Prime and maximal ideals

An ideal I in a commutative ring R is *prime* if $I \neq R$ and $ab \in I$ implies that either $a \in I$ or $b \in I$. The ideal $I \subset R$ is *maximal* if $I \neq R$, and $I \subset J$ for some ideal $J \neq I$ implies $J = R$.

▶ **A maximal ideal is a prime ideal.**

Proof. Let I be a maximal ideal, and suppose it is not prime: let $ab \in I$ such that $a \notin I$ and $b \notin I$. Consider the ideals $(a) + I$ and $(b) + I$. These both contain I and are not equal to I. Since I is maximal, it follows that $(a) + I = R$ and $(b) + I = R$. The product ideal of $(a) + I$ and $(b) + I$ is then on the one hand equal to $(1) = R$ and on the other hand contained in I, implying $R = I$, a contradiction. □

The following characterizes prime and maximal ideals in terms of the quotient ring.

▶ **Let R be a commutative ring and $I \subset R$ a proper ideal. Then**

(i) I is prime if and only if the ring R/I is an integral domain.

(ii) I is maximal if and only if the ring R/I is a field.

Proof. (i) Suppose I is prime. Suppose $ab + I = (a + I)(b + I) = I$ in R/I. Then $ab \in I$. Since I is prime, one of a or b is in I. Then one of $a + I$ or $b + I$ is equal to I. This proves R/I is an integral domain. Conversely, suppose R/I is an integral domain. Now let $ab \in I$. Then $ab + I = (a + I)(b + I) = I$, and since R/I is an integral domain, one of $a + I$ or $b + I$ is equal to I. This implies one of a or b is in I. Thus I is prime.

(ii) Suppose I is maximal. We claim R/I is a field. This is equivalent to R/I having only the ideals $\{I\}$, R/I. Suppose R/I has a non-zero proper ideal J'. Then $J = \{a \in R : a + I \in J'\} \subset R$ is an ideal of R. To verify this: if $a, b \in J$, then $a + I, b + I \in J'$, and

$$(a + I) - (b + I) = (a - b) + I$$

is in J' because J' is an ideal; thus $a - b \in J$. Similarly, if $a \in J$ and $b \in R$ then $(a + I)(b + I) = ab + I$ is in J', hence $ab \in J$. Also, $0 \in J$. Thus J is an ideal. Next, since I is maximal and $I \subset J$, we must have either $J = I$ or $J = R$. If $J = I$ then $J' = \{I\}$, a contradiction. If $J = R$ then $J' = R/I$, contradicting our assumption that J' is proper.

Conversely, suppose R/I is a field. Suppose I is not maximal, and let $I \subset J$ with $I \neq J$ and $J \neq R$. Then consider $J' = \{a + I : a \in J\} \subset R/I$. This is an ideal of R/I. Since R/I is a field, it must be either zero or R/I. If J' is zero then $a + I = I$ for all $a \in J$ i.e. $a \in I$ for all $a \in J$; but this implies $J \subset I$, a contradiction. Thus $J' = R/I$. Then for all $a \in R$ we have $a + I = b + I$ for some $b \in J$, i.e. $a + c = b + d$ for some $c, d \in I$ and $b \in J$. Then $a = b + d - c \in J$. We have shown $J = R$, a contradiction. □

Examples

1. Consider an ideal $(n) \subset \mathbb{Z}$ where $n \geqslant 0$. Suppose it is a prime ideal. This means $ab \in (n)$ implies one of a or b is in (n). Note $ab \in (n)$ if and only if $ab = nk$ for some $k \in \mathbb{Z}$, i.e. n divides ab. So (n) is prime if and only if "n divides ab" implies "n divides a or b". For this to be true n must be a prime number. Thus the prime ideals of $\mathbb{Z}$ are

$$(2), \quad (3), \quad (5), \quad (7), \quad (11), \quad \ldots$$

We recover that $\mathbb{Z}_n$ is an integral domain if and only if n is prime.

We also know from earlier lectures that $\mathbb{Z}_n$ is a field if and only if n is prime. This implies that the prime ideals in $\mathbb{Z}$ are also the maximal ideals.

2. Consider the ring $\mathbb{C}[x, y]$. The ideal $I = (x^2 + y^2 - 1)$ is prime, but not maximal. To see it is prime, suppose $f(x, y)g(x, y) = (x^2 + y^2 - 1)h(x, y)$. Then since $x^2 + y^2 - 1$ is irreducible (cannot be factored over $\mathbb{C}[x, y]$) it must divide one of $f(x, y)$ or $g(x, y)$. To see that it is not a maximal ideal, note that we have an inclusion

$$(x^2 + y^2 - 1) \subset (x - 1, y)$$

because $x^2 + y^2 - 1 = (x + 1)(x - 1) + (y)(y)$. This inclusion is proper since $y \notin I$. Also, $(x - 1, y) \neq \mathbb{C}[x, y]$. Thus I is not maximal. However, $(x - 1, y)$ is maximal.

3. In the ring $\mathbb{Z}[x]$, the ideal (2) is prime but not maximal. To see this, consider

$$\phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}_2[x]$$

defined by $\phi(f(x)) = f(x) \pmod 2$, i.e. take the coefficients mod 2. Then ϕ is a homomorphism and the kernel is the principal ideal $(2) \subset \mathbb{Z}[x]$. By the 1st Isomorphism Theorem,

$$\mathbb{Z}[x]/\ker(\phi) = \mathbb{Z}[x]/(2) \cong \mathbb{Z}_2[x]$$

Now $\mathbb{Z}_2[x]$ is an integral domain, but not a field (e.g. x is not invertible). Thus (2) is a prime ideal that is not maximal. On the other hand, we may consider the homomorphism

$$\psi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}_2$$

defined by $\psi(f(x)) = f(0) \pmod 2$. Then you can check that the kernel is the ideal $(2, x) = (2) + (x)$. Since $\mathbb{Z}_2$ is a field, this ideal is maximal.