Rings and geometry

In this lecture we continue studying ideals in rings, and also explain the connection of ring theory to the geometry of algebraic varieties.

Let R be a commutative ring and consider the ring of polynomials $R[x, y]$. In fact for the time being we will take $R = \mathbb{R}$ to be the real numbers, so that we may easily draw pictures.

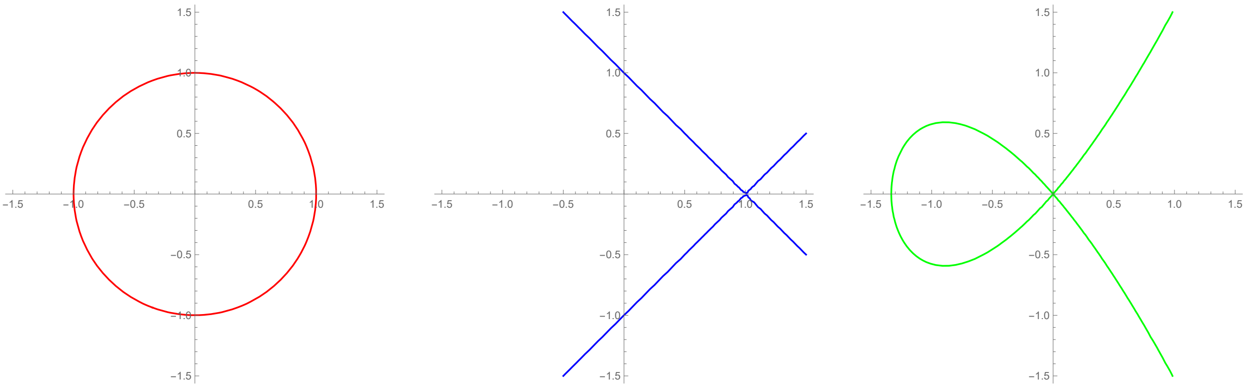
An (algebraic) *variety* in $\mathbb{R}^2$ is a set $V = V(S) \subset \mathbb{R}^2$ which consists of points (x, y) in the plane $\mathbb{R}^2$ that are the common roots of a given collection of polynomials $S \subset R[X]$:

$$V(S) = \{(x_0, y_0) \in \mathbb{R}^2 : f(x_0, y_0) = 0 \text{ for all } f \in S\} \subset \mathbb{R}^2$$

For example, consider the following subsets of $\mathbb{R}[x, y]$:

$$\begin{aligned} S_1 &= \{x^2 + y^2 - 1\} \\ S_2 &= \{(x + y - 1)(x - y - 1)\} \\ S_3 &= \{y^2 - x^2(x + \tfrac{4}{3})\} \\ S_4 &= \{x, y + 1\} \end{aligned}$$

The corresponding varieties $V(S_1)$, $V(S_2)$, $V(S_3)$ are shown below, while $V(S_4) = \{(0, -1)\}$:



Given a subset $S \subset R$ in any ring, we can also associate an ideal to S, called $I(S) \subset R$: it is the smallest ideal containing S. Explicitly, we can define this ideal as follows:

$$I(S) = \{\sum_{i=1}^{n} r_i f_i : \; r_i \in R, f_i \in S\}$$

Each of the examples above yields such an ideal:

$$\begin{aligned} I(S_1) &= (x^2 + y^2 - 1) \\ I(S_2) &= ((x + y - 1)(x - y - 1)) \\ I(S_3) &= (y^2 - x^2(x + \tfrac{4}{3})) \\ I(S_4) &= (x, y + 1) = (x) + (y + 1) \end{aligned}$$

(See below for the sum of ideals.) We have the following (note the reversal of inclusion!):

- **For subsets $S \subset T$ in $\mathbb{R}[x, y]$ we have $V(T) \subset V(S)$.**

Proof. Let $(x_0, y_0) \in V(T)$. This means $f(x_0, y_0) = 0$ for all polynomials $f \in T$. Since $S \subset T$ it is then certainly true that $f(x_0, y_0) = 0$ for all $f \in S$. Thus $(x_0, y_0) \in V(S)$. □

The next observation is that the variety corresponding to a set $S \subset \mathbb{R}[x, y]$ of polynomials is the same as the variety associated to the ideal $I(S)$.

▶ **For $S \subset \mathbb{R}[x, y]$ we have $V(S) = V(I(S))$.**

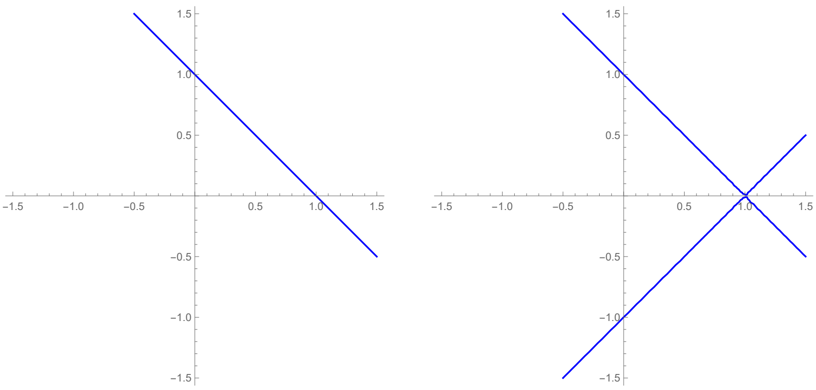
Proof. As $S \subset I(S)$, the previous result gives us $V(I(S)) \subset V(S)$.

For the reverse inclusion, suppose $(x_0, y_0) \in V(S)$. Then $f(x_0, y_0) = 0$ for all $f \in S$. Let $f \in I(S)$. Then $f = \sum_{i=1}^n r_i f_i$ for some $r_i \in \mathbb{R}[x, y]$ and $f_i \in S$. Then

$$f(x_0, y_0) = \sum_{i=1}^n r_i(x_0, y_0) f_i(x_0, y_0) = \sum_{i=1}^n r_i(x_0, y_0) 0 = 0$$

Thus $f(x_0, y_0) = 0$ for all $f \in I(S)$, and $(x_0, y_0) \in V(I(S))$. Thus $V(S) \subset V(I(S))$. □

For example, for the ideals $I = (x + y - 1)$ and $I_2 = I(S_2) = ((x + y - 1)(x - y - 1))$ we have $I_2 \subset I$ and so $V(I) \subset V(I_2)$. On the left below is the variety $V(I)$, on the right is $V(I_2)$.



Given ideals $I, J \subset R$ in a commutative ring, we define the sum, intersection, and product:

$$\begin{aligned} I + J &= \{a + b : \ a \in I, b \in J\} \\ I \cap J &= \{a : \ a \in I, a \in J\} \\ IJ &= \{\textstyle\sum a_i b_i : \ a_i \in I, b_j \in J\} \end{aligned}$$

These are again ideals in R, as you can check. If $I = (a)$ and $J = (b)$ are principal, then in fact $IJ = (a)(b) = (ab)$. We now explore the geometric interpretations of these operations.

▶ **For ideals $I, J \subset \mathbb{R}[x, y]$ we have $V(I + J) = V(I) \cap V(J)$.**

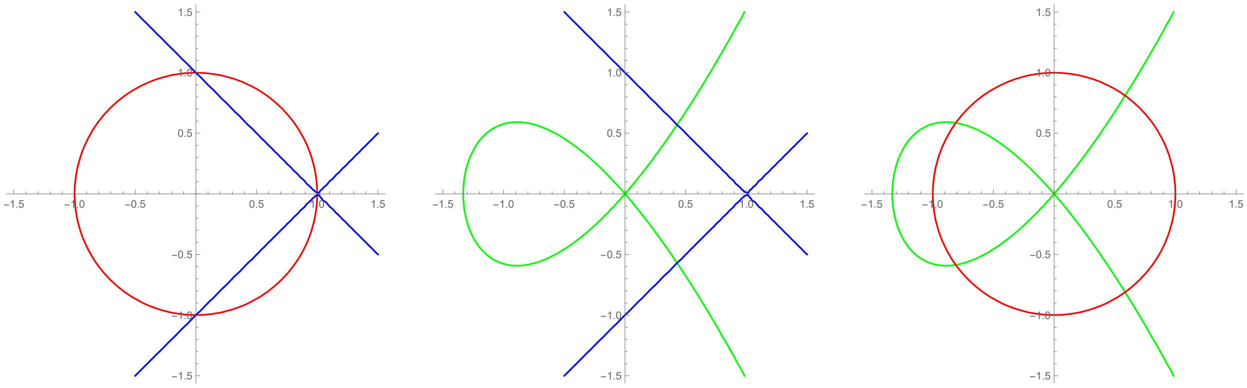
Proof. Let $(x_0, y_0) \in V(I) \cap V(J)$. Then $f(x_0, y_0) = 0$ for all $f \in I$ and $f \in J$. Let $f \in I + J$. Then $f = f' + f''$ where $f' \in I$, $f'' \in J$. We compute

$$f(x_0, y_0) = f'(x_0, y_0) + f''(x_0, y_0) = 0 + 0 = 0$$

and therefore $(x_0, y_0) \in V(I + J)$, and we conclude $V(I) \cap V(J) \subset V(I + J)$.

For the reverse inclusion, since $I \subset I + J$ we have $V(I + J) \subset V(I)$. Similarly, $V(I + J) \subset V(J)$. Thus $V(I + J) \subset V(I) \cap V(J)$. All together we obtain the statement. □

The conclusion we draw from this is that the sum of ideals corresponds to the *intersection* of varieties. For example, consider the ideals $I_1 = I(S_1), I_2 = I(S_2), I_3 = I(S_3)$ from earlier.



Then $I_1 + I_2$ is the ideal which is the sum of the principal ideals $(x^2 + y^2 - 1)$ and $((x + y - 1)(x - y - 1))$. It is not a principal ideal itself. However the variety $V(I_1 + I_2)$ is very simple:

$$V(I_1 + I_2) = V(I_1) \cap V(I_2) = \{(0, 1), (0, -1), (1, 0)\} \subset \mathbb{R}^2$$

It consists of the 3 points of intersection between the circle $V(I_1)$ and the two lines $V(I_2)$. Similarly, we see that $V(I_2 + I_3)$ is 2 points, and $V(I_3 + I_1)$ is 4 points.

- **For ideals $I, J \subset \mathbb{R}[x, y]$ we have $V(I \cap J) = V(I) \cup V(J)$.**

The proof is similar to the previous one. Thus the variety associated to the intersection of two ideals is the union of the two varieties of each ideal. As a consequence, $V(I_1 \cap I_2)$, $V(I_2 \cap I_3)$, $V(I_3 \cap I_1)$ are the three pictures above (not just the intersection points!).

In fact, the same property holds for the product of ideals!

- **For ideals $I, J \subset \mathbb{R}[x, y]$ we have $V(IJ) = V(I) \cup V(J)$.**

Note in general $IJ \neq I \cap J$. For example, consider $I = (x)$ And $J = (xy)$ in $\mathbb{R}[x, y]$. Then $IJ = (x^2y)$ while $I \cap J = (xy) = J$. In general, $IJ \subset I \cap J$, but not the other way around.

Thus the above two results give instances of the phenomenon that $V(I) = V(J)$ for two ideals $I \neq J$. In other words, the map given by

$$\{\text{ideals in } \mathbb{R}[x,y]\} \longrightarrow \{\text{varieties in } \mathbb{R}^2\}$$

$$I \longmapsto V(I)$$

is not 1-1. (From our earlier discussion, it is onto.) In fact it is very far from being 1-1! Take any polynomial $f \in \mathbb{R}[x,y]$ which has no real solutions to $f(x,y) = 0$. Examples are $f(x,y) = 1$ and $f(x,y) = x^2 + 1$. Then for any such polynomial,

$$V(f) = \varnothing,$$

since $f(x_0, y_0) = 0$ does not hold for any $(x_0, y_0) \in \mathbb{R}^2$. However the ideals $(1) = \mathbb{R}[x,y]$ and $(1 + x^2)$ are not the same.

To fix this last sort of problem, we can work over a field such as $\mathbb{C}$ which has more solutions. Then for the ideal $(1 + x^2) \subset \mathbb{C}[x,y]$ the variety

$$V(1 + x^2) = \{(x_0, y_0) \in \mathbb{C}^2 : 1 + x_0^2 = 0\} = \{(i, y_0) : y_0 \in \mathbb{C}\} \cup \{(-i, y_0) : y_0 \in \mathbb{C}\}$$

is two disjoint copies of $\mathbb{C}$ contained inside $\mathbb{C}^2$. Thus it is distinguished from $V(1) = \varnothing$.

Still, even after replacing $\mathbb{R}$ with $\mathbb{C}$, we do not have a 1-1 correspondence between ideals and varieties. To remedy this, we focus on a particular class of ideals which does make the assignment $I \mapsto V(I)$ into a 1-1 correspondence.

▶ **For any ideal $I \subset R$ in a commutative ring R, the *radical* of I is defined to be**

$$\sqrt{I} = \{a \in R : \ a^n \in I \text{ for some positive } n \in \mathbb{Z}\}$$

It is instructive to verify that $\sqrt{I} \subset R$ is an ideal. A *radical ideal* is an ideal of the form $\sqrt{I}$.

For example, consider $I = (x^2) \subset \mathbb{R}[x,y]$. Then $\sqrt{I} = (x)$. On the other hand, $\sqrt{(x)} = (x)$. A key property is the following: for ideals I, J we have $\sqrt{IJ} = \sqrt{I \cap J}$.

▶ **The following assignment is a 1-1 and onto correspondence:**

$$\{\text{radical ideals in } \mathbb{C}[x,y]\} \longrightarrow \{\text{varieties in } \mathbb{C}^2\}$$

$$I \longmapsto V(I)$$

This sets up a "dictionary" between certain ideals, which are algebraic objects, and varieties, which are geometric objects. The dictionary converts sums of ideals into intersections of varieties, and intersections of ideals into unions of varieties.

This lecture is essentially an introduction to the subject of *Algebraic Geometry*, which is the study of algebraic varieties in a much broader setting.