

# THESIS PROJECT

## 1) Basic definitions

Lorentzian geometry is the mathematical framework that supports some of the most important theories in modern physics, like general relativity and string theory.

From a purely mathematical point of view, a *Lorentzian manifold*  $M$  is a smooth manifold endowed with a symmetric and non-degenerate bilinear form  $g$ , called the *metric*, of signature  $(-, +, \dots, +)$ .

The main difference between Lorentzian and Riemannian geometry is the existence of non zero tangent vectors  $X$  of length 0, that is,  $g(X, X) = 0$ . Such vectors are called *null*. We will also say that  $X \in T_p M$  is *timelike* or *spacelike* if  $g(X, X) < 0$  or  $g(X, X) > 0$  respectively. If  $X$  is timelike or null we call it a *causal* vector.

The set  $\{X \in T_p M : g(X, X) = 0\}$  is called the *null cone* at  $p \in M$ . Null cones are very important objects in Lorentzian geometry. In fact, it can be shown that the collection of null cones determines the metric  $g$  up to a conformal factor.

Each null cone disconnects the set of timelike vectors into two components. A consistent continuous choice over all  $M$  of one of these components is called a *time orientation* and the tangent vectors belonging to the chosen components are then called *future pointing*. Naturally, we say  $M$  is time orientable if it has a time orientation.

In what follows, a *space-time*  $M$  is a connected, time orientable, 4-dimensional Lorentzian manifold.

In the context of general relativity, every point in a space-time  $M$  represents an *event*. The coordinate corresponding to the negative sign of  $g$  is interpreted as the *time*, while the remaining three coordinates are thought as *spatial* coordinates.

## 2) Causality

The concept of *causality* is crucial in the study of Lorentzian manifolds.

Let  $\gamma: [a, b] \rightarrow M$  be a smooth curve. We say that  $\alpha$  is a timelike (spacelike, null, causal) curve provided that  $\gamma'(t)$  is a timelike (spacelike, null, causal) tangent vector for each  $t \in [a, b]$ .

We say that  $p \in M$  *chronologically (causally) precedes*  $q \in M$  if there is a timelike (causal) path from  $p$  to  $q$ . Then for each  $p \in M$  we define

$$\begin{aligned}\mathcal{I}^+(p) &= \{q \in M : p \text{ chronologically precedes } q\} \\ \mathcal{J}^+(p) &= \{q \in M : p \text{ causally precedes } q\}\end{aligned}\tag{1}$$

Such sets are called the *chronological and causal future of  $p$*  respectively.  $\mathcal{I}^-(p)$  and  $\mathcal{J}^-(p)$  are defined time dually.

On physical grounds, it is desirable for a space-time not to have any closed timelike curves. Such a space-time is said to satisfy the *chronology condition*. If this condition is not met, then all kinds of paradoxes arise: for instance someone could take a trip from which he returns before his departure.

Causality is related to the topology of the manifold. For instance, all compact Lorentz manifolds have a closed timelike curve. Thus, compact Lorentz manifolds have traditionally held less interest for relativists .

In addition to the chronological condition, there is a range of stronger conditions restricting causal pathologies. Now we define two of them:

We say  $M$  is *strongly causal* if for each  $p \in M$ , there exist arbitrary small neighborhoods  $U$  of  $p$  such that any causal curve which starts in  $U$  and then leaves  $U$  never returns to  $U$ .  $M$  is *globally hyperbolic* if it is strongly causal and for any  $p, q \in M$  the sets of the form  $\mathcal{J}^+(p) \cap \mathcal{J}^-(q)$  are compact.

### 3) The Einstein Equations

Cornerstone in the theory of general relativity are the Einstein equations, given in coordinates by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi T_{ij} \quad (2)$$

where  $g$  is a Lorentzian metric on  $M$ ,  $R_{ij}$  the components of its Ricci tensor,  $R$  the scalar curvature and  $\Lambda$  a constant called the *cosmological constant*, whereas on the right hand side we have a physical quantity, the energy-momentum tensor  $T_{ij}$ .

This is a system of partial differential equations on the unknowns  $g_{ij}$ . The high complexity of the system permits to find exact solutions only when strong symmetry assumptions are made. Thus, when dealing with more general scenarios one is unable to find exact solutions, hence the study of the asymptotic behavior of solutions becomes of capital importance.

This notion can be made more precise:

A set  $S$  is called *achronal* if no two points can be joined by a timelike path. A *Cauchy surface*  $S$  for  $M$  is an achronal  $C^0$  hypersurface  $S$  in  $M$  which is met by every inextendible causal curve in  $M$ . If  $S$  is a Cauchy surface for  $M$  then it can be shown that  $M$  is homeomorphic to  $\mathbb{R} \times S$ . Moreover, the homeomorphism can be arranged so that  $\{t = \text{constant}\} \times S$  is a Cauchy surface for all  $t$  [13].

Now, if we take a slice  $\{t = \text{constant}\}$  -that is, a frozen picture of the universe- and consider some initial data on this surface (the metric and the second fundamental form it inherits from  $M$ ) and let the system evolve under the flow given by the Einstein Equations we end up with an Initial Value Problem. Then the ultimate fate of the universe might be thought as the state of the system when  $t$  approaches infinity.

### 4) Conformal Infinity

Let us introduce now the notion of conformal infinity as defined by Penrose, since it will enable us to deal with asymptotics in a nice way.

Assume that there exists a space-time-with-boundary  $(\tilde{M}, \tilde{g})$  and a smooth function  $\Omega$  on  $\tilde{M}$  such that

- $M$  is the interior of  $\tilde{M}$ . Hence  $\tilde{M} = M \cup \mathcal{J}$  where  $\mathcal{J} = \partial\tilde{M}$ .
- $\tilde{g} = \Omega^2 g$ , where  $\Omega > 0$  on  $M$ ,  $\Omega = 0$  on  $\mathcal{J}$  and  $d\Omega \neq 0$  along  $\mathcal{J}$ .

In this case, we call  $\mathcal{J}$  the conformal boundary of  $M$ .

With the above conventions, we say that  $M$  is of *Minkowski (de Sitter, anti de Sitter) type* if  $\mathcal{J}$  is null (spacelike, timelike) respectively.

### 5) The Rigidity Philosophy.

This technique consist on comparing the geometry of a general manifold  $M$  with that of a simply connected model space  $M_K$  of constant curvature  $K$ . A typical conclusion is that  $M$  retains particular geometrical properties under certain strict bounds. Once this has been established, it is usually possible to conclude that  $M$  retains topological properties of  $M_K$  as well.

The distinction between strict and weak bounds is important, since this may reflect the difference between the geometry of say the sphere and that of Euclidean space. However, it is often the case that a conclusion which becomes false when one relaxes the condition of strict inequality to weak inequality can be shown to fail only under very special circumstances. Results like this are known as *rigidity theorems* and usually require delicate global arguments [4].

Recall that the model spaces for general relativity of constant scalar curvature  $K = 0, 1, -1$  are *Minkowsky space, de Sitter space* and *anti de Sitter space* respectively, and all of them imbed conformally, in the sense of Penrose, into the *Einstein static cylinder*.

## 6) Splitting Theorems

One of the principal examples of the rigidity philosophy arise in the context of Splitting Theorems. As an example, let us recall a Riemannian example, the celebrated

**Cheeger-Gromoll Splitting Theorem:** *Let  $(N, h)$  be a complete Riemannian manifold of dimension  $n \geq 2$  which satisfies the curvature condition  $\text{Ric}(X, X) \geq 0$  for all  $X \in T(M)$  and which contains a line (i.e. a complete geodesic  $c: \mathbb{R} \rightarrow (N, h)$  joining two different ends of  $N$  and minimizing distance between any two of its points). Then  $(N, h)$  may be written uniquely as an isometric product  $N_1 \times \mathbb{R}^k$  where  $N_1$  contains no lines and  $\mathbb{R}^k$  is given the standard flat metric [5].*

Now, it follows from previous work of Gromoll and Meyer that a complete Riemannian manifold of dimension  $n \geq 2$  such that  $\text{Ric}(X, X) > 0$  for all nonzero  $X \in T(M)$  is connected at infinity [14].

Rigidity comes in the following manner: Suppose now that  $\text{Ric}(X, X) \geq 0$  and that  $(N, h)$  **fails** to be connected at infinity. Then, since  $N$  is assumed to be complete, there exists a line joining any two different ends of  $N$ . Then the Cheeger-Gromoll Splitting Theorem asserts that  $(N, h)$  is *isometric* to a product manifold. In this sense  $(N, h)$  is “special”.

In 1982, Yau posed the problem of obtaining the Lorentzian analogue of the Cheeger-Gromoll Splitting Theorem [21]. This was accomplished in a joint effort by Beem, Ehrlich, Markovsen and Galloway [2,3], Eschenburg [7], and Newman [16] in a series of papers. The theorems states the following:

**Lorentzian Splitting Theorem:** *Suppose that the space-time  $M$  satisfies the following conditions:*

- *$M$  is timelike geodesically complete.*
- *$M$  obeys the Strong Energy Condition,  $\text{Ric}(X, X) \geq 0$  for all timelike  $X \in T(M)$ .*
- *$M$  has a timelike line.*

*Then  $M$  splits isometrically along the line, i.e.  $(M, g)$  is isometric to  $(\mathbb{R} \times V, -dt^2 \oplus h)$ , where  $(V, h)$  is a complete Riemannian manifold.*

## 7) Current Research

Recently, Galloway was able to prove a very important result regarding the rigidity of null geodesically complete space-times:

**The Null Splitting Theorem** *Suppose  $(M, g)$  is a spacetime satisfying*

- *$M$  is null geodesically complete.*
- *$M$  obeys the null energy condition:  $\text{Ric}(X, X) \geq 0$  for all null vectors  $X$ .*
- *$M$  has a null line  $\eta$ .*

*Then  $\eta$  is contained in a smooth closed achronal totally geodesic null hypersurface  $S$  [11,12].*

As an application of the Null Splitting Theorem we have a rigidity theorem for spaces of de Sitter type.

Recall that a space of de Sitter type has a spacelike conformal boundary  $\mathcal{J}$ . This boundary decomposes into two disjoint sets  $\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$  called *future* and *past conformal infinity*.

One last definition: A space-time  $M$  of de Sitter type is *asymptotically simple* provided that each inextendible null geodesic in  $M$  has a future end point in  $\mathcal{J}^+$  and a past endpoint on  $\mathcal{J}^-$ .

Let us turn our attention to the Einstein Equations. In the *vacuum case* (i.e. when  $T_{ij} = 0$ ) The equations become  $R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 0$  so after tracing and scaling we end up with the simpler equations

$$R_{ij} = \lambda g_{ij} \tag{3}$$

where  $\lambda$  is a constant. From this equation, it is clear that a space-time satisfying the vacuum Einstein equations satisfies the null energy condition.

We can now state the rigidity theorem for space-times of de Sitter type:

**Theorem:** Suppose  $M$  is an asymptotically simple space-time of de Sitter type satisfying the vacuum Einstein equation

$$\text{Ric} = \lambda g$$

with  $\lambda > 0$ . If  $M$  contains a null line then  $M$  is isometric to de Sitter space [12].

This theorem is important for a number of reasons. For instance, according to the work of Friedrich [8], the set of asymptotically simple solutions to the equation (3) with  $\lambda > 0$ , is open in the set of maximal globally hyperbolic solutions with compact spatial sections. Thus by the previous theorem, in conjunction with the work of Friedrich, a sufficiently small perturbation in the Cauchy data on a fixed Cauchy hypersurface of de Sitter space will in general destroy **all** the null lines of de Sitter space. While one would expect many of the null lines to be destroyed, it is somewhat surprising that **none** of the null lines persist, especially because of the well known fact that all null geodesics in de Sitter space are in fact null lines [15].

My current work focuses in possible generalizations of this rigidity result. The main lines are the following:

- I) Consider now the more general non vacuum case of the Einstein equations (i.e.  $T \neq 0$ ). In this case, some extra assumptions are needed.
- II) Weaken the hypothesis of the theorem. For instance, it has been conjectured that the result holds if the hypothesis of asymptotical simplicity is replaced by maximal global hyperbolicity.

## 6.8) References.

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