THE FUNDAMENTAL THEOREM OF ALGEBRA

The Fundamental Theorem of Algebra was first proved by Gauss (at a very young age!). There are now many proofs of this theorem known. Here we give one which uses a number of things we've done in Math 25.

Let F be a field, and let $F[x]$ denote the set of all polynomials in one variable with coefficients in F. As usual, we denote the field of complex numbers by $\mathbf{C}$. By a *root* of a polynomial $P(x) \in F[x]$, we mean an element $\lambda \in F$ such that $P(\lambda) = 0$. According to the *Factor theorem* (see Curtis, section 20), $P(\lambda) = 0$ iff $x - \lambda$ divides $P(x)$.

Theorem 1: (Fundamental Theorem of Algebra) If $P(z) \in \mathbf{C}[z]$ has degree $n \geq 1$, then there exists a nonzero $a \in \mathbf{C}$ and elements $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ such that

$$P(z) = a(z - \lambda_1) \cdots (z - \lambda_n).$$

To prove the Fundamental Theorem of Algebra (henceforth called the FTA), it suffices to prove the seemingly weaker statement:

Theorem $1'$**:** Every nonconstant polynomial $P(z) \in \mathbf{C}[z]$ has a complex root.

Proof that Theorem $1'$ implies Theorem 1:

We use induction on the degree n of P. If $n = 1$, then $P(z) = az + b$ for some $a, b \in \mathbf{C}$ with $a \neq 0$, and we're done by setting $\lambda_1 = -b/a$.
Now assume that the FTA is true for all polynomials of degree $\leq n - 1$. Let P be a polynomial of degree n, and let λ_n be a complex root of P (which is guaranteed to exist by Theorem $1'$). By the Factor theorem, $z - \lambda_n$ divides P, i.e., there exists $Q(z) \in \mathbf{C}[z]$ of degree $n - 1$ such that $P(z) = (z - \lambda_n)Q(z)$. The proof follows by applying the inductive hypothesis to $Q(z)$. QED

To prove Theorem $1'$, we need to review five facts. For the rest of these notes, $P(z)$ will denote a nonzero polynomial of degree $n \geq 1$ with complex coefficients, and we will think of P as giving us a continuous map from $\mathbf{C}$ to $\mathbf{C}$, or equivalently from $\mathbf{R}^2$ to $\mathbf{R}^2$.

1. If $n \geq 2$ and $A \subset \mathbf{R}^n$ is a finite set of points, then $\mathbf{R}^n - A$ is path connected, hence connected. [We leave this as an exercise. Note that this would be false for $n = 1$.]

2. If $P(z) \in \mathbf{C}[z]$ is a nonzero polynomial of degree n, then P has *at most* n complex roots. [This is easily proved using the induction and the factor theorem.]

3. If S denotes the set of roots of $P'(z)$ (the complex derivative of $P(z)$), and S' denotes $P(S)$, then

$$P : \mathbf{R}^2 - S \to \mathbf{R}^2 - S'$$

   is an *open mapping* (the image of every open set is open). [This follows from Math 25(b) HW # 4, Problem 6(e), which was itself a consequence of the inverse function theorem and the Cauchy–Riemann equations.]

4. In addition,

$$P : \mathbf{R}^2 - S \to \mathbf{R}^2 - S'$$

   is a *closed mapping* (the image of every closed set is closed). [This follows from Math 25(b) HW # 5, Problem 4, since the inverse image of a bounded set in $\mathbf{R}^2 - S'$ must be bounded.]

5. If X is a connected metric space, then the only subsets of X which are both open and closed are $\emptyset$ and X. [This follows from Math 25(a) HW # 8, Problem 3(c).]

Proof of Theorem $1'$:
Since $n \geq 1$, $P'(z)$ is not identically zero. By (3) and (4), $P(\mathbf{R}^2 - S)$ is both open and closed in $\mathbf{R}^2 - S'$. By (2), S is finite, so S' is finite, and therefore by (1), $\mathbf{R}^2 - S'$ is connected. By (5), $P : \mathbf{R}^2 - S \to \mathbf{R}^2 - S'$ is *surjective.* Since $S' = P(S)$, $P : \mathbf{R}^2 \to \mathbf{R}^2$ is surjective. Therefore 0 is in the image, so P has a root. QED