \frac{\partial \hat{f}}{\partial u^i}(u^1, u^2), \quad (p = x(u^1, u^2)) \]
\[ \nabla_{X} f = \sum_{i} X^i \frac{\partial \hat{f}}{\partial u^i}(u^1, u^2), \]
or simply,
\[ \nabla_{X} f = \sum_{i} X^i \frac{\partial \hat{f}}{\partial u^i} \]
\[ = X^1 \frac{\partial \hat{f}}{\partial u^1} + X^2 \frac{\partial \hat{f}}{\partial u^2}. \quad (*) \]

**Ex.** Let \( X = x_1 \). Since \( x_1 = 1 \cdot x_1 + 0 \cdot x_2, \) \( X^1 = 1 \) and \( X^2 = 0 \). Hence the above equation implies, \( \nabla_{x_1} f = \frac{\partial \hat{f}}{\partial u^1} \). Similarly, \( \nabla_{x_2} f = \frac{\partial \hat{f}}{\partial u^2} \). I.e.,
\[ \nabla_{x_i} f = \frac{\partial \hat{f}}{\partial u^i}, \quad i = 1, 2. \]

The following proposition summarizes some basic properties of directional derivatives in surfaces.

**Proposition**

1. \( \nabla_{(aX+bY)} f = a\nabla_{X} f + b\nabla_{Y} f \)
2. \( \nabla_{X} (f + g) = \nabla_{X} f + \nabla_{X} g \)
3. \( \nabla_{X} fg = (\nabla_{X} f)g + f(\nabla_{X} g) \)

**Exercise 5.1.** Prove this proposition.

Vector fields along a surface.

A vector field along a surface \( M \) is a rule which assigns to each point of \( M \) a vector at that point,
\[ x \in M \rightarrow Y(x) \in T_x \mathbb{R}^3 \]
N.B. \( Y(x) \) need not be tangent to \( M \).

Analytically vector fields along a surface \( M \) are described by mappings.

\[
Y : M \to \mathbb{R}^3 \\
Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \in T_x \mathbb{R}^3
\]

Components of \( Y \): \( Y^i : M \to \mathbb{R}, \ i = 1, 2, 3 \). We say \( Y \) is smooth if its component functions are smooth.

The directional derivative of a vector field along \( M \) is defined in a manner similar to the directional derivative of a function defined on \( M \).

Given a vector field along \( M \), \( Y : M \to \mathbb{R}^3 \), for \( p \in M, X \in T_p M \), the directional derivative of \( Y \) in the direction \( X \), denoted \( \nabla_X Y \), is defined as,

\[
\nabla_X Y = \frac{d}{dt} Y \circ \sigma(t)|_{t=0}
\]

where \( \sigma : (-\epsilon, \epsilon) \to M \) is a smooth curve in \( M \) such that \( \sigma(0) = p \) and \( \frac{d\sigma}{dt}(0) = X \).

I.e. to compute \( \nabla_X Y \), restrict \( Y \) to \( \sigma \) to obtain a vector valued function of \( t \) - then differentiate with respect to \( t \).

**Fact** If \( Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \) then,

\[
\nabla_X Y = (\nabla_X Y^1, \nabla_X Y^2, \nabla_X Y^3)
\]
Proof: Exercise.

Surface Coordinate Expression. Let \( x: U \to M \) be a proper patch in \( M \) containing \( p \). Let \( X \in T_p M, \ X = \sum_i x^i x_i. \) An argument like that for functions on \( M \) shows,

\[
\nabla_X Y = \sum_{i=1}^{2} X^i \frac{\partial \hat{Y}}{\partial u^i}(u^1, u^2), \quad (p = x(u^1, u^2))
\]

where \( \hat{Y} = Y \circ x: U \to \mathbb{R}^3 \) is \( Y \) expressed in terms of coordinates.

**Exercise 5.2.** Derive the expression above for \( \nabla_X Y \). In particular, show

\[
\nabla_{x^i} Y = \frac{\partial \hat{Y}}{\partial u^i}, \quad i = 1, 2.
\]

Some basic properties are described in the following proposition.

**Proposition**

1. \( \nabla_{aX+bY} Z = a \nabla_X Z + b \nabla_Y Z \)
2. \( \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \)
3. \( \nabla_X (fY) = (\nabla_X f)Y + f \nabla_X Y \)
4. \( \nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \)
The Weingarten Map and the 2nd Fundamental Form.

We are interested in studying the \textit{shape} of surfaces in $\mathbb{R}^3$. Our approach (essentially due to Gauss) is to study how the unit normal to the surface “wiggles” along the surface.

The objects which describe the shape of $M$ are:

1. The \textit{Weingarten Map}, or \textit{shape} operator. For each $p \in M$ this is a certain linear transformation $L : T_p M \to T_p M$.

2. The \textit{second fundamental form}. This is a certain bilinear form $\mathcal{L} : T_p M \times T_p M \to \mathbb{R}$ associated in a natural way with the Weingarten map.

We now describe the Weingarten map. Fix $p \in M$. Let $n : W \to \mathbb{R}^3$, $p \in W \to n(p) \in T_p \mathbb{R}^3$, be a smooth \textit{unit normal} vector field defined along a neighborhood $W$ of $p$.

\begin{center}
\includegraphics[width=0.5\textwidth]{weingarten_map}
\end{center}

\textbf{Remarks}

1. $n$ can always be constructed by introducing a proper patch $x : U \to M$, $x = x(u^1, u^2)$ containing $p$:

$$\hat{n} = \frac{x_1 \times x_2}{|x_1 \times x_2|},$$

$\hat{n} : U \to \mathbb{R}^3$, $\hat{n} = \hat{n}(u^1, u^2)$. Then, $n = \hat{n} \circ x^{-1} : x(U) \to \mathbb{R}^3$ is a smooth unit normal v.f. along $x(U)$.

2. The choice of $n$ is not quite unique: $n \to -n$; choice of $n$ is unique “up to sign”
3. A smooth unit normal field $n$ always exists in a neighborhood of any given point $p$, but it may not be possible to extend $n$ to all of $M$. This depends on whether or not $M$ is an orientable surface.

Ex. Möbius band.

**Lemma.** Let $M$ be a surface, $p \in M$, and $n$ be a smooth unit normal vector field defined along a neighborhood $W \subset M$ of $p$. Then for any $X \in T_pM$, $\nabla_X n \in T_pM$.

**Proof.** It suffices to show that $\nabla_X n$ is perpendicular to $n$. $|n| = 1 \Rightarrow \langle n, n \rangle = 1 \Rightarrow$

\[
\nabla_X \langle n, n \rangle = \nabla_X 1
\]

\[
\langle \nabla_X n, n \rangle + \langle n, \nabla_X n \rangle = 0
\]

\[
2\langle \nabla_X n, n \rangle = 0
\]

and hence $\nabla_X n \perp n$.

**Def.** Let $M$ be a surface, $p \in M$, and $n$ be a smooth unit normal v.f. defined along a nbd $W \subset M$ of $p$. The Weingarten Map (or shape operator) is the map $L : T_pM \to T_pM$ defined by,

$L(X) = -\nabla_X n$.

**Remarks**

1. The minus sign is a convention – will explain later.
2. $L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0}$
Lemma: \( L : T_p M \to T_p M \) is a linear map, i.e.,
\[
L(aX + bY) = aL(X) + bL(Y).
\]
for all \( X, Y, \in T_p M, \ a, b \in \mathbb{R}. \)

Proof. Follows from properties of directional derivative,
\[
L(aX + bY) = -\nabla_{aX + bY} n = -[a\nabla_X n + b\nabla_Y n] = a(-\nabla_X n) + b(-\nabla_Y n) = aL(X) + bL(Y).
\]

Ex. Let \( M \) be a plane in \( \mathbb{R}^3 \):
\[
M : ax + by + cz = d
\]
Determine the Weingarten Map at each point of \( M \). Well,
\[
n = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} = \left( \frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda} \right),
\]
where \( \lambda = \sqrt{a^2 + b^2 + c^2} \). Hence,
\[
L(X) = -\nabla_X n = -\nabla_X \left( \frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda} \right)
\]
\[
= -\left( \nabla_X \frac{a}{\lambda}, \nabla_X \frac{b}{\lambda}, \nabla_X \frac{c}{\lambda} \right) = 0.
\]
Therefore \( L(X) = 0 \ \forall \ X \in T_p M, \) i.e. \( L \equiv 0. \)

Ex. Let \( M = S^2_r \) be the sphere of radius \( r \), let \( n \) be the outward pointing unit normal. Determine the Weingarten map at each point of \( M \).
Fix \( p \in S_r^2 \), and let \( X \in T_p S_r^2 \). Let \( \sigma : (-\varepsilon, \varepsilon) \to S_r^2 \) be a curve in \( S_r^2 \) such that \( \sigma(0) = p, \frac{d\sigma}{dt}(0) = X \). Then,

\[
L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0} .
\]

But note, \( n \circ \sigma(t) = n(\sigma(t)) = \frac{\sigma(t)}{|\sigma(t)|} \). Hence,

\[
L(X) = -\frac{d}{dt} \frac{\sigma(t)}{r}|_{t=0} = -\frac{1}{r} \frac{d\sigma}{dt}|_{t=0}
\]

\[
L(X) = -\frac{1}{r} X.
\]

for all \( X \in T_p M \). Hence, \( L = -\frac{1}{r} \text{id} \), where \( \text{id} : T_p M \to T_p M \) is the identity map, \( \text{id}(X) = X \).

**Remark.** If we had taken the inward pointing normal then \( L = \frac{1}{r} \text{id} \).

**Def.** For each \( p \in M \), the *second fundamental form* is the bilinear form \( \mathcal{L} = T_p M \times T_p M \to \mathbb{R} \) defined by,

\[
\mathcal{L}(X, Y) = \langle L(X), Y \rangle
\]

\[
= -\langle \nabla_X n, Y \rangle.
\]

\( \mathcal{L} \) is indeed *bilinear*, e.g.,

\[
\mathcal{L}(aX + bY, Z) = \langle L(aX + bY), Z \rangle
\]

\[
= \langle aL(X) + bL(Y), Z \rangle
\]

\[
= a\langle L(X), Z \rangle + b\langle L(Y), Z \rangle
\]

\[
= a\mathcal{L}(X, Z) + b\mathcal{L}(Y, Z).
\]

**Ex.** \( M = \text{plane}, \ \mathcal{L} \equiv 0 \):

\[
\mathcal{L}(X, Y) = \langle L(X), Y \rangle = \langle 0, Y \rangle = 0.
\]

**Ex.** The sphere \( S^2_r \) of radius \( r \), \( \mathcal{L} : T_p S^2_r \times T_p S^2_r \to \mathbb{R} \),

\[
\mathcal{L}(X, Y) = \langle L(X), Y \rangle
\]

\[
= \langle -\frac{1}{r} X, Y \rangle
\]

\[
= -\frac{1}{r} \langle X, Y \rangle
\]
Hence, $L = -\frac{1}{r}( \cdot, \cdot )$. Multiple of the first fundamental form!

Coordinate expressions

Let $x : U \to M$ be a patch containing $p \in M$. Then $\{x_1, x_2\}$ is a basis for $T_p M$. We express $L : T_p M \to T_p M$ and $L : T_p M \times T_p M \to \mathbb{R}$ with respect to this basis. Since $L(x_j) \in T_p M$, we have,

$$L(x_j) = L^1_j x_1 + L^2_j x_2, \quad j = 1, 2$$

$$= \sum_{i=1}^2 L^i_j x_i.$$  

The numbers $L^i_j$, $1 \leq i, j \leq 2$, are called the components of $L$ with respect to the coordinate basis $\{x_1, x_2\}$. The $2 \times 2$ matrix $[L^i_j]$ is the matrix representing the linear map $L$ with respect to the basis $\{x_1, x_2\}$.

**Exercise 5.3** Let $X \in T_p M$ and let $Y = L(X)$. In terms of components, $X = \sum_j X^j x_j$ and $Y = \sum_i Y^i x_i$. Show that

$$Y^i = \sum_j L^i_j X^j, \quad i = 1, 2,$$

which in turn implies the matrix equation,

$$\begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} = [L^i_j] \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}. $$

This is the Weingarten map expressed as a matrix equation.

Introduce the unit normal field along $W = x(U)$ with respect to the patch $x : U \to M$,

$$\hat{n} = \frac{x_1 \times x_2}{|x_1 \times x_2|}, \quad \hat{n} = \hat{n}(u^1, u^2),$$

$$n = \hat{n} \circ x^{-1} : W \to \mathbb{R}.$$  

Then by Exercise 5.2,

$$L(x_j) = -\nabla_{x_j} n = -\frac{\partial \hat{n}}{\partial u^j}.$$  

Setting $n_j = \frac{\partial \hat{n}}{\partial u^j}$ we have

$$n_j = -L(x_j)$$

$$n_j = -\sum_i L^i_j x_i, \quad j = 1, 2 \quad \text{(The Weingarten equations.)}$$
These equations can be used to compute the components of the Weingarten map. However, in practice it turns out to be more useful to have a formula for computing the components of the second fundamental form.

**Components of $\mathcal{L}$:**

The components of $\mathcal{L}$ with respect to $\{x_1, x_2\}$ are defined as,

$$L_{ij} = \mathcal{L}(x_i, x_j), \quad 1 \leq i, j \leq 2.$$

By bilinearity, the components completely determine $\mathcal{L}$,

$$\mathcal{L}(X, Y) = \mathcal{L}(\sum_i X^i x_i, \sum_j Y^j x_j) = \sum_{i,j} X^i Y^j \mathcal{L}(x_i, x_j) = \sum_{i,j} L_{ij} X^i Y^j.$$

The following proposition provides a useful formula for computing the $L_{ij}$’s.

**Proposition.** The components $L_{ij}$ of $\mathcal{L}$ are given by,

$$L_{ij} = \langle \hat{n}, x_{ij} \rangle,$$

where $x_{ij} = \frac{\partial^2 x}{\partial u^j \partial u^i}$.

**Remark.** Henceforth we no longer distinguish between $n$ and $\hat{n}$, i.e., lets agree to drop the “^”, then,

$$L_{ij} = \langle n, x_{ij} \rangle,$$

**Proof:**

![Diagram](image-url)
Along \( x(U) \) we have, \( \langle n, \frac{\partial x}{\partial u^i} \rangle = 0 \), and hence, 
\[
\frac{\partial}{\partial u^i}\langle n, \frac{\partial x}{\partial u^j} \rangle = 0 \\
\langle \frac{\partial n}{\partial u^i}, \frac{\partial x}{\partial u^j} \rangle + \langle n, \frac{\partial^2 x}{\partial u^i \partial u^j} \rangle = 0 \\
\langle \frac{\partial n}{\partial u^i}, \frac{\partial x}{\partial u^j} \rangle = -\langle n, \frac{\partial^2 x}{\partial u^i \partial u^j} \rangle = -\langle n, \frac{\partial^2 x}{\partial u^j \partial u^i} \rangle,
\]
or, using shorthand notation,
\[
\langle n_i, x_j \rangle = -\langle n, x_{ij} \rangle.
\]
But,
\[
L_{ij} = \mathcal{L}(x_i, x_j) = \langle L(x_i), x_j \rangle \\
= -\langle n_i, x_j \rangle,
\]
and hence \( L_{ij} = \langle n, x_{ij} \rangle \).
\[\square\]

Observe,
\[
L_{ij} = \langle n, x_{ij} \rangle \\
= \langle n, x_{ji} \rangle \quad \text{(mixed partials equal!)} \\
L_{ij} = L_{ji}, \quad 1 \leq i, j \leq 2.
\]

In other words, \( \mathcal{L}(x_i, x_j) = \mathcal{L}(x_j, x_i) \).

**Proposition.** The second fundamental form \( \mathcal{L} : T_pM \times T_pM \to \mathbb{R} \) is symmetric, i.e.
\[
\mathcal{L}(X, Y) = \mathcal{L}(Y, X) \quad \forall \ X, Y \in T_pM.
\]

**Exercise 5.4** Prove this proposition by showing \( \mathcal{L} \) is symmetric iff \( L_{ij} = L_{ji} \) for all \( 1 \leq i, j \leq 2 \).

**Relationship between \( L^j_i \) and \( L_{ij} \)**
\[
L_{ij} = \mathcal{L}(x_i, x_j) = \mathcal{L}(x_j, x_i) \\
= \langle L(x_j), x_i \rangle = \langle \sum_k L^k_j x_k, x_i \rangle \\
= \sum_k L^k_j \langle x_k, x_i \rangle \\
L_{ij} = \sum_k g_{ik} L^k_j, \quad 1 \leq i, j \leq 2
\]
Classical tensor jargon: $L_{ij}$ obtained from $L^{k}_{j}$ by “lowering the index $k$ with the metric”. The equation above implies the matrix equation

$$[L_{ij}] = [g_{ij}][L^i_j].$$

**Geometric Interpretation of the 2nd Fundamental Form**

**Normal Curvature.** Let $s \rightarrow \sigma(s)$ be a unit speed curve lying in a surface $M$. Let $p$ be a point on $\sigma$, and let $n$ be a smooth unit normal v.f. defined in a nbd $W$ of $p$. The normal curvature of $\sigma$ at $p$, denoted $\kappa_n$, is defined to be the component of the curvature vector $\sigma'' = T'$ along $n$, i.e.,

$$\kappa_n = \text{normal component of the curvature vector} = \langle \sigma'', n \rangle = \langle T', n \rangle = |T'| |n| \cos \theta = \kappa \cos \theta,$$

where $\theta$ is the angle between the curvature vector $T'$ and the surface normal $n$. If $\kappa \neq 0$ then, recall, we can introduce the principal normal $N$ to $\sigma$, by the equation, $T' = \kappa N$; in this case $\theta$ is the angle between $N$ and $n$.

![Diagram](image)

**Remark:** $\kappa_n$ gives a measure of how much $\sigma$ is bending in the direction perpendicular to the surface; it neglects the amount of bending tangent to the surface.

**Proposition.** Let $M$ be a surface, $p \in M$. Let $X \in T_p M$, $|X| = 1$ (i.e. $X$ is a unit tangent vector). Let $s \rightarrow \sigma(s)$ be any unit speed curve in $M$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then

$$\mathcal{L}(X,X) = \text{normal curvature of } \sigma \text{ at } p = \langle \sigma'', n \rangle.$$
Proof. Along $\sigma$,
\[
\langle \sigma'(s), n \circ \sigma(s) \rangle = 0, \quad \text{for all } s
\]
\[
\frac{d}{ds} \langle \sigma', n \circ \sigma \rangle = 0
\]
\[
\langle \sigma'', n \circ \sigma \rangle + \langle \sigma', \frac{d}{ds} n \circ \sigma \rangle = 0.
\]

At $s = 0$,
\[
\langle \sigma'', n \rangle + \langle X, \nabla_X n \rangle = 0
\]
\[
\langle \sigma'', n \rangle = -\langle X, \nabla_X n \rangle
\]
\[
\kappa_n = \langle X, L(X) \rangle
\]
\[
\kappa_n = \langle L(X), X \rangle
\]
\[
= \mathcal{L}(X, X).
\]

Remark: the sign convention used in the definition of the Weingarten map ensures that $\mathcal{L}(X, X) = +\kappa_n$ (rather than $-\kappa_n$).

Corollary. All unit speed curves lying in a surface $M$ which pass through $p \in M$ and have the same unit tangent vector $X$ at $p$, have the same normal curvature at $p$. That is, the normal curvature depends only on the tangent direction $X$.

Thus it makes sense to say:
\[
\mathcal{L}(X, X) \text{ is the normal curvature in the direction } X.
\]

Given a unit tangent vector $X \in T_p M$, there is a distinguished curve in $M$, called the normal section at $p$ in the direction $X$. Let,
\[
\Pi = \text{plane through } p \text{ spanned by } n \text{ and } X.
\]

$\Pi$ cuts $M$ in a curve $\sigma$. Parameterize $\sigma$ wrt arc length, $s \to \sigma(s)$, such that $\sigma(0) = p$ and $\frac{d\sigma}{ds}(0) = X$:
By definition, $\sigma$ is the normal section at $p$ in the direction $X$. By the previous proposition,

$$L(X, X) = \text{normal curvature of the normal section } \sigma$$
$$= \langle \sigma'' , n \rangle = \langle T' , n \rangle$$
$$= \kappa \cos \theta ,$$

where $\theta$ is the angle between $n$ and $T'$. Since $\sigma$ lies in $\Pi$, $T'$ is tangent to $\Pi$, and since $T'$ is also perpendicular to $X$, it follows that $T'$ is a multiple of $n$. Hence, $\theta = 0$ or $\pi$, which implies that $L(X, X) = \pm \kappa$.

Thus we conclude that,

$$L(X, X) = \text{signed curvature of the normal section at } p \text{ in the direction } X.$$ 

**Principal Curvatures.**

The set of unit tangent vectors at $p$, $X \in T_p M$, $|X| = 1$, forms a circle in the tangent plane to $M$ at $p$. Consider the function from this circle into the reals,

$$X \rightarrow \text{normal curvature in direction } X$$
$$X \rightarrow L(X, X).$$

The principal curvatures of $M$ at $p$, $\kappa_1 = \kappa_1(p)$ and $\kappa_2 = \kappa_2(p)$, are defined as follows,

$$\kappa_1 = \text{the maximum normal curvature at } p$$
$$= \max_{|X|=1} L(X, X)$$

$$\kappa_2 = \text{the minimum normal curvature at } p$$
$$= \min_{|X|=1} L(X, X)$$

This is the geometric characterization of principal curvatures. There is also an important algebraic characterization.
Some Linear Algebra

Let $V$ be a vector space over the reals, and let $\langle \ , \ \rangle : V \times V \to \mathbb{R}$ be an inner product on $V$; hence $V$ is an inner product space. Let $L : V \to V$ be a linear transformation. Our main application will be to the case: $V = T_pM$, $\langle \ , \ \rangle =$ induced metric, and $L =$ Weingarten map.

$L$ is said to be self adjoint provided

$$\langle L(v), w \rangle = \langle v, L(w) \rangle \quad \forall \ v, w \in V .$$

Remark. Let $V = \mathbb{R}^n$, with the usual dot product, and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $[L^i_j] =$ matrix representing $L$ with respect to the standard basis, $e_1 : (1, 0, ..., 0)$, etc. Then $L$ is self-adjoint if and only if $[L^i_j]$ is symmetric $[L^i_j] = [L^j_i]$.

Proposition. The Weingarten map $L : T_pM \to T_pM$ is self adjoint, i.e.

$$\langle L(X), Y \rangle = \langle X, L(Y) \rangle \quad \forall \ X, Y \in T_pM ,$$

where $\langle \ , \ \rangle =$ 1st fundamental form.

Proof. We have,

$$\langle L(X), Y \rangle = \mathcal{L}(X,Y) = \mathcal{L}(Y,X)$$

$$= \langle L(Y), X \rangle = \langle X, L(Y) \rangle .$$

Self adjoint linear transformations have very nice properties, as we now discuss. For this discussion, we restrict attention to 2-dimensional vector spaces, $\dim V = 2$.

A vector $v \in V$, $v \neq 0$, is called an eigenvector of $L$ if there is a real number $\lambda$ such that,

$$L(v) = \lambda v .$$

$\lambda$ is called an eigenvalue of $L$. The eigenvalues of $L$ can be determined by solving

$$\det(A - \lambda I) = 0 \quad (\ast)$$

where $A$ is a matrix representing $L$ and $I =$ identity matrix. The equation $(\ast)$ is a quadratic equation in $\lambda$, and hence has at most 2 real roots; it may have no real roots.

Theorem (Fundamental Theorem of Self Adjoint Operators) Let $V$ be a 2-dimensional inner product space. Let $L : V \to V$ be a self-adjoint linear map. Then $V$ admits an orthonormal basis consisting of eigenvectors of $L$. That is, there exists an orthonormal basis $\{e_1, e_2\}$ of $V$ and real numbers $\lambda_1, \lambda_2$, $\lambda_1 \geq \lambda_2$ such that

$$L(e_1) = \lambda_1 e_1, \quad L(e_2) = \lambda_2 e_2 ,$$
i.e., $e_1$ and $e_2$ are eigenvectors of $L$ and $\lambda_1, \lambda_2$ are the corresponding eigenvalues. Moreover the eigenvalues are given by

$$
\lambda_1 = \max_{|v|=1} \langle L(v), v \rangle,
\lambda_2 = \min_{|v|=1} \langle L(v), v \rangle.
$$

**Proof.** See handout from Do Carmo.

**Remark on orthogonality of eigenvectors.** Let $e_1, e_2$ be eigenvectors with eigenvalues $\lambda_1, \lambda_2$. If $\lambda_1 \neq \lambda_2$, then $e_1$ and $e_2$ are necessarily orthogonal, as seen by the following,

$$
\lambda_1 \langle e_1, e_2 \rangle = \langle L(e_1), e_2 \rangle = \langle e_1, L(e_2) \rangle = \lambda_2 \langle e_1, e_2 \rangle,
$$

$\Rightarrow (\lambda_1 - \lambda_2) \langle e_1, e_2 \rangle = 0 \Rightarrow \langle e_1, e_2 \rangle = 0$. On the other hand, if $\lambda_1 = \lambda_2 = \lambda$ then $L(v) = \lambda v$ for all $v$. Hence any o.n. basis is a basis of eigenvectors.

We now apply these facts to the Weingarten map,

$$
L : T_p M \to T_p M, \\
\mathcal{L} : T_p M \times T_p M \to \mathbb{R}, \quad \mathcal{L}(X,Y) = \langle L(X), Y \rangle.
$$

Since $L$ is self adjoint, and, by definition,

$$
\kappa_1 = \max_{|X|=1} \mathcal{L}(X,X) = \max_{|X|=1} \langle L(X), X \rangle,
\kappa_2 = \min_{|X|=1} \mathcal{L}(X,X) = \min_{|X|=1} \langle L(X), X \rangle,
$$

we obtain the following.

**Theorem.** The principal curvatures $\kappa_1, \kappa_2$ of $M$ at $p$ are the eigenvalues of the Weingarten map $L : T_p M \to T_p M$. There exists an orthonormal basis $\{e_1, e_2\}$ of $T_p M$ such that

$$
L(e_1) = \kappa_1 e_1, \quad L(e_2) = \kappa_2 e_2,
$$

i.e., $e_1, e_2$ are eigenvectors of $L$ associated with the eigenvalues $\kappa_1, \kappa_2$, respectively. The eigenvectors $e_1$ and $e_2$ are called **principal directions**.

Observe that,

$$
\kappa_1 = \kappa_1 \langle e_1, e_1 \rangle = \langle L(e_1), e_1 \rangle = \mathcal{L}(e_1, e_1),
\kappa_2 = \kappa_2 \langle e_2, e_2 \rangle = \langle L(e_2), e_2 \rangle = \mathcal{L}(e_2, e_2),
$$

i.e., the principal curvature $\kappa_1$ is the normal curvature in the principal direction $e_1$, and similarly for $\kappa_2$. 

91
Now, let $A$ be the matrix associated to the Weingarten map $L$ with respect to the orthonormal basis $\{e_1, e_2\}$; thus,

\[
L(e_1) = \kappa_1 e_1 + 0 e_2 \\
L(e_2) = 0 e_1 + \kappa_2 e_2
\]

which implies,

\[
A = \begin{bmatrix}
\kappa_1 & 0 \\
0 & \kappa_2
\end{bmatrix}.
\]

Then,

\[
\det L = \det A = \kappa_1 \kappa_2 \\
\text{tr } L = \text{tr } A = \kappa_1 + \kappa_2.
\]

**Definition.** The *Gaussian curvature* of $M$ at $p$, $K = K(p)$, and the *mean curvature* of $M$ at $p$, $H = H(p)$ are defined as follows,

\[
K = \det L = \kappa_1 \kappa_2 \\
H = \text{tr } L = \kappa_1 + \kappa_2.
\]

**Remarks.** The Gaussian curvature is the more important of the two curvatures; it is what is meant by the *curvature* of a surface. A famous discovery by Gauss is that it is intrinsic – in fact can be computed in terms of the $g_{ij}$'s (This is not obvious!). The *mean curvature* (which has to do with minimal surface theory) is *not* intrinsic. This can be easily seen as follows. Changing the normal $n \rightarrow -n$ changes the sign of the Weingarten map,

\[
L_{-n} = -L_n.
\]

This in turn changes the sign of the principal curvatures, hence $H = \kappa_1 + \kappa_2$ changes sign, but $K = \kappa_1 \kappa_2$ does not change sign.
Some Examples

**Ex.** For $S_r^2$ = sphere of radius $r$, compute $\kappa_1, \kappa_2, K, H$ (Use outward normal).

**Geometrically:** $p \in S_r^2, \ X \in T_pM, \ |X| = 1,$

$$L(X, X) = \pm \text{curvature of normal section in direction } X$$
$$= -\text{curvature of great circle}$$
$$= -\frac{1}{r}.$$

Therefore

$$\kappa_1 = \max_{|X|=1} L(X, X) = -\frac{1}{r},$$

$$\kappa_2 = \min_{|X|=1} L(X, X) = -\frac{1}{r},$$

$$K = \kappa_1 \kappa_2 = \frac{1}{r^2} > 0, \quad H = \kappa_1 + \kappa_2 = -\frac{2}{r}.$$

**Algebraically:** Find eigenvalues of Weingarten map: $L : T_pM \to T_pM$. We showed previously,

$$L = -\frac{1}{r} \text{id}, \quad \text{ i.e.,}$$

$$L(X) = -\frac{1}{r}X \quad \text{for all } X \in T_pM.$$

Thus, with respect to any orthonormal basis $\{e_1, e_2\}$ of $T_pM$,

$$L(e_i) = -\frac{1}{r}e_i \quad i = 1, 2.$$
Therefore, \( \kappa_1 = \kappa_2 = -\frac{1}{r}, \ K = \frac{1}{r^2}, \ H = -\frac{2}{r} \).

**Ex.** Let \( M \) be the cylinder of radius \( a \): \( x^2 + y^2 = a^2 \). Compute \( \kappa_1, \kappa_2, K, H \). (Use the inward pointing normal)

**Geometrically:**

\[
\mathcal{L}(X_1, X_1) = \pm \text{curvature of normal section in direction } X_1 \\
= + \text{curvature of circle of radius } a \\
= \frac{1}{a},
\]

\[
\mathcal{L}(X_2, X_2) = \pm \text{curvature of normal section in direction } X_2 \\
= \text{curvature of line} \\
= 0.
\]

In general, for \( X \neq X_1, X_2 \),

\[
\mathcal{L}(X, X) = \text{curvature of ellipse through } p.
\]

The curvature is between 0 and \( \frac{1}{a} \), and thus,

\[
0 \leq \mathcal{L}(X, X) \leq \frac{1}{a}.
\]
We conclude that,

\[ \kappa_1 = \max_{|X|=1} \mathcal{L}(X, X) = \mathcal{L}(X_1, X_1) = \frac{1}{a}, \]

\[ \kappa_2 = \min_{|X|=1} \mathcal{L}(X, X) = \mathcal{L}(X_2, X_2) = 0. \]

Thus, \( K = 0 \) (cylinder is flat!) and \( H = \frac{1}{a} \).

**Algebraically:** Determine the eigenvalues of the Weingarten map. By a rotation and translation we may take \( p \) to be the point \( p = (a, 0, 0) \). Let \( e_1, e_2 \in T_p M \) be the tangent vectors \( e_1 = (0, 1, 0) \) and \( e_2 = (0, 0, 1) \).

To compute \( L(e_1) \), consider the circle,

\[ \sigma(s) = (a \cos \left( \frac{s}{a} \right), a \sin \left( \frac{s}{a} \right), 0) \]

Note that \( \sigma(0) = p \) and \( \sigma'(0) = e_1 \). Thus,

\[ L(e_1) = -\nabla_{e_1} n \]

\[ = -\frac{d}{ds} n(\sigma(s))_{|s=0} \]

But,

\[ n(\sigma(s)) = -\frac{\sigma(s)}{|\sigma(s)|} = -\frac{\sigma(s)}{a} \]

\[ = -\left( \cos \left( \frac{s}{a} \right), \sin \left( \frac{s}{a} \right), 0 \right) \]
Therefore,

\[ L(e_1) = \frac{d}{ds} \left( \cos \left( \frac{s}{a} \right), \sin \left( \frac{s}{a} \right), 0 \right) \bigg|_{s=0} \]

\[ = \frac{1}{a} \left( -\sin \left( \frac{s}{a} \right), \cos \left( \frac{s}{a} \right), 0 \right) \bigg|_{s=0} \]

\[ = \frac{1}{a} (0, 1, 0) \]

\[ L(e_1) = \frac{1}{a} e_1 \]

Thus, \( e_1 \) is an eigenvector with eigenvalue \( \frac{1}{a} \). Similarly (exercise!),

\[ L(e_2) = 0 = 0 \cdot e_2 \]

i.e., \( e_2 \) is an eigenvector with eigenvalue 0. (Note; \( e_2 \) is tangent to a vertical line in the surface, along which \( n \) is constant.)

We conclude that, \( \kappa_1 = \frac{1}{a}, \kappa_2 = 0, K = 0, H = \frac{1}{a} \).

**Ex.** Consider the saddle surface, \( M: z = y^2 - x^2 \), Compute \( \kappa_1, \kappa_2, K, H \) at \( p = (0, 0, 0) \).

\[ L(e_1, e_1) = \pm \text{ curvature of normal section in direction of } e_1 \]

\[ = + \text{ curvature of } z = y^2 \]

The curvature is given by,

\[ \kappa = \frac{\left| \frac{d^2 z}{dy^2} \right|}{\left[ 1 + \left( \frac{dz}{dy} \right)^2 \right]^{3/2}} = 2 \]
and so, $\mathcal{L}(e_1, e_1) = 2$. Similarly, $\mathcal{L}(e_2, e_2) = -2$. Observe,

$$\mathcal{L}(e_2, e_2) \leq \mathcal{L}(X, X) \leq \mathcal{L}(e_1, e_1)$$

Therefore, $\kappa_1 = 2$, $\kappa_2 = -2$, $K = -4$, and $H = 0$ at $(0, 0, 0)$.

**Exercise 5.5.** For the saddle surface $M$ above, consider the Weingarten map $L : T_pM \to T_pM$ at $p = (0, 0, 0)$. Compute $L(e_1)$ and $L(e_2)$ directly from the definition of the Weingarten map to show,

$$L(e_1) = 2e_1 \text{ and } L(e_2) = -2e_2.$$  

Hence, $-2$ and 2 are the eigenvalues of $L$, which means $\kappa_1 = 2$ and $\kappa_2 = -2$.

**Remark.** We have computed the quantities $\kappa_1$, $\kappa_2$, $K$, and $H$ of the saddle surface only at a single point. To compute these quantities at all points, we will need to develop better computational tools.

**Significance of the sign of Gaussian Curvature**

We have,

$$K = \det L = \kappa_1 \kappa_2.$$  

1. $K > 0 \iff \kappa_1$ and $\kappa_2$ have the same sign $\iff$ the normal sections in the principal directions $e_1, e_2$ both bend in the same direction,

   \[ K > 0 \text{ at } p. \]

   **Ex.** $z = ax^2 + by^2$, $a, b$ have the same sign (elliptic paraboloid). At $p = (0, 0, 0)$, $K = 4ab > 0$.

2. $K < 0 \iff \kappa_1$ and $\kappa_2$ have opposite signs $\iff$ normal sections in principle directions $e_1$ and $e_2$ bend in opposite directions,

   \[ K < 0 \text{ at } p. \]

   **Ex.** $z = ax^2 + by^2$, $a, b$ have opposite sign (hyperbolic paraboloid). At $p = (0, 0, 0)$, $K = 4ab < 0$.  

97
Thus, roughly speaking,

\[ K > 0 \text{ at } p \Rightarrow \text{surface is “bowl-shaped” near } p \]
\[ K < 0 \text{ at } p \Rightarrow \text{surface is “saddle-shaped” near } p \]

This rough observation can be made more precise, as we now show. Let \( M \) be a surface, \( p \in M \). Let \( e_1, e_2 \) be principal directions at \( p \). Choose \( e_1, e_2 \) so that \( \{e_1, e_2, n\} \) is a positively oriented orthonormal basis.

By a translation and rotation of the surface, we can assume, (see the figure),

1. \( p = (0, 0, 0) \)
2. \( e_1 = (1, 0, 0), \ e_2 = (0, 1, 0), \ n = (0, 0, 1) \) at \( p \)
3. Near \( p = (0, 0, 0) \), the surface can be described by an equation of form, \( z = f(x, y) \), where \( f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is smooth and \( f(0, 0) = 0 \).

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{center}

**Claim:**

\[ z = \frac{1}{2} \kappa_1 x^2 + \frac{1}{2} \kappa_2 y^2 + \text{higher order terms} \]

**Proof.** Consider the Taylor series about \((0, 0)\) for functions of two variables,

\[ z = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2 + \text{higher order terms}. \]

We must compute 1st and 2nd order partial derivatives of \( f \) at \((0, 0)\). Introduce the Monge patch,

\[ x = u \]
\[ y = v \]
\[ z = f(u, v) \]

i.e. \( \mathbf{x}(u, v) = (u, v, f(u, v)) \).
We have,

\[ x_1 = x_u = (1, 0, f_u), \]
\[ x_2 = x_v = (0, 1, f_v), \]
\[ n = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{x_u \times x_v}{|x_u \times x_v|} \]
\[ = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}} \]

At \((u, v) = (0, 0)\): \(n = (0, 0, 1) \Rightarrow f_u = f_v = 0, \Rightarrow x_1 = (1, 0, 0) = e_1 \) and \(x_2 = (0, 1, 0) = e_2\).

Recall, the components of the 2nd fundamental form \(L_{ij} = L(x_i, x_j)\) may be computed from the formula,

\[ L_{ij} = \langle n, x_{ij} \rangle, \quad x_{ij} = \frac{\partial^2 x}{\partial u^i \partial u^j}. \]

In particular, \(L_{11} = \langle n, x_{11} \rangle\), where \(x_{11} = x_{uu} = (0, 0, f_{uu})\).

At \((u, v) = (0, 0)\): \(L_{11} = \langle n, x_{11} \rangle = (0, 0, 1) \cdot (0, 0, f_{uu}(0, 0)) = f_{uu}(0, 0)\).

Therefore, \(f_{uu}(0, 0) = L_{11} = L(x_1, x_1) = L(e_1, e_1) = \kappa_1\). Similarly,

\[ f_{vv}(0, 0) = L(e_2, e_2) = \kappa_2 \]
\[ f_{uv}(0, 0) = L(e_1, e_2) = \langle L(e_1), e_2 \rangle = \lambda_1(e_1, e_2) = 0. \]

Thus, setting \(x = u, \ y = v\), we have shown,

\[ f_x(0, 0) = f_y(0, 0) = 0 \]
\[ f_{xx}(0, 0) = \kappa_1, \ f_{yy}(0, 0) = \kappa_2, \ f_{xy}(0, 0) = 0, \]

which, substituting in the Taylor expansion, implies,

\[ z = \frac{1}{2} \kappa_1 x^2 + \frac{1}{2} \kappa_2 y^2 + \text{higher order terms}. \]

**Computational Formula for Gaussian Curvature.**

We have,

\[ K = \text{Gaussian curvature} = \det L = \det[L_{ij}]. \]
From the equation at the top of p. 14,

\[
L_{ij} = [g_{ij}]L^i_j, \\
\det[L_{ij}] = \det[g_{ij}]\det[L^i_j] = \det[g_{ij}] \cdot K
\]

Hence,

\[
K = \frac{\det[L_{ij}]}{\det[g_{ij}]}, \quad g_{ij} = \langle x_i, x_j \rangle, \quad L_{ij} = \langle n, x_{ij} \rangle.
\]

Further,

\[
\det[L_{ij}] = \det \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = L_{11}L_{22} - L_{12}^2,
\]

since \( L_{12} = L_{21} \), and similarly,

\[
\det[g_{ij}] = g_{11}, g_{22} - g_{12}^2.
\]

Thus,

\[
K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}, g_{22} - g_{12}^2}.
\]

**Ex.** Compute the Gaussian curvature of the saddle surface \( z = y^2 - x^2 \).

Introduce the Monge patch, \( \mathbf{x}(u, v) = (u, v, v^2 - u^2) \).

Compute metric components \( g_{ij} \):

\[
\begin{align*}
\mathbf{x}_u &= (1, 0, -2u), \quad \mathbf{x}_v = (0, 1, 2v), \\
g_{uu} &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = (1, 0, -2u) \cdot (1, 0, -2u) \\
&= 1 + 4u^2. \\
g_{vv} &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + 4v^2, \\
g_{uv} &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = -4uv.
\end{align*}
\]

Thus,

\[
\det[g_{ij}] = g_{uu}g_{vv} - g_{uv}^2
\]

\[
= (1 + 4u^2)(1 + 4v^2) - 16u^2v^2
\]

\[
= 1 + 4u^2 + 4v^2.
\]

100
Compute the second fundamental form components \( L_{ij} \):

We use, \( L_{ij} = \langle n, x_{ij} \rangle \). We have,

\[
n = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{(2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}},
\]

and,

\[
x_{uu} = (0, 0, -2), \quad x_{vv} = (0, 0, 2), \quad x_{uv} = (0, 0, 0).
\]

Then,

\[
L_{uu} = \langle n, x_{uu} \rangle = \frac{-2}{\sqrt{1 + 4u^2 + 4v^2}},
\]

\[
L_{vv} = \langle n, x_{vv} \rangle = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}},
\]

\[
L_{uv} = \langle n, x_{uv} \rangle = 0.
\]

Thus,

\[
\det[L_{ij}] = L_{uu}L_{vv} - L_{uv}^2 = \frac{-4}{1 + 4u^2 + 4v^2},
\]

and therefore,

\[
K(u, v) = \frac{\det[L_{ij}]}{\det[g_{ij}]} = \frac{-4}{1 + 4u^2 + 4v^2} \cdot \frac{1}{1 + 4u^2 + 4v^2}.
\]

Hence the saddle surface \( z = y^2 - x^2 \) has Gaussian curvature function,

\[
K(x, y) = \frac{-4}{(1 + 4x^2 + 4y^2)^2}.
\]

Observe that \( K < 0 \) and, \( K = \frac{-4}{(1 + 4r^2)^2} \sim \frac{1}{r^4} \), where \( r = \sqrt{x^2 + y^2} \) is the distance from the z-axis. As \( r \to \infty \), \( K \to 0 \) rapidly.

**Exercise 5.6.** Consider the surface \( M \) which is the graph of \( z = f(x, y) \). Show that the Gaussian curvature \( K = K(x, y) \) is given by,

\[
K(x, y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}
\]

where \( f_x = \frac{\partial f}{\partial x}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2} \), etc.
Exercise 5.7. Let $M$ be the torus of large radius $R$ and small radius $r$ described in Exercise 3.3. Using the parameterization,

$$x(t, \theta) = ((R + r \cos t) \cos \theta, (R + r \cos t) \sin \theta, r \sin t)$$

show that the Gaussian curvature $K = K(t, \theta)$ is given by,

$$K = \frac{\cos t}{r(R + r \cos t)}.$$ 

Where on the torus is the Gaussian curvature negative? Where is it positive?

Exercise 5.8. Derive the following expression for the mean curvature $H$,

$$H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11}}{g_{11}g_{22} - g_{12}^2}$$

The principal curvatures $\kappa_1$ and $\kappa_2$ at a point $p \in M$ are the normal curvatures in the principal directions $e_1$ and $e_2$. The normal curvature in any direction $X$ is determined by $\kappa_1$ and $\kappa_2$ as follows.

If $X \in T_p M$, $|X| = 1$ then $X$ can be expressed as (see the figure),

$$X = \cos \theta e_1 + \sin \theta e_2.$$

Proposition (Euler’s formula). The normal curvature in the direction $X$ is given by,

$$\mathcal{L}(X, X) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta,$$

where $\kappa_1, \kappa_2$ are the principal curvatures, and $\theta$ is the angle between $X$ and the principal direction $e_1$.

Proof. Use the shorthand, $c = \cos \theta$, $s = \sin \theta$. Then $X = ce_1 + se_2$, and

$$L(X) = L(ce_1 + se_2)$$

$$= cL(e_1) + sL(e_2)$$

$$= c\kappa_1 e_1 + s\kappa_2 e_2.$$
Therefore,
\[ \mathcal{L}(X, X) = \langle L(X), X \rangle \]
\[ = \langle c \kappa_1 e_1 + s \kappa_2 e_2, ce_1 + se_2 \rangle \]
\[ = c^2 \kappa_1 + s^2 \kappa_2. \]

**Exercise 5.9.** Assuming \( \kappa_1 > \kappa_2 \), determine where (i.e., for which values of \( \theta \)) the function,
\[ \kappa(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta, \quad 0 \leq \theta \leq 2\pi \]
achieves its maximum and minimum. The answer shows that the principal directions \( e_1, e_2 \) are unique, up to sign, in this case.

**Gauss Theorema Egregium**

The Weingarten map,
\[ L(X) = -\nabla_X n \]
is an *extrinsically* defined object - it involves the normal to the surface. There is no reason to suspect that the determinant of \( L \), the Gaussian curvature, is *intrinsic*, i.e. can be computed from measurements taken in the surface. But Gauss carried out some courageous computations and made the extraordinary discovery that, in fact, the Gaussian curvature \( K \) *is* intrinsic - i.e., can be computed from the \( g_{ij} \)'s. This is the most important result in the subject - albeit not the prettiest! If this result were not true then the subject of differential geometry, as we know it, would not exist.

We now embark on the same path - courageously carrying out the same computation.

**Some notation.** Introduce the "inverse" metric components, \( g^{ij} \), \( 1 \leq i, j \leq 2 \), by
\[ [g^{ij}] = [g_{ij}]^{-1}, \]
i.e. \( g^{ij} \) is the \( i-j \)th entry of the inverse of the matrix \( [g_{ij}] \). Using the formula,
\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]
we can express \( g^{ij} \) explicitly in terms of the \( g_{ij} \), e.g.
\[ g^{11} = \frac{g_{22}}{g_{11}g_{22} - g_{12}^2}, \quad \text{etc.} \]

Note, in an orthogonal coordinate system, i.e., a proper patch in which \( g_{12} = \langle x_1, x_2 \rangle = 0 \), we have simply,
\[ g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{12} = g^{21} = 0. \]
By definition of inverse, we have
\[
[g_{ij}][g^{ij}] = I
\]
where \(I = \text{identity matrix} = [\delta_{ij}]\), where \(\delta_{ij}\) is the Kronecker delta (cf., Chapter 1),
\[
\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}
\]
and so,
\[
[g_{ij}][g^{ij}] = [\delta_{ij}].
\]
The product formula for matrices then implies,
\[
\sum_k g_{ik} g^{kj} = \delta_{ij}
\]
or, by the Einstein summation convention,
\[
g_{ik} g^{kj} = \delta_{jk}.
\]

Now, let \(M\) be a surface and \(x : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3\) be any proper patch in \(M\). Then,
\[
x = x(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2)),
\]
\[
x_i = \frac{\partial x}{\partial u^i} = \left( \frac{\partial x}{\partial u^i}, \frac{\partial y}{\partial u^i}, \frac{\partial z}{\partial u^i} \right),
\]
\[
x_{ij} = \frac{\partial^2 x}{\partial u^j \partial u^i} = \left( \frac{\partial^2 x}{\partial u^j \partial u^i}, \frac{\partial^2 y}{\partial u^j \partial u^i}, \frac{\partial^2 z}{\partial u^j \partial u^i} \right).
\]

We seek useful expressions for these second derivatives. At any point \(p \in x(U)\), \(\{x_1, x_2, n\}\) form a basis for \(T_p \mathbb{R}^3\). Since at \(p\), \(x_{ij} \in T_p \mathbb{R}^3\), we can write,
\[
x_{ij} = \Gamma_{ij}^1 x_1 + \Gamma_{ij}^2 x_2 + \lambda_{ij} n,
\]
\[
x_{ij} = \sum_{\ell=1}^2 \Gamma_{ij}^\ell x_\ell + \lambda_{ij} n.
\]
or, making use of the Einstein summation convention,
\[
x_{ij} = \Gamma_{ij}^\ell x_\ell + \lambda_{ij} n. \quad (\ast)
\]

We obtain expressions for \(\lambda_{ij}, \Gamma_{ij}^\ell\). Dotting (\ast) with \(n\) gives,
\[
\langle x_{ij}, n \rangle = \Gamma_{ij}^\ell \langle x_\ell, n \rangle + \lambda_{ij} \langle n, n \rangle
\]
\[
\Rightarrow \lambda_{ij} = \langle x_{ij}, n \rangle = \langle n, x_{ij} \rangle
\]
\[
\lambda_{ij} = L_{ij}.
\]
Dotting (*) with \( x_k \) gives,
\[
\langle x_{ij}, x_k \rangle = \Gamma^\ell_{ij} \langle x_\ell, x_k \rangle + \lambda_{ij} \langle n, x_k \rangle \\
\langle x_{ij}, x_k \rangle = \Gamma^\ell_{ij} g_{tk}.
\]
Solving for \( \Gamma^\ell_{ij} \),
\[
\langle x_{ij}, x_k \rangle g^{km} = \Gamma^\ell_{ij} g_{tk} g^{km} \\
= \Gamma^\ell_{ij} \delta^m_\ell \\
\langle x_{ij}, x_k \rangle g^{km} = \Gamma^m_{ij}
\]
Thus,
\[
\Gamma^\ell_{ij} = g^{kt} \langle x_{ij}, x_k \rangle
\]
**Claim.** The quantity \( \langle x_{ij}, x_k \rangle \) is given by,
\[
\langle x_{ij}, x_k \rangle = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)
\]
\[
= \frac{1}{2} (g_{ik,j} + g_{jk,i} - g_{ij,k})
\]
**Proof of Claim.** We use Gauss’ trick of permuting indices.
\[
g_{ij,k} = \frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k} \langle x_i, x_j \rangle
\]
\[
= \langle \frac{\partial x_i}{\partial u^k}, x_j \rangle + \langle x_i, \frac{\partial x_j}{\partial u^k} \rangle
\]
\[
(1) \quad g_{ij,k} = \langle x_{ik}, x_j \rangle + \langle x_i, x_{jk} \rangle \\
(j \leftrightarrow k) \quad (2) \quad g_{jk,i} = \langle x_{ij}, x_k \rangle + \langle x_i, x_{ki} \rangle \\
(i \leftrightarrow j) \quad (3) \quad g_{jk,i} = \langle x_{ji}, x_k \rangle + \langle x_j, x_{ki} \rangle
\]
Then (2) + (3) − (1) gives:
\[
g_{ik,j} + g_{jk,i} - g_{ij,k} = 2 \langle x_{ij}, x_k \rangle
\]
Thus,
\[
\Gamma^\ell_{ij} = \frac{1}{2} g^{kt} (g_{ik,j} + g_{jk,i} - g_{ij,k})
\]
**Remark.** These are known as the Christoffel symbols.
Summarizing, we have,
\[ x_{ij} = \Gamma^\ell_{ij} x_\ell + L_{ij} n, \quad \text{Gauss Formula} \]
where \( L_{ij} \) are the components of the 2nd fundamental form and \( \Gamma^\ell_{ij} \) are the Christoffel symbols as given above. Let us also recall the Weingarten equations (p. 11),
\[ n_j = -L^i_j x_i \]

**Remark.** The vector fields \( x_1, x_2, n \), play a role in surface theory roughly analogous to the Frenet frame for curves. The two formulas above for the partial derivatives of \( x_1, x_2, n \) then play a role roughly analogous to the Frenet formulas.

Now, Gauss takes things one step further and computes the 3rd derivatives,
\[ x_{ijk} = \frac{\partial}{\partial u^k} x_{ij} \]

Thus,
\[ x_{ijk} = (\Gamma^\ell_{ij,k} + \Gamma^m_{ij} \Gamma^\ell_{mk} - L_{ij} L^\ell_k) x_\ell + (L_{ij,k} + \Gamma^\ell_{ij} L^\ell_k) n, \]
and interchanging indices \((j \leftrightarrow k)\),
\[ x_{ikj} = (\Gamma^\ell_{ik,j} + \Gamma^m_{ik} \Gamma^\ell_{mj} - L_{ik} L^\ell_j) x_\ell + (L_{ik,j} + \Gamma^\ell_{ik} L^\ell_j) n. \]

Now, \( x_{ikj} = x_{ijk} \) implies
\[ \Gamma^\ell_{ik,j} + \Gamma^m_{ik} \Gamma^\ell_{mj} - L_{ik} L^\ell_j = \Gamma^\ell_{ij,k} + \Gamma^m_{ij} \Gamma^\ell_{mk} - L_{ij} L^\ell_k = 0 \]
or,
\[ \Gamma^\ell_{ik,j} - \Gamma^\ell_{ij,k} + \Gamma^m_{ik} \Gamma^\ell_{mj} - \Gamma^m_{ij} \Gamma^\ell_{mk} = L_{ik} L^\ell_j - L_{ij} L^\ell_k. \]

These are the components of the famous Riemann curvature tensor. **Observe:** \( R^\ell_{ijk} \) are intrinsic, i.e. can be computed from the \( g_{ij} \)'s (involve 1st and 2nd derivatives of the \( g_{ij} \)'s).
We arrive at,

\[ R^\ell_{ijk} = L^\ell_{ik}L^\ell_j - L^\ell_{ij}L^\ell_k \quad \text{The Gauss Equations.} \]

**Gauss’ Theorem Egregium.** The Gaussian curvature of a surface is intrinsic, i.e. can be computed in terms of the \( g_{ij} \)’s.

**Proof.** This follows from the Gauss equations. Multiply both sides by \( g_{m\ell} \),

\[ g_{m\ell}R^\ell_{ijk} = L^\ell_{ik}g_{m\ell}L^\ell_j - L^\ell_{ij}g_{m\ell}L^\ell_k. \]

But recall (see p. 13),

\[ L^\ell_{mj} = g_{m\ell}L^\ell_j. \]

Hence,

\[ g_{m\ell}R^\ell_{ijk} = L^\ell_{ik}L^\ell_{mj} - L^\ell_{ij}L^\ell_{mk}. \]

Setting \( i = k = 1, m = j = 2 \) we obtain,

\[ g_{2\ell}R^\ell_{121} = L^{11}_1L^{22}_2 - L_{12}L_{21} = \det[L_{ij}]. \]

Thus,

\[ K = \frac{\det[L_{ij}]}{\det[g_{ij}]} \]

\[ K = \frac{g_{2\ell}R^\ell_{121}}{g}, \quad g = \det[g_{ij}]. \]

**Comment.** Gauss’ Theorema Egregium can be interpreted in a slightly different way in terms of isometries. We discuss this point here very briefly and very informally.

Let \( M \) and \( N \) be two surfaces. A one-to-one, onto map \( f : M \to N \) that preserves lengths and angles is called an isometry. (This may be understood at the level of tangent vectors: \( f \) takes curves to curves, and hence velocity vectors to velocity vectors. \( f \) is an isometry \( \iff \) \( f \) preserves angle between velocity vectors and preserve length of velocity vectors.)

**Ex.** The process of wrapping a piece of paper into a cylinder is an isometry.

**Theorem** Gaussian curvature is a bending invariant, i.e. is invariant under isometries, by which we mean: if \( f : M \to N \) is an isometry then

\[ K_N(f(p)) = K_M(p), \]
i.e., the Gaussian curvature is the same at corresponding points.

**Proof**  
$f$ preserves lengths and angles. Hence, in appropriate coordinate systems, the metric components for $M$ and $N$ are the same. By the formula for $K$ above, the Gaussian curvature will be the same at corresponding points.

**Application 1.** The cylinder has Gaussian curvature $K = 0$ (because a plane has zero Gaussian curvature).

**Application 2.** No piece of a plane can be bent into a piece of a sphere without distorting lengths (because $K_{\text{plane}} = 0$, $K_{\text{sphere}} = \frac{1}{r^2}$, $r =$radius).

**Theorem (Riemann).** Let $M$ be a surface with vanishing Gaussian curvature, $K = 0$. Then each $p \in M$ has a neighborhood which is isometric to an open set in the Euclidean plane.

**Exercise 5.10.** Although The Gaussian curvature $K$ is a “bending invariant”, show that the principal curvatures $\kappa_1, \kappa_2$ are not. I.e., show that the principle curvatures are not in general invariant under an isometry. (Hint: Consider the bending of a rectangle into a cylinder).
Chapter 6. Geodesics and the Gauss-Bonnet Theorem

Geodesics in surfaces

Straight lines in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1) Kinematical description: lines are curves of zero acceleration

$$\sigma = p + t \nu$$

$$\frac{d\sigma}{dt} = \nu$$

$$\frac{d^2\sigma}{dt^2} = 0$$

2) Lines are curves of zero curvature

$$s \rightarrow \sigma(s)$$

$$\tau = \frac{d\sigma}{ds}$$

$$\kappa = |\tau'|$$

$$\kappa = |\sigma''|$$

$$\kappa = 0 \Rightarrow \sigma'' = 0$$

3) Lines are shortest curves

What does it mean for this curve to be a geodesic?

$$\frac{d^2\sigma}{dt^2} = 0$$

Can't be our definition because great circles on the sphere are geodesics, but they have acceleration.
\[ \frac{d^2 \sigma}{dt^2} \perp M \] at each point is the correct idea.

**Def:** A geodesic in M is a curve \( \gamma \rightarrow \sigma(t) \) in M s.t. \( \frac{d^2 \sigma}{dt^2} \perp M \) at each point of \( \gamma \).

**Ex.** \( M = S^2 : x^2 + y^2 + z^2 = 1 \)

Equator: \( x^2 + y^2 = 1, z = 0 \)

\[
\begin{align*}
X &= \cos t \\quad \gamma \\
y &= \sin t \\
z &= 0
\end{align*}
\]

\( \sigma(t) = (\cos t, \sin t, 0) \)

\[
\begin{align*}
\frac{d\sigma}{dt} &= (-\sin t, \cos t, 0) \\
\frac{d^2\sigma}{dt^2} &= (-\cos t, -\sin t, 0)
\end{align*}
\]

\[
\begin{align*}
\frac{d^2\sigma}{dt^2} &= -\sigma \perp S^2 \\
\frac{d^2\sigma}{dt^2} &\perp S^2
\end{align*}
\]

\[ \Rightarrow \sigma(t) = (\cos t, \sin t, 0) \text{ geodesic} \]
Def (Rephrased):

\[ X = X^T + X^\perp \]

(\(X\) can be decomposed into two vectors, one tangential \& one perpendicular to the plane \(\langle p, a, e \rangle\))

\[ \text{tan} \ T_p \mathbb{R}^3 \rightarrow T_p M \quad (\text{tan is just an operation}) \]

\[ \text{tan} \ (X) = X^T \]

\[ \{ \bar{x}_1, \bar{x}_2, n \} \]

\[ X = X^1 \bar{x}_1 + X^2 \bar{x}_2 + X^3 n \]

\[ \text{covariant acceleration of motion of } t \rightarrow \sigma(t) \text{ in } M' \]

\[ \frac{\partial}{\partial t} \frac{d\sigma}{dt} = \text{tan} \left( \frac{d^2\sigma}{dt^2} \right) \]

(This is a projection of the acceleration)

Fact: \( t \rightarrow \sigma(t) \) is a geodesic iff its covariant acceleration vanishes; i.e.,

\[ \frac{\partial}{\partial t} \frac{d\sigma}{dt} = 0 \quad \text{geodesic eqn} \]

\[ \frac{d}{dt} \frac{d\sigma}{dt} = 0 \Leftrightarrow \text{tan} \left( \frac{d\sigma}{dt} \right) = 0 \Leftrightarrow \frac{d^2\sigma}{dt^2} \perp M \]
A geodesic \( t \to \sigma(t) \) in \( M \) is necessarily a curve of constant speed

\[
\frac{ds}{dt} = \text{constant}
\]

**Proof:**

\[
\left| \frac{ds}{dt} \right| = \sqrt{\frac{d\sigma}{dt} \cdot \frac{d\sigma}{dt}}
\]

WTS: \( \frac{d\sigma}{dt} \cdot \frac{d\sigma}{dt} = \text{constant} \)

\[
\frac{d}{dt} \left( \frac{d\sigma}{dt} , \frac{d\sigma}{dt} \right) = \frac{d^2\sigma}{dt^2} + \frac{d\sigma}{dt} \cdot \frac{d^2\sigma}{dt^2} = 2 \left( \frac{d\sigma}{dt} \cdot \frac{d\sigma}{dt} \right) = 0
\]

because \( \frac{d^2\sigma}{dt^2} \) is \( \perp \) to \( M \) & \( \frac{d\sigma}{dt} \) is constant \( \perp \) to \( M \)

\[
\Rightarrow \left| \frac{ds}{dt} \right| = c = \frac{ds}{dt} \Rightarrow s = ct
\]

a) \( c = 0 \)

\[
\left| \frac{ds}{dt} \right| = 0 \Rightarrow \frac{d\sigma}{dt} = 0 \Rightarrow \sigma(t) = p \quad \text{(*trivial geodesic*)}
\]

b) \( c \neq 0 \) \( (c > 0) \)

\( t \to \sigma(t) \) regular curve

\( s \to \sigma(s) \)

\[
\frac{ds}{dt} = \frac{ds}{ds} \frac{d\sigma}{dt} = c \frac{d\sigma}{ds}
\]

\[
\frac{d\sigma}{dt} = c \frac{d\sigma}{ds}
\]

\[
\frac{d^2\sigma}{ds^2} = c^2 \frac{d^2\sigma}{ds^2}
\]

Fact: A unit speed curve \( \Gamma \) in \( M \) is a geodesic if \( \sigma' \perp M \)

A simple way to identify geodesics (this condition is sufficient but not necessary)

\( s \to \sigma(s) \)

Fact: Suppose \( \Pi \) is a plane that intersects \( M \) orthogonally at every point of \( M \). Then the curve of intersection, \( s \to \sigma(s) \), is a geodesic
A great circle is the intersection of a plane passing through the origin & a sphere. The plane meets the sphere orthogonally.

Fact above \(\Rightarrow\) all great circles geodesics

Note: circles of latitudes are not geodesics (except for the equator)

Surfaces of Revolution:

\[ \sigma : x = r(t) \quad \sigma'(t) = (r'(t), 0, 0) \]

Claim 1 (every meridian is a geodesic)

every meridian is the intersection of the surface of revolution of a plane through the z-axis
2) Circles of latitude are geodesics at points where the profile curve has vertical tangents (\(r'(\xi) = 0\)).

Geodesic curvature

\[ s \to \sigma(s) \in \mathbb{M} \]

\( T = \text{unit tangent} \)
\( n = \text{unit normal to } \sigma \)
\( S = \text{intrinsic (surface) normal} \)

\[ S = T \times n \quad (|S| = 1) \]

(\( S \) is unique up to sign)

(in his notes, he has \( S = n \times T \) but here he switched to make pictures easier to draw)

\[ T = T(s) \]
\[ S = S(s) \]
\[ n = n(s) \]

\( \{ T(s), S(s), n(s) \} \) orthornormal frame of vectors based at \( \sigma(s) \)

\( X \in T_\sigma \mathbb{R}^3 \)

\[ X = \langle x, T \rangle T + \langle x, S \rangle S + \langle x, n \rangle n \]

\[ x = \sigma'' \]

\[ \sigma'' = \langle \sigma'', T \rangle T + \langle \sigma'', S \rangle S + \langle \sigma'', n \rangle n \]

\[ \langle \sigma'', T \rangle = \langle T', T \rangle = 0 \]
Recall: \( K_n = \) normal component of \( \sigma'' \)
\[
K_n = \langle \sigma'', n \rangle
\]

\((K_n = \mathcal{K}(T, T))\)

\[
\langle \sigma'', n \rangle = K_n
\]

\(\sigma'' = \langle \sigma'', S \rangle S + K_n n\)

\[K_g = \text{geodesic curvature}\]

\[= \text{component of the curvature vector tangent to } M\]

\[K_g = \langle \sigma'', S \rangle\]

\[
\sigma'' = K_g S + K_n n
\]

**Proposition:** \( S \mapsto \sigma(s) \) in \( M \). \( \sigma \) is a geodesic iff \( K_g = 0 \).

**Proof:** \( S \mapsto \sigma(s) \) is a geodesic \( \iff \sigma'' \perp M \iff \sigma'' \perp n \iff K_g = 0\)

**Relationship between** \( K, K_n, K_g \)

\[
\langle \sigma'', \sigma'' \rangle = \langle K_g S + K_n n, K_g S + K_n n \rangle = K_g^2 \langle S, S \rangle + K_n^2 \langle n, n \rangle + 2K_g K_n \langle S, n \rangle = \langle \sigma'', \sigma'' \rangle = |\sigma''|^2
\]

\[K^2 = K_g^2 + K_n^2\]
$S \rightarrow \sigma(s)$ be a unit speed curve in $M$

$$\frac{D}{ds} \frac{ds}{ds} = \tan \left( \frac{d^2 \sigma}{ds^2} \right)$$

$$= \tan \left( K_g S, K_n n \right)$$

$$\frac{D}{ds} \frac{ds}{ds} = K_g S$$

$$\left| \frac{D}{ds} \frac{ds}{ds} \right| = |K_g S| = |K_g|$$

$$|K_g| = \left| \frac{D}{ds} \frac{ds}{ds} \right|$$

"we will show next time that this is intrinsic"
Geodesics

\[ \text{Def: } t \to \sigma(t) \in M \sigma \text{ is a geodesic iff } \frac{d^2\sigma}{dt^2} \perp M \left( \frac{d^2\sigma}{dt^2} = \lambda n \right) \text{ at each point of } \sigma. \]

Covariant acceleration: an intrinsic quantity

\[ \frac{D}{dt} \frac{d\sigma}{dt} = \tan \left( \frac{d\sigma}{dt^2} \right) \]

\[ t \to \sigma(t) \text{ is a geodesic iff } \frac{D}{dt} \frac{d\sigma}{dt} = 0 \]

geodesic equation

Observe: Geodesics are necessarily constant speed curves. \[ \left| \frac{d\sigma}{dt} \right| = \text{const} \neq 0 \]

A unit speed curve \( s \to \sigma(s) \) is a geodesic iff its curvature vector \( \sigma'' \), \( \sigma'' \perp \sigma' \), is perpendicular to \( M \) at each point of \( \sigma \)

\[ s = \tau \times n \]

\[ \sigma'' = \kappa_g s + \kappa_n n \]

\[ \kappa_n = \text{normal curvature} \]

\[ \kappa_g = \text{geodesic curvature} \]

Fact: \( s \to \sigma(s) \) is a geodesic iff \( \kappa_g = 0 \).

\[ \kappa^2 = \kappa_g^2 + \kappa_n^2 \]

Geodesics in Coordinates

\[ \frac{D}{dt} \frac{d\sigma}{dt} = 0 \]

\[ \frac{D}{dt} \frac{d\sigma}{dt} = \tan \left( \frac{d\sigma}{dt^2} \right) \]

proper patch: \( \mathcal{F} : \mathbb{R}^2 \to M \subset \mathbb{R}^3 \)

\[ \mathcal{F} = \mathcal{F}(x, y) \]
$$\frac{d^2 u^k}{dt^2} = \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \frac{dx^k}{dt}$$

$$= \sum_k \left[ \frac{d^2 u^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \right] \frac{dx^k}{dt} + \frac{dx^k}{dt} \left( \frac{dx^k}{dt} \right)$$

$$\frac{d}{dt} \frac{dx^k}{dt} = \tan \left( \frac{dx^k}{dt} \right) = \frac{\sum_k \left[ \frac{d^2 u^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \right] \frac{dx^k}{dt}}{\sum_k \left( \frac{dx^k}{dt} \right)^2}$$

$\rightarrow \sigma(t)$ is a geodesic, i.e. $\frac{d}{dt} \frac{dx^k}{dt} = 0$

$$\Rightarrow \frac{d^2 u^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0, \ k=1,2$$

geodesic equations

$\sigma(t) = \bar{x}(u^1(t), u^2(t))$

$$\frac{d^2 u^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0, \ k=1,2$$

Basic existence & uniqueness theorems for systems of ODE's imply the following:

Proposition: Given $p \in M, X \in T_p M$, $\exists$ a unique geodesic $t \rightarrow \sigma(t)$ satisfying

$\sigma(0) = p$, $\frac{d\sigma}{dt}(0) = X$

$C_r(p) =$ geodesic circle of radius $r$, centered at $p$

**Fact:** Provided $r$ is small enough, $C_r(p)$ will be a smooth curve
$D_r(p) =$ geodesic disk of radius $r$ centered at $p$
$= \text{region bounded by } C_r(p)$

$C_r(p)$ = Euclidean circle of radius $r$
$T_r(p)$ = Euclidean disk of radius $r$

$L(C_r(p)) = 2\pi r$ (if we are in the plane)

$A(D_r(p)) = \pi r^2$ (if we are in the plane)

$S_r =$ sphere of radius $R$

$x^2 + y^2 + z^2 = R^2$

$C_r(p)$ = latitude

$L(C_r(p)) = 2\pi l \theta = 2\pi R \sin \theta = 2\pi r \sin (\frac{r}{R})$

Maclaurin Series:

$\sin \theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin (\frac{r}{R}) = \frac{r}{R} - \frac{1}{6} \frac{r^3}{R^3} + \ldots$

$L(C_r(p)) = 2\pi R \left( \frac{r}{R} - \frac{1}{6} \frac{r^3}{R^3} + \ldots \right) = 2\pi r - \frac{\pi}{3} \left( \frac{r}{R} \right)^3 + \text{HDFT in } r$

$K^2 =$ Gaussian curvature
In general,

\[ L(c, (p)) = 2\pi r - \frac{\pi}{3} k(p) r^3 + \text{HOT} \]

\[ A(D, (p)) = \pi r^2 - \frac{\pi}{12} k(p) r^4 + \text{HOT} \quad (\star) \]

Exercise: Verify $\star$ for $S_k^2$

\[ r_0 < r_0 < r_0 \]
Gauss-Bonnet Theorem

Angle Excess Theorem

**Def.** A triangle $T$ in $M$ is a simple region in $M$ bounded by 3 smooth curve segments.

Simple: it sits inside a proper patch (bounded by a simple non-intersecting closed curve).

**Def.** A geodesic triangle is a triangle whose sides are geodesics.

Comment: A geodesic triangle in $M$ is a Euclidean triangle where its sides are straight lines. For a Euclidean triangle, $A + B + C = \pi$.

**Angle Excess Theorem** For a geodesic triangle,

\[ A + B + C = \pi + \frac{1}{2} \int K ds \]

**Ex.** $M = \mathbb{R}^2$ plane

$K = 0$

$A + B + C = \pi$

**Ex.** $M = S^2_R$

\[ A + B + C = \frac{3}{2} \pi \] (guess from picture)

\[ \frac{1}{2} \int K ds = \frac{1}{2} \iint \frac{1}{R^2} dA = \frac{1}{R^2} \iint dA = \frac{1}{R^2} \text{Area}(T) \]

\[ \frac{1}{R^2} \cdot \frac{4\pi R^2}{2} = \frac{\pi}{2} \]

\[ \Rightarrow A + B + C = \pi + \frac{\pi}{2} = \frac{3\pi}{2}, \text{ just like we guessed} \]
\[ K > 0 \Rightarrow A + B + C > \pi \Rightarrow \text{fat triangles} \]

\[ K < 0 \Rightarrow A + B + C < \pi \Rightarrow \text{skinny triangles} \]

**Topology of Surfaces**

- Sphere
-surface of a potato
- dumbell

From the point of view of topology, these are all equivalent surfaces:

- Torus = surface of a doughnut

- Coffee cup

**Euler Number (Characteristic)**

This number determines if two surfaces are of the same topological type.
(determines if they are diffeomorphisms)

If two surfaces have the same Euler number then they are topologically equivalent.
Definition: A triangulation of $M$ is a decomposition of $M$ into a finite number of triangles $T_1, T_2, \ldots, T_n$ such that
\[ \bigcup_{i=1}^{n} T_i = M \]
and
\[ T_i \cap T_j = \emptyset \text{ or } T_i \cap T_j \text{ is either a common edge or a vertex} \]

Fact: Every compact (closed & bounded) surface can be triangulated.

Example:
Consider a tetrahedron.

\[ X(M) = \text{Euler number} \]
\[ = V - E + F \]

$V$ = number of vertices
$E$ = number of edges
$F$ = number of faces

Example: For the sphere above,
$F = 4, E = 6, V = 4$
$X(M) = 4 - 6 + 4 = 2$

Fact: The Euler number does not depend on the particular triangulation.
$X(M) = \text{topological invariant}$

Fact: Two surfaces are diffeomorphic (have the same topological type) if and only if they have the same Euler characteristic.
\( \chi(T^2) = 0 \) (we can try to compute this on our own or just trust him)
\[ \chi(T^2) \neq \chi(S^2) \]

**Genn & Classification of Compact Surfaces in \( \mathbb{R}^3 \)**

1) \( S^2 \)

Add a handle \( \bigcirc \) - hollow cylinder - tube & assume it's flexible & glue ends of tube to holes in the sphere. This gives 2)

2) Sphere with one handle

Add a handle to get 3)

3) Sphere with two handles

Continuing in this way we can construct a sphere with \( g \) handles, where \( g = \text{genus} = \# \) of handles attached

\( g \) is always a non-negative integer

This is a list of all possible topological types

**Theorem:** Every compact surface in \( \mathbb{R}^3 \) is diffeomorphic to a sphere with \( g \) handles, for some \( g \).
Fact: Let $M^2$ be a compact surface of genus $g$. Then $\chi(M) = 2(1-g)$.

- ex $M = S^2$
  - $\chi(M) = 2$
  - $g = 0$
- ex $M = T^2$
  - $\chi(M) = 0$
  - $g = 1$

Gauss-Bonnet Theorem: If $M$ is a compact surface in $\mathbb{R}^3$ then

$$\int_M K\,dA = 2\pi \chi = 4\pi (1-g)$$

Corollary: Suppose $M$ has everywhere positive Gaussian curvature, $K > 0$. Then $\int_M K\,dA > 0$.

$\Rightarrow 4\pi (1-g) > 0 \Rightarrow g = 0$. Therefore, $M$ is diffeomorphic to a sphere.
Proof (of Gauss-Bonnet): Consequence of the angle excess theorem.

\[ T = \text{geodesic triangle} \]

\[ A + B + C = \pi + \int K ds \]

Triangulate \( M \) into geodesic triangles \( T_1, T_2, \ldots, T_n \)

\[ \int K ds = \frac{\pi}{\text{area}} \int K ds = \sum_{i=1}^{n} \left[ \left( A_i + B_i + C_i \right) - \pi \right] = \sum_{i=1}^{n} \left( A_i + B_i + C_i \right) - n\pi \]

\[ F = n \]

\[ \int K ds = \sum_{i=1}^{n} \left( A_i + B_i + C_i \right) - \pi F \]

Claim: \( \sum_{i=1}^{n} A_i + B_i + C_i = 2\pi \times \text{vertices} \)

\[ \sum_{i=1}^{n} A_i + B_i + C_i = \text{sum of all angles over all triangles} \]

\[ \sum_{i=1}^{n} \text{sum of all angles around each vertex} = 2\pi \Rightarrow \sum_{i=1}^{n} A_i + B_i + C_i = 2\pi \]

\[ \int K ds = 2\pi V - \pi F = 2\pi V - 3\pi F + 2\pi F \]

Claim: \( 3F = 2E \)

Count edges:

1) First try

\[ E = 3F \]

\[ \text{but each edge belongs to each edge} \]

2) Correct

\[ E = \frac{3E}{2} \Rightarrow 2E = 3F \]
\[ \int dS = 2\pi \nu - 3\pi F + 2\pi F - 2\pi \nu - 2\pi \varepsilon + 2\pi F = 2\pi (\nu - \varepsilon + F) \]
\[ = 2\pi \chi(M) \]
Proof of angle excess Theorem

Geodesic Polar Coordinates

\{e_1, e_2\} orthonormal tangent vectors at p.

Exponential polar map: \( \bar{X}(r, \theta) = \) point in M reached by travelling along the geodesic starting at p in the direction \( u_0 \) a distance \( r \).

\( r, \theta = \) geodesic polar coordinates

\( U = \{ (r, \theta) : 0 < r < \infty, 0 < \theta < 2\pi \} \)

\( \bar{X} : U \to M \)

Fact: For \( \varepsilon \) sufficiently small, \( \bar{X} : U \to M \) is a proper patch

\( r \)-curves: \( \theta = \theta_0 \)

\( r \to \bar{X}(r, \theta_0) \)

radial geodesics emanating from p

\( \theta \)-curves: \( r = r_0 \)

\( \theta \to \bar{X}(r_0, \theta) \)

geodesic circles

\( g_{ij} \)'s in these coordinate system?

\( g_{rr}, g_{\theta \theta}, g_{r \theta}, g_{\theta r} \)

\( \frac{\partial \bar{X}}{\partial r} = \bar{X}_r, \quad \frac{\partial \bar{X}}{\partial \theta} = \bar{X}_\theta, \quad \bar{X} = \bar{X}(r, \theta) \)

\( g_{rr} = <\bar{X}_r, \bar{X}_r> = |\bar{X}_r|^2 = 1 \) (since \( r \)-coordinate)
Grassmann Lemma: \( g_{\theta \theta} = 0 \), i.e. geodesic polar coordinate system is an orthogonal coordinate system.

WTS: \( \langle \bar{X}_r, \bar{X}_\theta \rangle = 0 \)

\[
\frac{\partial}{\partial r} \langle \bar{X}_r, \bar{X}_\theta \rangle = \langle \bar{X}_{r r}, \bar{X}_\theta \rangle + \langle \bar{X}_r, \bar{X}_{r \theta} \rangle
\]

\( \bar{X}_r \) is normal to the surface for a geodesic
\( \bar{X}_r \) is normal \( \perp \) tangent \( \Rightarrow \) \( \langle \bar{X}_{r r}, \bar{X}_\theta \rangle = 0 \)

\[
\frac{2}{\partial r} \langle \bar{X}_r, \bar{X}_\theta \rangle = \langle \bar{X}_{r r}, \bar{X}_{r \theta} \rangle = \langle \bar{X}_r, \bar{X}_{r \theta} \rangle = \frac{1}{2} \frac{\partial}{\partial \theta} |\bar{X}_r|^2 = 0
\]

\( \Rightarrow \langle \bar{X}_r, \bar{X}_\theta \rangle = c = \text{constant} \)

Claim: \( c = 0 \)

\[
|\langle \bar{X}_r, \bar{X}_\theta \rangle| = |\bar{X}_r| |\bar{X}_\theta| \cos \theta
\]

\[
|\langle \bar{X}_r, \bar{X}_\theta \rangle| \leq |\bar{X}_\theta|
\]

\[
|\bar{X}_\theta| \rightarrow 0 \quad \text{as} \ r \rightarrow 0 \Rightarrow |\langle \bar{X}_r, \bar{X}_\theta \rangle| \rightarrow 0
\]

\( \theta \rightarrow \bar{X}(r, \theta) \)
\( \theta \rightarrow \bar{X}(0, \theta) = \rho \)
\( \frac{\partial}{\partial \theta} (0, \theta) = 0 \)

This is a lot of hand waving

\[
[g_{ij}] = \begin{bmatrix}
1 & 0 \\
0 & g_{\theta \theta}
\end{bmatrix}
\]

\[
ds^2 = \sum g_{ij} \, du^i \, du^j
\]

\[
d s^2 = g_{rr} \, dr^2 + 2 g_{r \theta} \, dr \, d\theta + g_{\theta \theta} \, d\theta^2
\]

\[
ds^2 = dr^2 + g_{\theta \theta} \, d\theta^2
\]
\[ f = \sqrt{g_{\theta\theta}} = |x_{\theta}| \]
\[ f^2 = g_{\theta\theta} \]
\[ ds^2 = dr^2 + f^2 d\theta^2 \]
\[ f = f(r, \theta) \]

**Gaussian Curvature in Frenetic Polar Coordinates**

\[ K = \frac{g_{\theta\theta} R_{\theta\phi\theta\phi}}{g} \]
\[ g = \det [g_{ij}] \]
\[ R = \text{Poincare curvature tensor} \]

in an orthogonal coordinate system, \( x = x(u', u^2) \), \( g_{uu} = g_{u^2} = 0 \)

\( K \) simplifies to:
\[ K = -\frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u} \left( \frac{\partial^2}{\partial u^2} \right) \right] \]

Now specialize this to geodesic polar coordinates

\[ x = x(r, \theta) \]
\[ r = u', \quad \theta = u^2 \]

\[ K = -\frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial^2}{\partial \theta^2} \right) \right] \]

\[ g_{rr} = 1 \Rightarrow \frac{\partial g_{rr}}{\partial \theta} = 0 \]

\( \Rightarrow K = -\frac{1}{\sqrt{g}} \left( \frac{\partial^2}{\partial r \partial r} \right) \)

\[ g = \det [g_{ij}] = g_{\theta\theta} \]
\[ g = f^2 \Rightarrow \sqrt{g} = f \]

\( \Rightarrow K = -\frac{1}{f^2} \left( \frac{\partial^2}{\partial r^2} \right) \]
\[ K = -\frac{1}{f^2} \frac{\partial^2 f}{\partial r^2} \]
ex. In the $x-y$ plane

geodesic polar coordinates are just regular polar coordinates

$\mathbf{r} : x = r \cos \theta$
$y = r \sin \theta$

$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$

$ds^2 = dr^2 + r^2 d\theta^2$

$g_{\theta \theta} = r^2 \Rightarrow f = r$

$\chi = -\frac{1}{f} \frac{d^2 \chi}{dr^2} = -\frac{1}{r} \frac{d^2 r}{dr^2} = 0$

ex. $S^2 = \text{sphere of radius } 1$

graphic coordinates $\theta, \phi$

on the unit sphere $\theta$ measures the length of the geodesic
since it measures the arc from the north pole

$\theta, \phi = \text{geodesic polar coordinates}$

$f \rightarrow r$
$\phi \rightarrow \theta$

$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$

$= dr^2 + \sin^2 r d\theta^2$

$= \sin^2 r dr^2$

$K = -\frac{1}{\sin^2 r} (-\sin r)$

$K = 1$

$\frac{d^2 \phi}{dr^2} + Kf = 0 \quad \text{Jacobi Equation}$

$p = 0$

IC: $\lim_{r \to 0} f = 0$

$\lim_{r \to 0} \frac{df}{dr} = 1$ (near $p_0$, $f \approx r + \text{HOT}$, $\frac{df}{dr} \approx 1 + \text{HOT}$, but as $r \to 0$, $\frac{df}{dr} = 1$)
Proof of Excess Angle Theorem: Consider geodesic triangle $T$. Make one vertex the center of geodesic polar coordinates.

Parameterize the arc length $s$ and express in terms of geodesic polar coordinates.

\[ s(s) = \mathbf{r}(r(s), \theta(s)) \]

\[ \frac{ds}{ds} = \frac{\partial \mathbf{r}}{\partial r} \frac{dr}{ds} + \frac{\partial \mathbf{r}}{\partial \theta} \frac{d\theta}{ds} \]

\[ s' = r \mathbf{e}_r + \theta \mathbf{e}_\theta ; \quad s = \frac{ds}{ds} \]

\[ l = \langle \sigma', \sigma' \rangle = \langle r' \mathbf{e}_r + \theta' \mathbf{e}_\theta, r' \mathbf{e}_r + \theta' \mathbf{e}_\theta \rangle = (r')^2 + (\theta')^2 f^2 \]

\[ l = (r')^2 + (\theta')^2 f^2 \]

\[ \langle \phi', \mathbf{e}_r \rangle = \left| \sigma' \right| \left| \mathbf{e}_r \right| \cos \phi \]

\[ \cos \phi = \langle \sigma', \mathbf{e}_r \rangle = \langle r' \mathbf{e}_r + \theta' \mathbf{e}_\theta, \mathbf{e}_r \rangle = r' \langle \mathbf{e}_r, \mathbf{e}_r \rangle + \theta' \langle \mathbf{e}_\theta, \mathbf{e}_r \rangle \]

\[ \cos \phi = r' \]

Plug this into:

\[ l = \cos^2 \phi + (\theta')^2 f^2 \]

\[ (\theta')^2 f^2 = 1 - \cos^2 \phi = \sin^2 \phi \]

\[ \theta' = \sin \phi \]

\[ \frac{d}{ds} \cos \phi = \frac{d}{ds} \langle \sigma', \mathbf{e}_r \rangle \]

\[ -\sin \phi \frac{d}{ds} \mathbf{e}_\theta = \langle \sigma'', \mathbf{e}_r \rangle + \langle \sigma', \frac{d}{ds} \mathbf{e}_r \rangle \approx 0 \]
\[-\sin \phi \phi' = \left< \sigma', \frac{d}{ds} \bar{x}_r \right>
\]
\[= \left< \sigma', \bar{x}_{rr} r' + \bar{x}_{r\theta} \theta' \right>
\]
\[= r' \left< \sigma', \bar{x}_{rr} \right> + \theta' \left< \sigma', \bar{x}_{r\theta} \right>
\]
\[-\sin \phi \phi' = \phi' \left< \bar{x}_r, \bar{x}_{r\theta} \right>
\]
\[
\left< \sigma', \bar{x}_{r\theta} \right> = \left< \sigma' \bar{x}_r \theta', \bar{x}_{r\theta} \right> = r' \left< \bar{x}_r, \bar{x}_{r\theta} \right> + \theta' \left< \bar{x}_r, \bar{x}_{r\theta} \right>
\]
\[= r' \left< \bar{x}_r, \bar{x}_{r\theta} \right> = r' \left< r \bar{x}_r, \bar{x}_{r\theta} \right>
\]
\[= r \left< \bar{x}_r, \bar{x}_{r\theta} \right> = r \left< \bar{x}_r, \bar{x}_{r\theta} \right> = 0
\]
\[
\frac{\partial}{\partial \theta} \left< \bar{x}_r, \bar{x}_{r\theta} \right> = \left< \bar{x}_{r\theta}, \bar{x}_r \right> + \left< \bar{x}_r, \bar{x}_{r\theta} \right> = 0
\]
\[
\frac{\partial}{\partial r} \left< \bar{x}_r, \bar{x}_{r\theta} \right> = 0
\]
\[
\frac{\partial^2}{\partial r^2} \left< \bar{x}_r, \bar{x}_{r\theta} \right> = 0
\]
\[
\frac{\partial}{\partial r} r^2 = 2 \left< \bar{x}_r, \bar{x}_{r\theta} \right>
\]
\[
\left< \sigma', \bar{x}_{r\theta} \right> = \theta' \frac{1}{2} \frac{\partial^2 r^2}{\partial \phi'^2}
\]
\[-\sin \phi \phi'' = (\theta')^2 \frac{1}{2} \frac{\partial^2 r^2}{\partial \phi'^2}
\]
\[\text{recall : } \theta' \phi = \sin \phi
\]
\[-\theta' \phi = (\theta')^2 \frac{1}{2} \frac{\partial^2 r^2}{\partial \phi'^2} \quad \theta' \phi = 0
\]
\[-\phi' = \theta' \frac{\partial}{\partial r}
\]
\[-\phi' = -\theta' \frac{\partial}{\partial r}
\]
\[
\frac{\partial \phi}{\partial s} = -\theta' \frac{\partial}{\partial r}
\]
\[
\frac{\partial \phi}{\partial s} = -\theta' \frac{\partial}{\partial r}
\]
\[
\frac{\partial \phi}{\partial \theta} = -\frac{\partial}{\partial r}
\]
x-y plane in polar coordinates

\[ [g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \]

\[ ds^2 = dr^2 + r^2 \, d\theta^2 \]

\[ f = c \]

\[ \left. f \right|_{r=0} = 0 \]

\[ \left. \frac{\partial f}{\partial r} \right|_{r=0} = 1 \]

\[ \lim_{r \to 0} f = 0, \quad \lim_{r \to 0} \frac{\partial f}{\partial r} = 1 \]

Gaussian Curvature in Geodesic Polar Coordinates:

\[ K = - \frac{1}{f} \frac{\partial^2 f}{\partial r^2} \]

\( \iff \frac{\partial f}{\partial r} + Kf = 0 \) (Jacobi's Equation)

Proof of Excess Angle Theorem: Consider a geodesic triangle \( \Gamma \) & place one vertex at the center of geodesic polar coordinates.

\[ \sigma(s) = X \left( r(s), \theta(s) \right), \quad 0 \leq s \leq l \]

\( \sigma : \quad \{ r = r(s) \} \)

\( \sigma : \quad \theta = \theta(s) \)

\( \sigma : \quad r = r(\theta), \quad \theta_0 \leq \theta \leq \theta_1 \)

Along \( \sigma \) let \( \phi \) = angle between \( \sigma \) & the radial geodesic through the point on \( \sigma \)

\[ \phi = \phi(\theta) \]

\[ \frac{d\phi}{d\theta} = -\frac{\partial f}{\partial r} \]

\[ \frac{\partial^2 \phi}{\partial \theta^2}(\theta) = -\frac{\partial f}{\partial r}(r(\theta), \theta) \]
\[ \int K \, ds \]

\[ ds = \sqrt{g} \, dr \, d\theta \]

\[ \sqrt{g} = f \]

\[ ds = f \, dr \, d\theta \]

\[ \int K \, ds = \int K \, f \, dr \, d\theta \]

\[ = - \int_{\theta_0}^{\theta_1} \frac{\partial f}{\partial r} \left. \right|_0^r \, d\theta = \int_{\theta_0}^{\theta_1} \left( \frac{\partial f}{\partial r} - 1 \right) \, d\theta = \int_{\theta_0}^{\theta_1} \left( 1 - \frac{2f}{r} \right) \, d\theta \]

\[ = \int_{\theta_0}^{\theta_1} d\theta - \int_{\theta_0}^{\theta_1} \frac{\partial f}{\partial r} \, d\theta \]

\[ = \theta \bigg|_{\theta_0}^{\theta_1} = \theta_1 - \theta_0 = A \]

\[ \frac{\partial f}{\partial r} = \frac{\partial \phi_1}{\partial \phi} = \phi_{\phi_1} \]

\[ = \phi_{\phi_1} = \phi_0 - \phi_1 = (\pi - B) - (-c) = \pi - B - C \]

\[ A - (\pi - B - C) = A + B + C - \pi \]

#4 from take-home last semester:

\[ \chi(n) \]

\[ M \]