1. Let G be a group. Show that a nonempty subset $H \subset G$ is a subgroup if and only if the following holds: for all $a, b \in H$, $ab^{-1} \in H$.

2. Let G be any group. Let $a, b \in G$ and let $x \in G$ be some unknown. Suppose

$$x^3 = e, \qquad x^2 b = ba.$$

   (a) Solve for x in terms of a and b. Show all steps.

   (b) Suppose G is $(\mathbb{Z}_5^\times, \times)$ and $a \equiv 4 \pmod 5$, $b \equiv 3 \pmod 5$. Find x.

3. Prove that the symmetric group S_n is nonabelian if and only if $n \geqslant 3$.

4. Consider the following set of real 3×3 matrices:

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : \ a, b, c \in \mathbb{R} \right\}$$

   (a) Show that G with the operation of matrix multiplication is a group.

   (b) Is the group G abelian? Explain.

5. List the left and right cosets of the subgroups in each of the following.

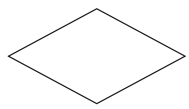
   (a) $3\mathbb{Z}$ in $\mathbb{Z}$

   (b) A_4 in S_4

   (c) $\langle 8 \rangle$ in $\mathbb{Z}_{24}$

   (d) $H = \{e, (123), (132)\}$ in S_4

6. Consider the following diamond shape situated inside the 2-dimensional plane:



   (a) Draw all the symmetries of this shape. How big is the group of symmetries?

   (b) Label the vertices with numbers $1, 2, 3, 4$. Write down the permutation of vertices corresponding to each symmetry.

   (c) Is the corresponding permutation group a subgroup of an alternating group?

7. Use Fermat's Little Theorem to show that if $p = 4n + 3$ is prime, then there is no solution to the equation $x^2 \equiv -1 \pmod p$.

8. (a) Consider $(142)(231)$ and $(54123)(24)$. Compute each as a product of disjoint cycles. What are the orders of these elements, and what are their parities?

   (b) Is $\{e,\ (12),\ (34),\ (12)(34),\ (45),\ (12)(45)\} \subset S_5$ a subgroup?

   (c) What are the possible orders of elements in the alternating group A_5?

9. Compute $7^{81} \pmod{30}$.

10. Let G be a finite cyclic group of order n generated by a. Show that if $b = a^k$ where k is relatively prime to n, then b generates G.