# SUPPLEMENTARY MATERIALS: EFFECTS OF ASYMPTOMATIC INFECTIONS ON THE SPATIAL SPREAD OF INFECTIOUS DISEASES*

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## SM1. Proof of Proposition 3.8.

*Proof.* It follows from the assumption on $L := L^I = L^A$ that the matrix $\hat{L} = CLC^{-1}$ is symmetric, irreducible, and essentially nonnegative but not necessarily Laplacian. Note that the two matrices

$$CV_{11}^{-1}C^{-1} = (CV_{11}C^{-1})^{-1} = (C(D_I - d_I L)C^{-1})^{-1} = (D_I - d_I\hat{L})^{-1} := \hat{V}_{11}^{-1},$$
$$CV_{22}^{-1}C^{-1} = (CV_{22}C^{-1})^{-1} = (C(D_A - d_A L)C^{-1})^{-1} = (D_A - d_A\hat{L})^{-1} := \hat{V}_{22}^{-1}$$

are symmetric and positive whenever $d_I > 0$ and $d_A > 0$. The fact

$$\begin{aligned} &C(F_{11}V_{11}^{-1} + F_{22}V_{22}^{-1})C^{-1} = (CF_{11}C^{-1})CV_{11}^{-1}C^{-1} + (CF_{22}C^{-1})CV_{22}^{-1}C^{-1} \\ =&F_{11}CV_{11}^{-1}C^{-1} + F_{22}CV_{22}^{-1}C^{-1} = F_{11}\hat{V}_{11}^{-1} + F_{22}\hat{V}_{22}^{-1} \end{aligned}$$

implies that $\mathcal{R}_0 = \rho(F_{11}V_{11}^{-1} + F_{22}V_{22}^{-1}) = \rho(F_{11}\hat{V}_{11}^{-1} + F_{22}\hat{V}_{22}^{-1})$.

Denote $\hat{V} = \operatorname{diag}(\hat{V}_{11}, \hat{V}_{22})$. Since $\mathcal{R}_0 = \rho(FV^{-1}) = \rho(F_{11}V_{11}^{-1} + F_{22}V_{22}^{-1}) = \rho(F_{11}\hat{V}_{11}^{-1} + F_{22}\hat{V}_{22}^{-1}) = \rho(F\hat{V}^{-1}) = \rho(\hat{V}^{-1}F)$, there exists a positive eigenvector $\hat{\boldsymbol{v}} = \binom{\hat{\boldsymbol{v}}_I}{\hat{\boldsymbol{v}}_A}$ such that

$$\hat{V}^{-1}F\hat{\boldsymbol{v}} = \mathcal{R}_0\hat{\boldsymbol{v}} \Longleftrightarrow \left(\frac{1}{\mathcal{R}_0}F - \hat{V}\right)\hat{\boldsymbol{v}} = \mathbf{0}.$$

Under assumptions $\theta_i = \theta$ and $\tau_i = \tau$ for all $i \in \Omega$, we can then proceed as the proof of Theorem 3.6 and obtain the derivative of $\mathcal{R}_0$ with respect to $d_I$ as follows

$$\mathcal{R}_0'(d_I) = \frac{\mathcal{R}_0^2}{\theta} \cdot \frac{(\hat{\boldsymbol{v}}_I)^{\mathrm{T}}\hat{L}\hat{\boldsymbol{v}}_I}{(\hat{\boldsymbol{v}}_I + \tau\hat{\boldsymbol{v}}_A)^{\mathrm{T}}B(\hat{\boldsymbol{v}}_I + \tau\hat{\boldsymbol{v}}_A)}.$$

The fact $s(\hat{L}) = s(L) = 0$ implies that the real symmetric matrix $\hat{L}$ is negative semidefinite and hence $\mathcal{R}_0'(d_I)$ is non-positive for any $d_I > 0$.

Suppose there exists some $\hat{d}_I \geq 0$ such that $\mathcal{R}_0'(\hat{d}_I) = 0$. It suffices to show that $\mathcal{R}_0(d_I) \equiv \mathcal{R}_0(\hat{d}_I)$ for any $d_I \in [0, \infty)$. Letting $\hat{\boldsymbol{\alpha}}_0 = C\boldsymbol{\alpha}_0 \gg \mathbf{0}$, we have $\hat{L}\hat{\boldsymbol{\alpha}}_0 = CLC^{-1}C\boldsymbol{\alpha}_0 = \mathbf{0}$. This implies that $\hat{\boldsymbol{\alpha}}_0$ is a right eigenvector of $\hat{L}$ associated with eigenvalue zero. Similar to the proof of Theorem A.1 in Gao [SM1], we claim

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that $(\hat{\boldsymbol{v}}_I)^{\mathrm{T}}\hat{L}\hat{\boldsymbol{v}}_I = 0$ if and only if $\hat{\boldsymbol{v}}_I$ is a multiple of $\hat{\boldsymbol{\alpha}}_0$. In fact, the spectrum of $\hat{L}$ takes the form $\sigma(\hat{L}) = \sigma(L) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ with $\lambda_0 = s(\hat{L}) = 0$ and $\lambda_i < 0$ for $1 \le i \le n-1$. Since $\hat{L}$ is a real symmetric matrix, there exists a basis of orthogonal eigenvectors, denoted by $\{\hat{\boldsymbol{\alpha}}_0, \hat{\boldsymbol{\alpha}}_1, \ldots, \hat{\boldsymbol{\alpha}}_{n-1}\}$, corresponding to eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, respectively. Denoting $\hat{\boldsymbol{v}}_I = \sum_{i=0}^{n-1} \xi_i \hat{\boldsymbol{\alpha}}_i$, we have

$$
\begin{aligned}
(\hat{\boldsymbol{v}}_I)^{\mathrm{T}}\hat{L}\hat{\boldsymbol{v}}_I &= \left(\sum_{i=0}^{n-1} \xi_i \hat{\boldsymbol{\alpha}}_i\right)^{\mathrm{T}} \hat{L} \left(\sum_{j=0}^{n-1} \xi_j \hat{\boldsymbol{\alpha}}_j\right) = \left(\sum_{i=0}^{n-1} \xi_i \hat{\boldsymbol{\alpha}}_i\right)^{\mathrm{T}} \sum_{j=0}^{n-1} \xi_j \hat{L}\hat{\boldsymbol{\alpha}}_j \\
&= \left(\sum_{i=0}^{n-1} \xi_i (\hat{\boldsymbol{\alpha}}_i)^{\mathrm{T}}\right) \sum_{j=0}^{n-1} \xi_j \lambda_j \hat{\boldsymbol{\alpha}}_j = \sum_{i,j} \xi_i \xi_j \lambda_j (\hat{\boldsymbol{\alpha}}_i)^{\mathrm{T}} \hat{\boldsymbol{\alpha}}_j = \sum_{i=0}^{n-1} \lambda_i \xi_i^2 (\hat{\boldsymbol{\alpha}}_i)^{\mathrm{T}} \hat{\boldsymbol{\alpha}}_i.
\end{aligned}
$$

So $(\hat{\boldsymbol{v}}_I)^{\mathrm{T}}\hat{L}\hat{\boldsymbol{v}}_I = 0$ if and only if $\xi_2 = \cdots = \xi_{n-1} = 0$, i.e., $\hat{\boldsymbol{v}}_I = \xi_0 \hat{\boldsymbol{\alpha}}_0$.

By repeating the process in the proof of Theorem 3.6, we have

$$
\mathcal{R}_0 \hat{V}_{11} \hat{\boldsymbol{v}}_I = (F_{11}\hat{V}_{11}^{-1} + F_{12}\hat{V}_{22}^{-1}F_{22}F_{12}^{-1})\hat{V}_{11}\hat{\boldsymbol{v}}_I. \tag{SM1.1}
$$

It follows from $\hat{V}_{11}\hat{\boldsymbol{\alpha}}_0 = D_I\hat{\boldsymbol{\alpha}}_0 - d_I\hat{L}\hat{\boldsymbol{\alpha}}_0 = D_I\hat{\boldsymbol{\alpha}}_0$ and assumption (i) that (SM1.1) becomes

$$
\begin{aligned}
\mathcal{R}_0(\hat{d}_I)D_I\hat{\boldsymbol{\alpha}}_0 &= F_{11}\hat{\boldsymbol{\alpha}}_0 + F_{22}\hat{V}_{22}^{-1}D_I\hat{\boldsymbol{\alpha}}_0 \\
&= CF_{11}C^{-1}\hat{\boldsymbol{\alpha}}_0 + F_{22}CV_{22}^{-1}C^{-1}D_I\hat{\boldsymbol{\alpha}}_0 \\
&= CF_{11}(D_I - d_IL^I)^{-1}(D_I - d_IL^I)C^{-1}\hat{\boldsymbol{\alpha}}_0 + CF_{22}V_{22}^{-1}D_IC^{-1}\hat{\boldsymbol{\alpha}}_0 \\
&= CF_{11}(D_I - d_IL^I)^{-1}(D_I - d_IL^I)\boldsymbol{\alpha}_0 + CF_{22}V_{22}^{-1}D_I\boldsymbol{\alpha}_0 \\
&= C(F_{11}(D_I - d_IL^I)^{-1} + F_{22}V_{22}^{-1})D_I\boldsymbol{\alpha}_0,
\end{aligned}
$$

that is,

$$
\mathcal{R}_0(\hat{d}_I)D_IC^{-1}\hat{\boldsymbol{\alpha}}_0 = \mathcal{R}_0(\hat{d}_I)D_I\boldsymbol{\alpha}_0 = (F_{11}(D_I - d_IL^I)^{-1} + F_{22}V_{22}^{-1})D_I\boldsymbol{\alpha}_0
$$

for any $d_I \ge 0$. Again by the Perron–Frobenius theorem, we have

$$
\mathcal{R}_0(\hat{d}_I) = \rho(F_{11}(D_I - d_IL^I)^{-1} + F_{22}V_{22}^{-1}) = \mathcal{R}_0(d_I),\ \forall d_I \ge 0.
$$

Furthermore, if $\gamma_i^I + \delta_i = \gamma_i^A$ for $i \in \Omega$, then

$$
\hat{V}_{22}\hat{\boldsymbol{\alpha}}_0 = D_A\hat{\boldsymbol{\alpha}}_0 = D_I\hat{\boldsymbol{\alpha}}_0 \Leftrightarrow \hat{V}_{22}^{-1}D_I\hat{\boldsymbol{\alpha}}_0 = \hat{\boldsymbol{\alpha}}_0
$$

and therefore

$$
\mathcal{R}_0(\hat{d}_I)D_I\hat{\boldsymbol{\alpha}}_0 = F_{11}\hat{\boldsymbol{\alpha}}_0 + F_{22}\hat{V}_{22}^{-1}D_I\hat{\boldsymbol{\alpha}}_0 = (F_{11} + F_{22})\hat{\boldsymbol{\alpha}}_0,
$$

which means that $\mathcal{R}_0^{(1)} = \cdots = \mathcal{R}_0^{(n)} = \mathcal{R}_0(\hat{d}_I)$. ∎

**SM2. Proof of Theorem 3.15.**

*Proof.* By Theorem 2.1, each entry of $(S_i^*, I_i^*, A_i^*, R_i^*)$ is bounded for any $d_I > 0$. Thus, up to a sequence of $d_I \to 0+$, we assume that

$$
(S_i^*, I_i^*, A_i^*, R_i^*) \to (\tilde{S}_i, \tilde{I}_i, \tilde{A}_i, \tilde{R}_i),\ i \in \Omega.
$$

Without loss of generality, we let $S_1^* = \min_{i\in\Omega} S_i^*$. It follows from (3.13a) that

$$\begin{aligned}0 =& d_S \sum_{j\in\Omega} L_{1j}^S S_j^* + \Lambda_1 - \beta_1 \frac{I_1^* + \tau_1 A_1^*}{N_1^*} S_1^* - \mu_1 S_1^* \\ \geq& \Lambda_1 - (\max\{1,\tau_1\}\beta_1 + \mu_1 - d_S L_{11}^S) S_1^*.\end{aligned}$$

Thus, with $L_{11}^S < 0$, it follows that $S_1^* \geq \Lambda_1/(\max\{1,\tau_1\}\beta_1 + \mu_1 - d_S L_{11}^S)$ for any $d_I > 0$. Hence $S_i^* \geq S_1^* > 0$ for $i \in \Omega$, giving $\tilde{S}_i > 0$ for $i \in \Omega$.

Claim 1: either $\tilde{\boldsymbol{I}} := (\tilde{I}_1, \dots, \tilde{I}_n) = \mathbf{0}$ or $\tilde{\boldsymbol{I}} \gg \mathbf{0}$. In fact, if $\tilde{I}_k = 0$ for some $k$, then the equation (3.13b) implies

$$\theta_k \beta_k \frac{\tilde{I}_k + \tau_k \tilde{A}_k}{\tilde{N}_k} \tilde{S}_k - (\mu_k + \gamma_k^I + \delta_i)\tilde{I}_k = \theta_k \beta_k \frac{\tau_k \tilde{A}_k}{\tilde{N}_k} \tilde{S}_k = 0$$

and therefore $\tilde{A}_k = 0$. Following (3.13c) and the irreducibility of $L^A$, we must have $\tilde{A}_i = 0$ for $i \in \Omega$. Similarly, the equation (3.13c) implies

$$(1-\theta_i)\beta_i \frac{\tilde{I}_i + \tau_i \tilde{A}_i}{\tilde{N}_i} \tilde{S}_i - (\mu_i + \gamma_i^A)\tilde{A}_i = (1-\theta_i)\beta_i \frac{\tilde{I}_i}{\tilde{N}_i} \tilde{S}_i = 0.$$

Therefore, $\tilde{I}_i = 0$ for $i \in \Omega$ and the claim is proved.

Claim 2: $\tilde{\boldsymbol{I}} \gg \mathbf{0}$. Suppose not, then $\tilde{\boldsymbol{I}} = \mathbf{0}$. From the proof of the first claim, we know $\tilde{A}_i = 0$ for $i \in \Omega$. The equation (3.13d) gives

$$\begin{aligned}&\sum_{i\in\Omega}\left( d_R \sum_{j\in\Omega} L_{ij}^R \tilde{R}_j + \gamma_i^I \tilde{I}_i + \gamma_i^A \tilde{A}_i - \mu_i \tilde{R}_i \right) \\ =& \sum_{i\in\Omega} (\gamma_i^I \tilde{I}_i + \gamma_i^A \tilde{A}_i - \mu_i \tilde{R}_i) = -\sum_{i\in\Omega} \mu_i \tilde{R}_i = 0,\end{aligned}$$

which implies $\tilde{R}_i = 0$ for $i \in \Omega$. Then the equation (3.13a) becomes

$$d_S \sum_{j\in\Omega} L_{ij}^S \tilde{S}_j + \Lambda_i - \mu_i \tilde{S}_i = 0, \ i \in \Omega$$

and therefore $\tilde{\boldsymbol{S}} := (\tilde{S}_1, \dots, \tilde{S}_n) = \boldsymbol{S}^0$. Namely, $E^* \to E_0$ as $d_I \to 0+$.

Define $\hat{I}_i = I_i^*/\varpi \in (0,1]$ and $\hat{A}_i = A_i^*/\varpi \in (0,1]$ for $i \in \Omega$ with $\varpi = \max_{i\in\Omega}\{I_i^*, A_i^*\}$. The equations (3.13b) and (3.13c) can be rewritten as

$$\begin{aligned}&d_I \sum_{j\in\Omega} L_{ij}^I \hat{I}_j + \theta_i \beta_i \frac{\hat{I}_i + \tau_i \hat{A}_i}{N_i^*} S_i^* - (\mu_i + \gamma_i^I + \delta_i)\hat{I}_i = 0, \ i \in \Omega, \\ &d_A \sum_{j\in\Omega} L_{ij}^A \hat{A}_j + (1-\theta_i)\beta_i \frac{\hat{I}_i + \tau_i \hat{A}_i}{N_i^*} S_i^* - (\mu_i + \gamma_i^A)\hat{A}_i = 0, \ i \in \Omega.\end{aligned} \tag{SM2.1}$$

Since $S_i^*/N_i^* \to 1$ as $d_I \to 0+$, it follows from (SM2.1) that

$$\hat{I}_i \to \bar{I}_i \quad \text{and} \quad \hat{A}_i \to \bar{A}_i, \quad \text{for} \quad i \in \Omega, \quad \text{as} \quad d_I \to 0+,$$

which satisfy

$$\begin{aligned}&\theta_i \beta_i (\bar{I}_i + \tau_i \bar{A}_i) - (\mu_i + \gamma_i^I + \delta_i)\bar{I}_i = 0, \ i \in \Omega, \\ &d_A \sum_{j=1}^n L_{ij}^A \bar{A}_j + (1-\theta_i)\beta_i (\bar{I}_i + \tau_i \bar{A}_i) - (\mu_i + \gamma_i^A)\bar{A}_i = 0, \ i \in \Omega,\end{aligned} \tag{SM2.2}$$

or equivalently,

$$(F - V(0))(\bar{\boldsymbol{I}}, \bar{\boldsymbol{A}})^{\mathrm{T}} = \boldsymbol{0} \quad \text{with} \quad V(0) = \operatorname{diag}(D_I, D_A - d_A L^A)$$

for some $\bar{\boldsymbol{I}} = (\bar{I}_1, \dots, \bar{I}_n) \geq \boldsymbol{0}$ and $\bar{\boldsymbol{A}} = (\bar{A}_1, \dots, \bar{A}_n) \geq \boldsymbol{0}$. The irreducibility and essential nonnegativity of $F - V(0)$ and $(\bar{\boldsymbol{I}}, \bar{\boldsymbol{A}})^{\mathrm{T}} > \boldsymbol{0}$ imply that $s(F - V(0)) = 0$. Thus $\mathcal{R}_0(0) = 1$, which contradicts the underlying assumption of the theorem. Since $\tilde{\boldsymbol{I}} \gg \boldsymbol{0}$, the equations (3.13c) and (3.13d) indicate that $\tilde{A}_i > 0$ and $\tilde{R}_i > 0$ for $i \in \Omega$, respectively. ∎

*Remark* SM2.1. For the limit of $E^*$ as $d_I \to 0+$, equation (3.13b) gives

$$\beta_i \frac{\tilde{I}_i + \tau_i \tilde{A}_i}{\tilde{N}_i} \tilde{S}_i = \frac{1}{\theta_i}(\mu_i + \gamma_i^I + \delta_i)\tilde{I}_i, \ i \in \Omega \tag{SM2.3}$$

and substituting it into (3.13a), (3.13c), and (3.13d) yields

$$\begin{aligned}
&d_S \sum_{j\in\Omega} L_{ij}^S \tilde{S}_j + \Lambda_i - \frac{1}{\theta_i}(\mu_i + \gamma_i^I + \delta_i)\tilde{I}_i - \mu_i \tilde{S}_i = 0, \ i \in \Omega, \\
&d_A \sum_{j\in\Omega} L_{ij}^A \tilde{A}_j + \frac{1-\theta_i}{\theta_i}(\mu_i + \gamma_i^I + \delta_i)\tilde{I}_i - (\mu_i + \gamma_i^A)\tilde{A}_i = 0, \ i \in \Omega, \\
&d_R \sum_{j\in\Omega} L_{ij}^R \tilde{R}_j + \gamma_i^I \tilde{I}_i + \gamma_i^A \tilde{A}_i - \mu_i \tilde{R}_i = 0, \ i \in \Omega.
\end{aligned} \tag{SM2.4}$$

Then (SM2.4) can be rewritten in matrix form as follows

$$\begin{aligned}
&\left(\operatorname{diag}(\mu_i) - d_S L^S\right) \tilde{\boldsymbol{S}} = \boldsymbol{\Lambda} - \operatorname{diag}(\theta_i^{-1}) D_I \tilde{\boldsymbol{I}}, \\
&\left(D_A - d_A L^A\right) \tilde{\boldsymbol{A}} = \left(\operatorname{diag}(\theta_i^{-1}) - \mathbb{I}_n\right) D_I \tilde{\boldsymbol{I}}, \\
&\left(\operatorname{diag}(\mu_i) - d_R L^R\right) \tilde{\boldsymbol{R}} = \operatorname{diag}(\gamma_i^I)\tilde{\boldsymbol{I}} + \operatorname{diag}(\gamma_i^A)\tilde{\boldsymbol{A}},
\end{aligned} \tag{SM2.5}$$

where $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)^{\mathrm{T}}$. Solving $\tilde{\boldsymbol{S}}$, $\tilde{\boldsymbol{A}}$, and $\tilde{\boldsymbol{R}}$ from (SM2.5), then substituting the results into (SM2.3) gives a system of quadratic equations in terms of $\tilde{\boldsymbol{I}}$.

## SM3. Proof of Proposition 3.17.

*Proof.* Under the above assumption, the total population of patch $i$ satisfies

$$\frac{dN_i}{dt} = \Lambda_i - \mu_i N_i + d \sum_{j\in\Omega} L_{ij} N_j, \ i \in \Omega,$$

which has a globally asymptotically stable positive equilibrium $\boldsymbol{N}^* = (N_1^*, \dots, N_n^*)$. That is, $\boldsymbol{N}^*$ is the unique positive solution to

$$\Lambda_i - \mu_i N_i^* + d \sum_{j\in\Omega} L_{ij} N_j^* = 0, \ i \in \Omega. \tag{SM3.1}$$

Since $N_i^*$ is bounded regardless of the selection of dispersal rate $d$ and connectivity matrix $L$, it follows from (SM3.1) that

$$\lim_{d\to\infty} (N_1^*(d), \dots, N_2^*(d)) = m\boldsymbol{\alpha}$$

for some $m > 0$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$ is the right positive eigenvector with eigenvalue zero of matrix $L$ satisfying $\sum_{i\in\Omega} \alpha_i = 1$. Summing up (SM3.1) over $i \in \Omega$ gives

$$\sum_{i\in\Omega}(\Lambda_i - \mu_i N_i^*(d)) = 0,\ \forall d \geq 0,$$

implying that

$$\lim_{d\to\infty}\left(\sum_{i\in\Omega}(\Lambda_i - \mu_i N_i^*(d))\right) = \sum_{i\in\Omega}\Lambda_i - \sum_{i\in\Omega}\mu_i m \alpha_i = 0,\ \text{i.e., } m = \sum_{i\in\Omega}\Lambda_i \Bigg/ \sum_{i\in\Omega}\mu_i\alpha_i\ .$$

Furthermore, the equilibrium equations (3.13) imply that

$$E^* \to (m^S\boldsymbol{\alpha}, m^I\boldsymbol{\alpha}, m^A\boldsymbol{\alpha}, m^R\boldsymbol{\alpha}) \ \text{ as } \ d \to \infty,$$

where $m^\natural > 0$ for $\natural \in \{S, I, A, R\}$ obey

$$\begin{aligned}
&m^S + m^I + m^A + m^R = m,\\
&\sum_{i\in\Omega}\left(\theta_i\beta_i\frac{m^I + \tau_i m^A}{m}m^S\alpha_i - (\mu_i + \gamma_i^I)m^I\alpha_i\right) = 0,\\
&\sum_{i\in\Omega}\left((1-\theta_i)\beta_i\frac{m^I + \tau_i m^A}{m}m^S\alpha_i - (\mu_i + \gamma_i^A)m^A\alpha_i\right) = 0,\\
&\sum_{i\in\Omega}\left(\gamma_i^I m^I\alpha_i + \gamma_i^A m^A\alpha_i - \mu_i m^R\alpha_i\right) = 0.
\end{aligned} \tag{SM3.2}$$

Denote $\tilde{m}^\natural = m^\natural/m \in (0, 1)$ for $\natural \in \{S, I, A, R\}$. Then (SM3.2) can be rewritten as

$$\tilde{m}^S + \tilde{m}^I + \tilde{m}^A + \tilde{m}^R = 1, \tag{SM3.3a}$$

$$(p_{22}\tilde{m}^I + p_{23}\tilde{m}^A)\tilde{m}^S - q_{22}\tilde{m}^I = 0, \tag{SM3.3b}$$

$$(p_{32}\tilde{m}^I + p_{33}\tilde{m}^A)\tilde{m}^S - q_{33}\tilde{m}^A = 0, \tag{SM3.3c}$$

$$p_{42}\tilde{m}^I + p_{43}\tilde{m}^A - p_{44}\tilde{m}^R = 0, \tag{SM3.3d}$$

where

$$\begin{aligned}
p_{22} &= \sum_{i\in\Omega}\theta_i\beta_i\alpha_i, & p_{23} &= \sum_{i\in\Omega}\theta_i\beta_i\tau_i\alpha_i, & q_{22} &= \sum_{i\in\Omega}(\mu_i + \gamma_i^I)\alpha_i,\\
p_{32} &= \sum_{i\in\Omega}(1-\theta_i)\beta_i\alpha_i, & p_{33} &= \sum_{i\in\Omega}(1-\theta_i)\beta_i\tau_i\alpha_i, & q_{33} &= \sum_{i\in\Omega}(\mu_i + \gamma_i^A)\alpha_i,\\
p_{42} &= \sum_{i\in\Omega}\gamma_i^I\alpha_i, & p_{43} &= \sum_{i\in\Omega}\gamma_i^A\alpha_i, & p_{44} &= \sum_{i\in\Omega}\mu_i\alpha_i.
\end{aligned}$$

It follows from (SM3.3b) and (SM3.3c) that

$$\frac{p_{22}\tilde{m}^I + p_{23}\tilde{m}^A}{p_{32}\tilde{m}^I + p_{33}\tilde{m}^A} = \frac{q_{22}\tilde{m}^I}{q_{33}\tilde{m}^A} \Leftrightarrow \frac{p_{22}\kappa + p_{23}}{p_{32}\kappa + p_{33}} = \frac{q_{22}}{q_{33}}\kappa$$

gives

$$\kappa = \frac{\tilde{m}^I}{\tilde{m}^A} = \frac{(p_{22}q_{33} - p_{33}q_{22}) + \sqrt{(p_{22}q_{33} - p_{33}q_{22})^2 + 4p_{32}q_{22}p_{23}q_{33}}}{2p_{32}q_{22}}. \tag{SM3.4}$$

Thus, (SM3.3c) implies that

$$\tilde{m}^S = \frac{q_{33}\tilde{m}^A}{p_{32}\tilde{m}^I + p_{33}\tilde{m}^A} = \frac{q_{33}}{p_{32}\kappa + p_{33}}. \tag{SM3.5}$$

Solving $\tilde{m}^A$ and $\tilde{m}^R$ from (SM3.3a) and (SM3.3d) yields

$$\begin{aligned} \tilde{m}^A &= \frac{(1-\tilde{m}^S)p_{44} - \tilde{m}^I(p_{42}+p_{44})}{p_{43}+p_{44}}, \\ \tilde{m}^R &= \frac{(1-\tilde{m}^S)p_{43} + \tilde{m}^I(p_{42}-p_{43})}{p_{43}+p_{44}}. \end{aligned} \tag{SM3.6}$$

Combining (SM3.4) and (SM3.6) gives

$$\tilde{m}^A = \frac{(1-\tilde{m}^S)p_{44}}{p_{43}+p_{44}+(p_{42}+p_{44})\kappa}.$$

Lastly, $\tilde{m}^I$ and $\tilde{m}^R$ are solvable using (SM3.4) and (SM3.6), respectively.

The overall nonsusceptible ratio at the limiting endemic equilibrium $E^*(\infty)$ is

$$\frac{m^S\sum_{i\in\Omega}\alpha_i}{(m^S+m^I+m^A+m^R)\sum_{i\in\Omega}\alpha_i} = \frac{m^S}{m} = \tilde{m}^S.$$

If $\theta_i = \theta$ or $\tau_i = \tau$ for all $i \in \Omega$, then $p_{22}p_{33} = p_{23}p_{32}$ and hence (SM3.4) becomes

$$\begin{aligned} \kappa &= \frac{(p_{22}q_{33} - p_{33}q_{22}) + \sqrt{(p_{22}q_{33}-p_{33}q_{22})^2 + 4p_{33}q_{22}p_{22}q_{33}}}{2p_{32}q_{22}} \\ &= \frac{(p_{22}q_{33} - p_{33}q_{22}) + (p_{22}q_{33}+p_{33}q_{22})}{2p_{32}q_{22}} = \frac{p_{22}q_{33}}{p_{32}q_{22}}. \end{aligned}$$

Substituting it into (SM3.5) yields

$$\frac{1}{\tilde{m}^S} = \left(\frac{p_{22}q_{33}}{q_{22}} + p_{33}\right)\frac{1}{q_{33}} = \frac{p_{22}}{q_{22}} + \frac{p_{33}}{q_{33}} = \mathcal{R}_{0I}(\infty) + \mathcal{R}_{0A}(\infty) = \mathcal{R}_0(\infty,\infty).$$

The last two equalities are due to Theorem 3.3. ∎

**SM4. Patch Model with Multiple Infectious Subgroups.** We generalize model (2.1) from two infectious groups to $m$ infectious groups. The number of individuals of infectious group $k$ in patch $i$ is denoted by $I_{ik}$ for $i \in \Omega = \{1,\dots,n\}$ and $k \in \Psi = \{1,\dots,m\}$. The transmission dynamics of the epidemic patch model with multiple infectious groups are described the following system of ordinary differential equations ($1 \le i \le n$)

$$\begin{aligned} \frac{dS_i}{dt} &= d_S\sum_{j\in\Omega}L_{ij}^S S_j + \Lambda_i - \beta_i\sum_{l\in\Psi}\tau_{il}I_{il}\frac{S_i}{N_i} - \mu_i S_i, \\ \frac{dI_{ik}}{dt} &= d_{I_k}\sum_{j\in\Omega}L_{ij}^{I_k}I_{jk} + \theta_{ik}\beta_i\sum_{l\in\Psi}\tau_{il}I_{il}\frac{S_i}{N_i} - (\mu_i+\gamma_{ik}+\delta_{ik})I_{ik}, k\in\Psi, \\ \frac{dR_i}{dt} &= d_R\sum_{j\in\Omega}L_{ij}^R R_j + \sum_{k\in\Psi}\gamma_{ik}I_{ik} - \mu_i R_i. \end{aligned} \tag{SM4.1}$$

Here $N_i = S_i + \sum_{k\in\Psi} I_{ik} + R_i$ is the total population size of patch $i$, the proportion of new infections in patch $i$ that progress to infectious group $k$ is $\theta_{ik}$ satisfying $0 \le \theta_{ik} \le 1$ and $\sum_{k\in\Psi} \theta_{ik} = 1$, and $d_{I_k}$ and $L^{I_k} = (L_{ij}^{I_k})_{n\times n}$ are the dispersal rate and connectivity matrix of infectious group $k$, respectively.

For convenience, suppose all connectivity matrices are irreducible. It is easy to check that the generalized model (SM4.1) still has a unique disease-free equilibrium $E_0 = (\boldsymbol{S}^0, \mathbf{0}, \dots, \mathbf{0}, \mathbf{0})$. Direct calculations give the incidence and transition matrices

$$F = (F_{kl})_{m\times m} \quad \text{and} \quad V = \operatorname{diag}(V_{11}, \dots, V_{mm}),$$

where the blocks

$$\begin{aligned} F_{kl} &= \operatorname{diag}(\theta_{1k}\beta_1\tau_{1l}, \dots, \theta_{nk}\beta_n\tau_{nl}), \\ V_{kk} &= \operatorname{diag}(\mu_1 + \gamma_{1k} + \delta_{1k}, \dots, \mu_n + \gamma_{nk} + \delta_{nk}) - d_{I_k} L^{I_k}. \end{aligned}$$

Following the next generation matrix method, the basic reproduction number of model (SM4.1) is defined as

$$\mathcal{R}_0 = \rho(FV^{-1}).$$

Proposition SM4.1. *The basic reproduction number of model* (SM4.1) *is*

$$\mathcal{R}_0 = \rho\left(\sum_{k\in\Psi} F_{kk}V_{kk}^{-1}\right) = \rho\left(\sum_{k\in\Psi} V_{kk}^{-1}F_{kk}\right).$$

*Proof.* We introduce

$$P = \operatorname{diag}(P_1, \dots, P_m),\ Q = (\mathbb{I}_n)_{m\times m} \quad \text{and} \quad R = \operatorname{diag}(R_1, \dots, R_m),$$

where

$$P_k = \operatorname{diag}(\theta_{1k}, \dots, \theta_{nk}) \quad \text{and} \quad R_l = \operatorname{diag}(\beta_1\tau_{1l}, \dots, \beta_n\tau_{nl}),\ k, l \in \Psi.$$

It is easy to verify that $F = PQR$ which implies that

$$FV^{-1} = PQRV^{-1} = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_m \end{pmatrix} \begin{pmatrix} \mathbb{I}_n & \cdots & \mathbb{I}_n \\ \vdots & \ddots & \vdots \\ \mathbb{I}_n & \cdots & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} W_1 & & \\ & \ddots & \\ & & W_m \end{pmatrix},$$

where $W_l = R_l V_{ll}^{-1}$ for $l \in \Psi$. Multiplying the $m$-th row of $\lambda\mathbb{I}_{mn} - FV^{-1}$ by $-P_kP_m^{-1}$ and then adding it to the $k$-th row for all $k \in \{1, \dots, m-1\}$ yield

$$\begin{aligned} &|\lambda\mathbb{I}_{mn} - FV^{-1}| \\ =& \begin{vmatrix} \lambda\mathbb{I}_n - P_1W_1 & -P_1W_2 & \cdots & -P_1W_{m-1} & -P_1W_m \\ -P_2W_1 & \lambda\mathbb{I}_n - P_2W_2 & \cdots & -P_2W_{m-1} & -P_2W_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -P_{m-1}W_1 & -P_{m-1}W_2 & \cdots & \lambda\mathbb{I}_n - P_{m-1}W_{m-1} & -P_{m-1}W_m \\ -P_mW_1 & -P_mW_2 & \cdots & -P_mW_{m-1} & \lambda\mathbb{I}_n - P_mW_m \end{vmatrix} \\ =& \begin{vmatrix} \lambda\mathbb{I}_n & 0 & \cdots & 0 & -\lambda P_1P_m^{-1} \\ 0 & \lambda\mathbb{I}_n & \cdots & 0 & -\lambda P_2P_m^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda\mathbb{I}_n & -\lambda P_{m-1}P_m^{-1} \\ -P_mW_1 & -P_mW_2 & \cdots & -P_mW_{m-1} & \lambda\mathbb{I}_n - P_mW_m \end{vmatrix}. \end{aligned}$$

Multiplying the $l$-th column of the above by $P_l P_m^{-1}$ and adding it to the $m$-th column for all $l \in \{1, \ldots, m-1\}$ give

$$|\lambda \mathbb{I}_{mn} - FV^{-1}| = \begin{vmatrix} \lambda \mathbb{I}_n & 0 & \cdots & 0 & 0 \\ 0 & \lambda \mathbb{I}_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda \mathbb{I}_n & 0 \\ -P_m W_1 & -P_m W_2 & \cdots & -P_m W_{m-1} & \lambda \mathbb{I}_n - \sum_{l \in \Psi} P_l W_l \end{vmatrix}.$$

Therefore, $|\lambda \mathbb{I}_{mn} - FV^{-1}| = \lambda^{(m-1)n} \left|\lambda \mathbb{I}_n - \sum_{l \in \Psi} P_l W_l\right|$ with $P_l W_l = P_l R_l V_{ll}^{-1} = F_{ll} V_{ll}^{-1}$. This completes the proof of the first equality, whereas the second equality can be proved by similarly considering $V^{-1}F = ((V^{-1}P)Q)R$. ◻

By the comparison principle and persistence theory, we can again establish sharp threshold dynamics in terms of $\mathcal{R}_0$ for model system (SM4.1). Lower and upper bounds on $\mathcal{R}_0$ similar to Theorem 3.3 can be obtained. Under certain conditions (similar to Theorem 3.6 and Proposition 3.8, e.g., $\theta_{ik} = \theta_k$ and $\tau_{ik} = \tau_k$ for all $i \in \Omega$, and $L^{I_k}$ is symmetric for all $k \in \Omega$), some monotone decreasing results on $\mathcal{R}_0$ with respect to dispersal rates can be expected. Theorem 3.10 holds if only one infectious group moves, whereas it fails if there are more than two infectious groups move. When only two patches are concerned, it is easy to verify that the proof of Proposition 3.11 is still valid (one only needs to remove the unnecessary restriction "$a_{11}a_{22} > a_{12}a_{21}$"). Therefore, for model (SM4.1), the basic reproduction number $\mathcal{R}_0$ is either strictly increasing, or strictly decreasing, or constant with respect to the dispersal rate of any given infectious group in a two-patch environment.

## REFERENCE

[SM1] D. Gao, *Travel frequency and infectious diseases*, SIAM J. Appl. Math., 79 (2019), pp. 1581–1606.