From the Shape of the Earth to the Shape of Black Holes: Aristotle to Hawking and Beyond

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Arts and Sciences Cooper Lecture
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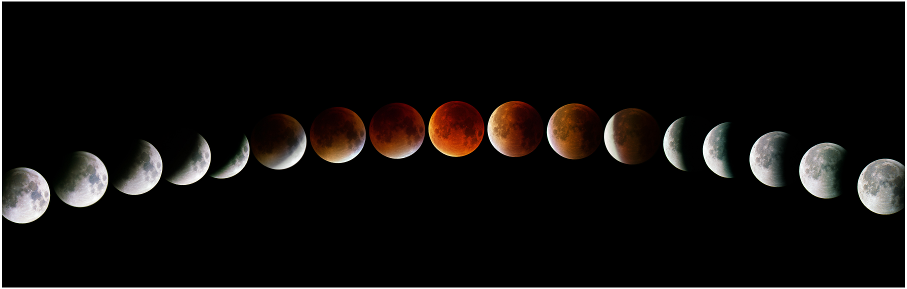
- Aristotle (384-322 BCE) pondered the shape of the earth.
- He sought reasons based on physical evidence that the surface of the earth is a sphere.
- He came up with several compelling arguments, one based on lunar eclipses.

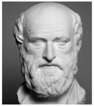
Introduction

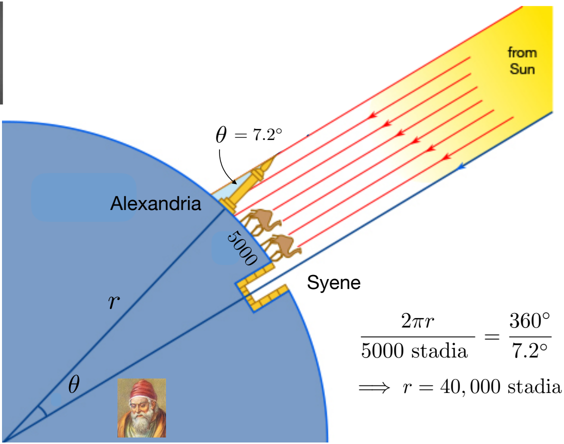


- Aristotle realized that a lunar eclipse occurs when the sun and moon are on opposite sides of the earth, and the moon enters the shadow cast by the earth.
- From the fact that the edge of the shadow is always a circular arc (through all stages of the eclipse), he inferred that the earth is round.

Introduction

- Eratosthenes (276 BC - 194 BC) devised a clever way, using some Euclidean geometry, to measure the radius of the earth.





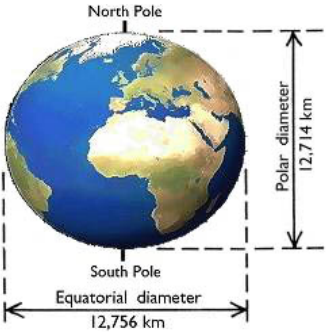
$$\frac{2\pi r}{5000 \text{ stadia}} = \frac{360°}{7.2°}$$

$$\implies r = 40,000 \text{ stadia}$$

- Assuming a reasonable value for the stadion (in terms of modern units), his answer was pretty darn accurate. Perhaps even more important is the simple but elegant geometry behind his method.

Introduction

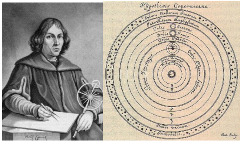
- Today we know that the Earth is not perfectly round. Due to the Earth's rotation, it bulges at the equator.



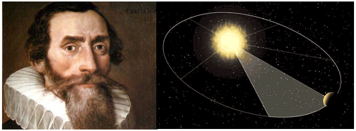
- The surface of the earth is a "stretched-out sphere". A mathematician would say that the surface of the earth is "topologically" a sphere.

Introduction

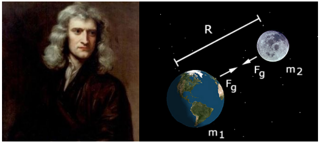
Fast forward in time...



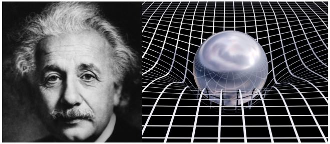
Nicolaus Copernicus (1473 - 1543)



Johannes Kepler (1571 - 1630)



Isaac Newton (1643 - 1727)



Albert Einstein (1879 - 1955)

Introduction

Stephen Hawking (1942 -)

- Hawking pondered the shape of black holes. His findings were reported in this 1972 paper on the theoretical foundations of black holes.

Commun. math. Phys. 25, 152—166 (1972)

Black Holes in General Relativity

S. W. Hawking

Institute of Theoretical Astronomy, University of Cambridge, Cambridge, England

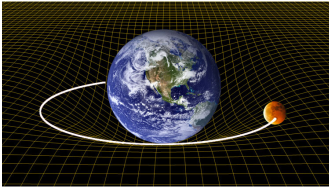
Received October 15, 1971

General relativity and black holes

- Einstein's General Theory of Relativity is a geometric theory of gravity.
- The field equations of General Relativity,

$$\boxed{R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}}$$

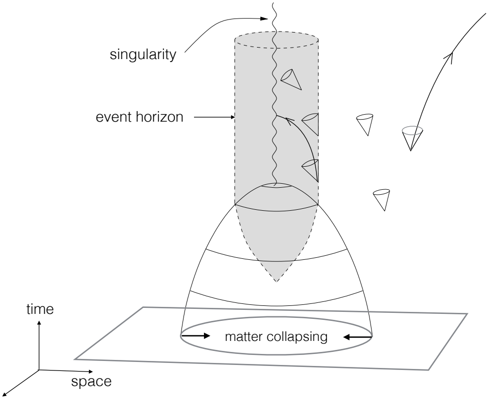
describe how our space-time universe curves in the presence of matter. This curvature is responsible for the effects of gravity.



Matter tells spacetime how to curve, and spacetime tells matter how to move. - John Wheeler

General relativity and black holes

- Black holes are certainly one of the most remarkable predictions of General Relativity.
- The following cartoon depicts the process of gravitational collapse and formation of a black hole.



- The shaded region is the black hole region. The boundary of this region is the black hole event horizon.

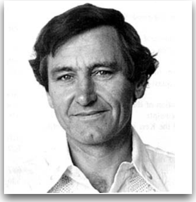
General relativity and black holes

- In 1916, Karl Schwarzschild obtained the first and extremely important exact solution to the Einstein equations.
- The solution was understood to represent the spacetime geometry (and hence gravitational field) outside of a spherical star.

- Only decades later (late 50's, 60's) was it realized that the Schwarzschild solution also describes a spherical black hole - more specifically a stationary (steady state) nonrotating spherical black hole.
- The term "black hole" was popularized by John Wheeler in the late 60's.

General relativity and black holes

- It was almost 50 years later, in 1963, that Roy Kerr discovered an exact solution to the Einstein equations that describes a stationary rotating black hole.



- The Kerr solution is a generalization of the Schwarzschild solution, in that when you set the rotation to zero, it reduces to the Schwarzschild solution.
- *No Hair Theorem:* By a series of mathematical results, it has been shown that the Kerr solution is the only stationary black hole solution to the Einstein equations.
- Largely for this reason, it is widely believed that "true" astrophysical black holes, resulting from gravitational collapse, "settle down" to a Kerr black hole.
- LIGO and the No Hair Theorem.

General relativity and black holes

- A basic step in the proof of the uniqueness of the Kerr solution ("No Hair Theorem") is Hawking's theorem on the shape of black holes: In his paper "Black Holes in General Relativity" he writes:

> *"It is shown that a stationary black hole must have topologically spherical boundary ... These results together with those of Israel and Carter go most of the way towards establishing the conjecture that any stationary black hole is a Kerr solution."*

- What does *topologically spherical* mean?

General relativity and black holes

- A basic step in the proof of the uniqueness of the Kerr solution ("No Hair Theorem") is Hawking's theorem on the shape of black holes: In his paper "Black Holes in General Relativity" he writes:

  > *"It is shown that a stationary black hole must have topologically spherical boundary ... These results together with those of Israel and Carter go most of the way towards establishing the conjecture that any stationary black hole is a Kerr solution."*

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Curvature and the Gauss-Bonnet theorem

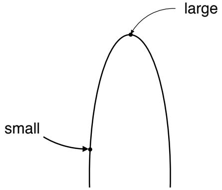
- Hawking's proof of his black hole topology theorem (that the surface of a black hole is topologically spherical) is a beautiful application of the Gauss-Bonnet Theorem.
- The Gauss-Bonnet formula is a remarkable formula that relates the *curvature* of a surface to its *topology* (topological shape).
- The great mathematician Carl Friedrich Gauss (1777-1855) introduced the notion of the curvature of a surface in the early part of the 19th century.

- *Historical side comment:* Bernhard Riemann (1826-1866), who was a student of Gauss, and, himsef a great mathematician, in his thesis generalized the curvature of surfaces to higher dimensional spaces. This turned out to be precisely the mathematical theory Einstein needed to formulate the General Theory of Relativity.

Curvature and the Gauss-Bonnet theorem

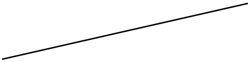
Curvature of curves in the plane.

- Curvature is a measure of bending:

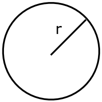


- This can be made precise by considering the rate at which a curve changes *direction* with respect to distance along the curve.

- Examples.
  - Straight line:
    $k = 0$



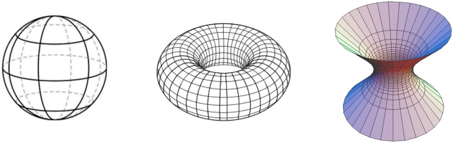
  - Circle of radius r:
    $k = \frac{1}{r}$



    So, small radius implies large curvature and vice versa.

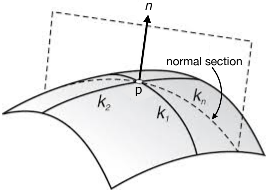
Curvature and the Gauss-Bonnet theorem

Curvature of surfaces in space.



- The curvature of a surface can be defined in terms of the curvature of plane curves.

  k_n = normal curvature = curvature of the normal section wrt the given normal plane.



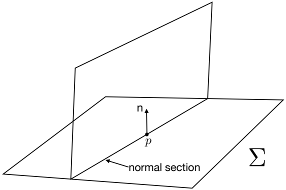
- Let k_1 and k_2 be the maximum and minimum normal curvatures, respectively.

  Then the *Gaussian* curvature at the point p is defined as,

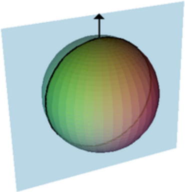
$$K = k_1 \cdot k_2$$

(Sign convention: $k_n > 0$ if normal section bends away from n, negative if it bends towards n.)

Some examples.



- Σ = a plane. All normal sections are lines.
  Hence $k_1 = k_2 = 0$, so,
  $K = k_1 \cdot k_2 = 0 \cdot 0 = 0$



- Σ = sphere of radius r. All normal sections are great circles of radius r.
  Hence, $k_1 = k_2 = \frac{1}{r}$, so,
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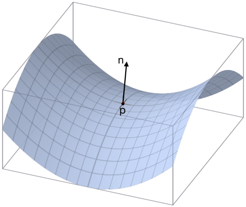
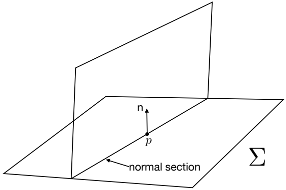


- Σ = "saddle surface". Some normal sections bend down, some bend up.
  Hence, $k_1 > 0$ and $k_2 < 0$, so,
  $K = k_1 \cdot k_2 < 0$.

Some examples.

- Σ = a plane. All normal sections are lines. Hence $k_1 = k_2 = 0$, so, $K = k_1 \cdot k_2 = 0 \cdot 0 = 0$



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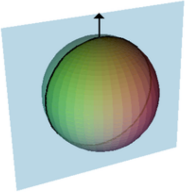
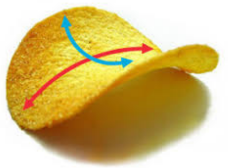


- Pringles have negative Gaussian curvature!



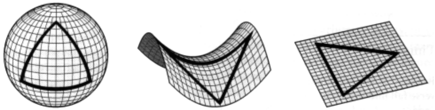
Curvature and the Gauss-Bonnet theorem

Gauss made a remarkable discovery about 'Gaussian curvature'.

Theorema Egregium

The Gaussian curvature of a surface is *intrinsic*, i.e., it can be computed from measurements taken solely within the surface.

- From such measurements two-dimensional creatures can determine whether they live in a flat or positively curved or negatively curved two-dimensional universe.



Positive Curvature Sum of angles $> 180°$

Negative Curvature Sum of angles $< 180°$

Flat Curvature Sum of angles $= 180°$

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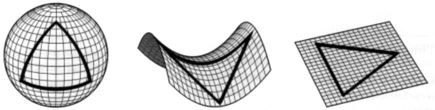
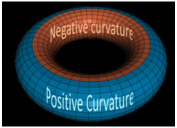


Positive Curvature	Negative Curvature	Flat Curvature
Sum of angles $> 180°$	Sum of angles $< 180°$	Sum of angles $= 180°$

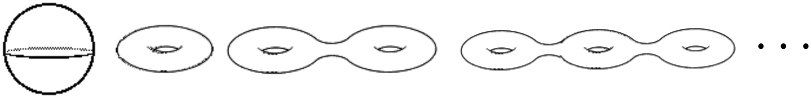
- In general, the curvature can vary from point to point:



Curvature and the Gauss-Bonnet theorem

Theorem (Classification of surfaces)

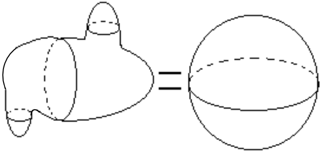
Every *<u>closed</u>* *surface in space is ‘topologically equivalent’ to one of the following surfaces.*

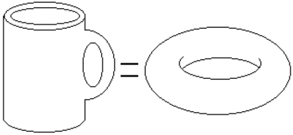


$g = 0$ $g = 1$ $g = 2$ $g = 3$

’g’ is called the genus of the surface.

<u>For example:</u>





Curvature and the Gauss-Bonnet theorem

Gauss-Bonnet Theorem

Let Σ be a closed surface, with Gaussian curvature K. Then,

$$\text{the average value of } K = \frac{4\pi(1-g)}{\text{Area of } \Sigma} \qquad (*)$$

- Check: If $\Sigma =$ sphere of radius r, then

$$\text{avg } K = \frac{4\pi(1-g)}{\text{Area of } \Sigma} = \frac{4\pi}{4\pi r^2} = \frac{1}{r^2}$$

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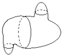
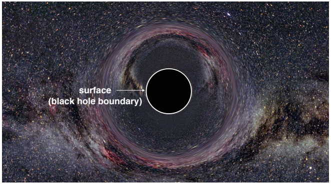
- Comment.
  - $(*)$ is equivalent to:

$$\iint_\Sigma K dA = 4\pi(1-g)$$

  - For example, whether $\Sigma =$  or $\Sigma =$  ,

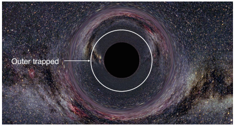
$$\iint_\Sigma K dA = 4\pi$$

Hawkings' black hole topology theorem: *In $3+1$-dimensions, the surface of a stationary (steady state) black hole is topologically a sphere.*



Idea of the proof:

Let $\bar{K}$ be the average Gaussian curvature of the black hole surface Σ. Hawking uses the Einstein equations and the positivity of mass-energy density, to show that if $\bar{K} \leq 0$, there would have to be an outer trapped surface outside the black hole:



But Hawking proves that outer trapped surfaces cannot occur outside the blackhole.

Hence, it must be that $\bar{K} > 0$. But by G.B., $\bar{K} = 4\pi(1-g)/(\text{Area of } \Sigma)$. It follows that $g = 0$, and so Σ is topologically a sphere.

Black holes in higher dimensional spacetimes

- The incompatibility between general relativity and quantum field theory is one of the main open problems in theoretical physics. String theory has been an attempt to resolve this.
- Because it requires extra spatial dimensions, string theory, and related developments, such as the AdS/CFT correspondence, have generated a great deal of interest in gravity in higher dimensions, and in particular, in higher dimensional black holes.
- One of the first questions to arise was:

  Does black hole uniqueness ("No hair theorem") hold in higher dimensions?

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  Does black hole uniqueness ("No hair theorem") hold in higher dimensions?

- With impetus coming from the development of string theory, in 1986, Myers and Perry constructed natural higher dimensional generalizations of the Kerr solution, which, in particular, have spherical horizon topology. (There are natural n-dimensional versions, S^n, of the two dimensional sphere S^2.)

Black holes in higher dimensional spacetimes

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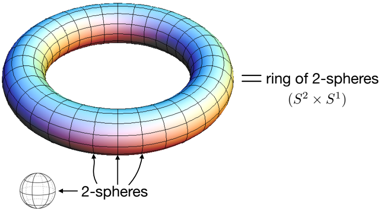
A Rotating Black Ring Solution in Five Dimensions

Roberto Emparan[1,*] and Harvey S. Reall[2]
[1]*Theory Division, CERN, CH-1211 Geneva 23, Switzerland*
[2]*Physics Department, Queen Mary College, Mile End Road, London E1 4NS, United Kingdom*
(Received 8 November 2001; published 21 February 2002)

- But in 2002, Emparan and Reall discovered a remarkable example of a $4+1$ dimensional stationary black hole spacetime having horizon topology a ring of two-dimensional spheres (the black ring):



- Thus in higher dimensions, black hole uniqueness does not hold and horizon topology need not be spherical.

Black holes in higher dimensional spacetimes

- The Emparan and Reall Black Ring caused a great surge of activity in the study of higher dimensional black holes.

  Question: What horizon topologies are allowed in higher dimensions? What restrictions are there?

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Digital Object Identifier (DOI) 10.1007/s00220-006-0019-z

A Generalization of Hawking's Black Hole Topology Theorem to Higher Dimensions

Gregory J. Galloway[1], Richard Schoen[2]

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E-mail: galloway@math.miami.edu
[2] Department of Mathematics, Stanford University, Stanford, CA 94305, USA

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E-mail: galloway@math.miami.edu
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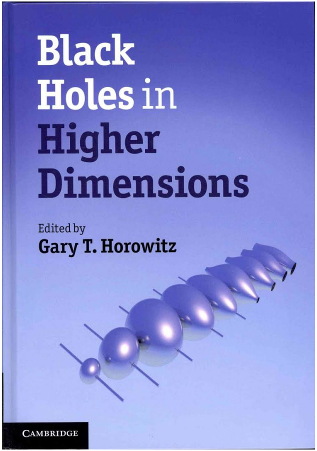
- The result we obtained is a geometric result about black holes in any spacetime dimension, which has topological consequences.
  - In standard $3+1$ spacetime dimensions our result recovers Hawking's theorem.
  - In $4+1$ spacetime dimensions our result gives a complete list of all possible horizon topologies, namely: S^3 , $S^2 \times S^1$, and certain spaces constructed from these.

For more on higher dimensional black holes:



Thank You!