1. Prove or disprove that the given map is a homomorphism. If it is a homomorphism, determine whether it is an isomorphism.

   (a) The map $\phi : (\mathbb{Z}_{20}, +) \to (\mathbb{Z}_{20}, +)$ given by $\phi(k \pmod{20}) = 8k \pmod{20}$.

   (b) The map $\phi : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ given by $\phi(k \pmod{10}) = k(k+4) \pmod{10}$.

   (c) The map $\phi : (\mathbb{C}^\times, \times) \to (\mathbb{C}^\times, \times)$ given by $\phi(z) = z^3$.

2. If H is a non-trivial normal subgroup of G, is it possible that $G/H \cong G$?

3. Define the following rotation and reflection matrices in $\mathrm{GL}_2(\mathbb{R})$, where $\theta \in \mathbb{R}$:

$$A_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   The matrices $G = \{A_\theta : \theta \in \mathbb{R}\} \cup \{BA_\theta : \theta \in \mathbb{R}\}$ form a group (no need to check this).

   (a) Prove or disprove that $H = \{A_\theta : \theta \in \mathbb{R}\}$ is a normal subgroup of G.

   (b) Is $\langle B \rangle$, the subgroup generated by B, a normal subgroup of G?

4. Prove or disprove whether the two groups listed are isomorphic.

   (a) S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_6$

   (b) $(\mathbb{Z}_3, +)$ and the complex numbers $\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\}$ with multiplication

   (c) Q_8 =quaternion group and G =symmetries of a square

   (d) $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$

   (e) $\mathbb{R}^\times$ and $\mathbb{C}^\times$

   (f) $A_4 \times \mathbb{Z}_2$ and $S_3 \times \mathbb{Z}_4$[1]

5. Use the method of Cayley's Theorem to produce an explicit injective homomorphism from the group $\mathbb{Z}_8^\times$ into a symmetric group.

6. Decide whether each statement is true or false. Give an explanation for each answer.

   (a) The determinant map from the ring $M_2(\mathbb{R})$ to the ring $\mathbb{R}$ is a homomorphism.

   (b) The set $M_2(\mathbb{Z}_2)$, of 2×2 matrices with entries in $\mathbb{Z}_2$, forms an integral domain.

   (c) Any ideal $I \subset R$ in a ring is closed under multiplication.

   (d) The subset $\{a + bi : a + b \equiv 0 \pmod{2}\}$ in the ring $\mathbb{Z}[\sqrt{2}]$ is an ideal.

   (e) The rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{-2}]$ are isomorphic.

[1] In a product of two groups $G_1 \times G_2$, the order of $(a, b) \in G_1 \times G_2$ is equal to $\mathrm{lcm}(\mathrm{ord}(a), \mathrm{ord}(b))$.