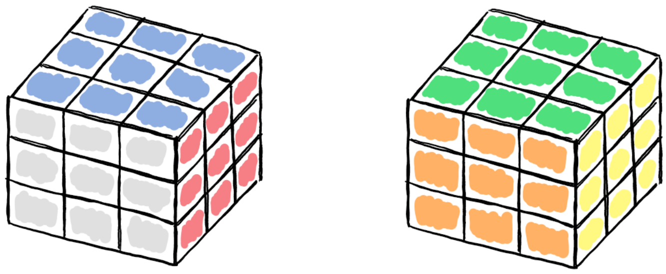
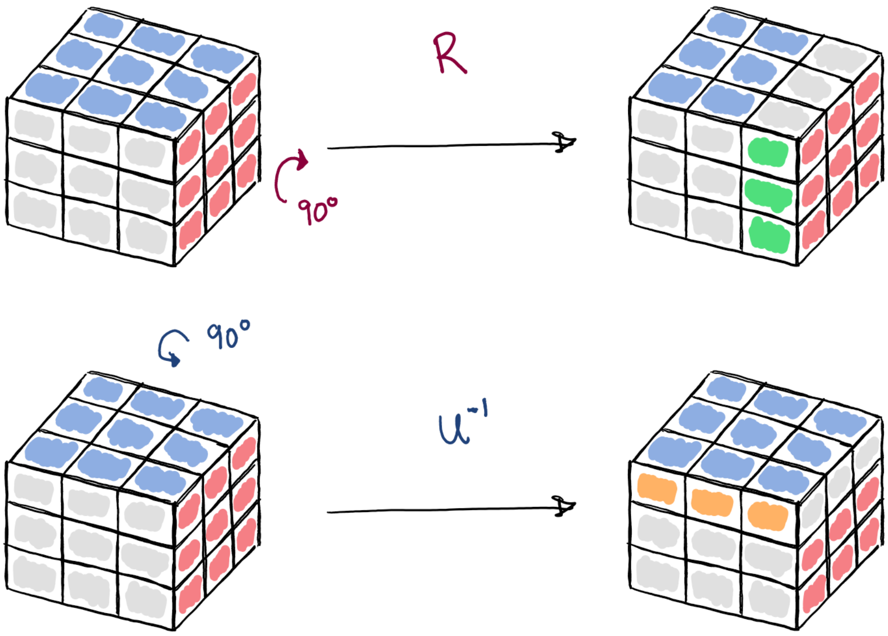
The Rubik's cube group

Today we will study the Rubik's cube using some of the group theory we have developed. During lecture, I'll hand out Rubik's cubes and you should play along. If you don't know how to solve the cube, however, be careful not to scramble it!

Let's begin with a top (on the left) and bottom (on the right) view of the Rubik's cube:



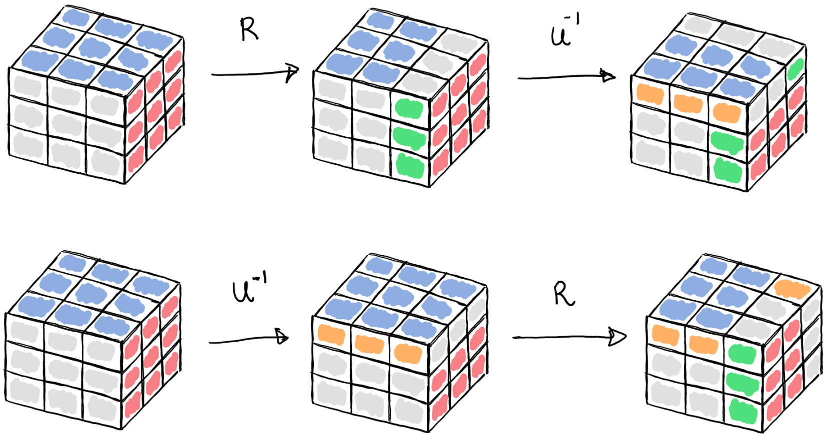
We label the 6 faces of the cube F (front), B (back), U (up), D (down), L (left), R (right). To each face we associate a symmetry of the cube which is a 90° clockwise rotation of that face, if you are facing the face. For example, we have the moves R and U^{-1}:



And R^2, for example, is 180° rotation of the right face. We write G for the *Rubik's cube group*, which is the group of symmetries generated by the moves F, B, U, D, L, R.

Our convention is that the expression $LU^{-1}RD$ means: first do D, then do R, then U^{-1}, and then L. Warning: This is opposite the order in which Rubik's cube enthusiasts write moves, but it goes better with our conventions in group theory.

Let us try to understand the group G. First, compare the following two sequences of moves:



This shows that $U^{-1}R \neq RU^{-1}$. In particular G is non-abelian.

However we can find non-trivial abelian subgroups. Here's an example. Consider the subgroup generated by R. This is clearly $\langle R \rangle = \{e, R, R^2, R^3\}$ as R is a 90° rotation of a face. So we get a cyclic subgroup of order 4, which is isomorphic to $\mathbb{Z}_4$. Similarly, $\langle L \rangle \cong \mathbb{Z}_4$. Note also that L and R *commute* as these two faces are not sharing any part of the cube. From this we get a subgroup $\langle R, L \rangle \subset G$ isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Let's look at the orders of some elements in G. Check with your cube the following:

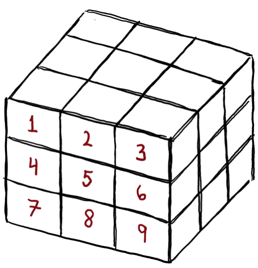
$$\text{ord}(R^{-1}UR) = 4, \qquad \text{ord}(UR^{-1}UR) = 5$$

As another example, you can check (after a few minutes!) that $\text{ord}(UR) = 105$. For a more extravagant example, one can check that

$$\text{ord}(B^{-1}UB^{-1}F^2R) = 1260,$$

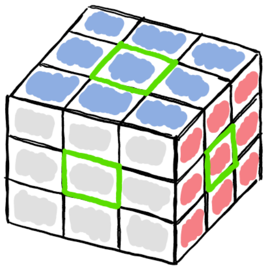
although I suggest you don't waste your time doing it! This order, 1260, is in fact the highest possible order of any element in G.

How should we better understand G? One way to is map it into a symmetric group, just as we've done for the triangle and the tetrahedron. Label the "sticker positions" of the cube:



In the picture we labelled one face, but continuing in this fashion we will get $(9)(6) = 54$ labels. Each move $a \in G$ is determined by how these sticker positions are permuted, and to each such a we get permutation in S_{54}. This gives a 1-1 homomorphism $G \to S_{54}$.

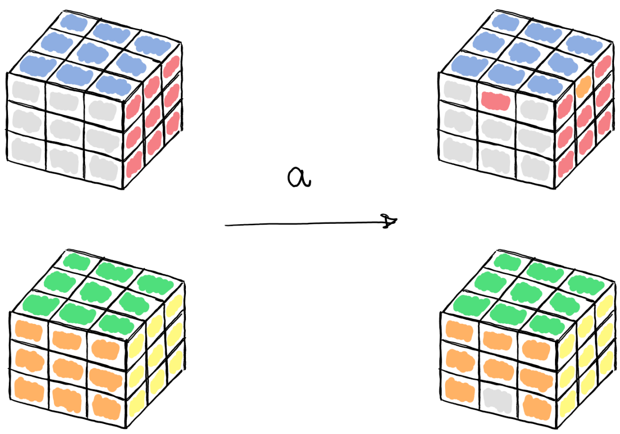
But we can do better! Note each $a \in G$ fixes the middle sticker position on each face.



So forget about these. We only label the other sticker positions, of which there are 48. This gives a homomorphism $G \to S_{48}$. Note that $48! \approx 10^{61}$. However it turns out $|G| = 8!12!2^{10}3^7 \approx 4 \times 10^{19}$, which is very large but much smaller than $|S_{48}| = 48!$.

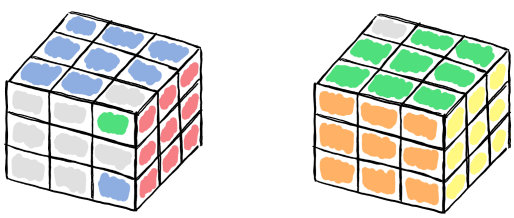
How can you solve the Rubik's cube? The basic strategy is as follows: (1) first, come up with some basic moves that permute only a few parts of the cube; (2) use these basic moves to progressively solve parts of the cube. (In reality it is best to just learn one of the algorithms that solves it.) We won't solve the cube but rather aim to understand G better.

But let's focus on (1) for a moment. Here is a "basic move" called an *edge 3-cycle*:



Both top and bottom views are shown. It is called an edge 3-cycle because exactly 3 pieces (subcubes) are permuted. A representation of $a \in G$ is given by $a = F^2UL^{-1}RF^2LR^{-1}UF^2$.

Having shown that we can permute 3 edge cubes as above, we might suspect we can permute any given set of subcubes as we like. Not true! For example, consider the configuration:



This configuration has exactly 2 corner cubes swapped. To show this is impossible, we introduce another homomorphism. Consider subcube positions which are not in the middles of faces. You can count there are 20 such non-middle subcube positions in the cube.

Next, label these subcube positions $1, \ldots, 20$ in any fashion. By seeing how a symmetry $a \in G$ permutes these 20 subcube positions, we obtain a homomorphism

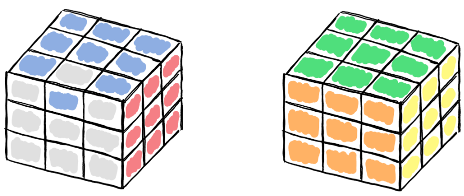
$$\phi : G \longrightarrow S_{20}$$

For example, take the subcubes starting on the right face at a corner and label them $1, \ldots, 8$ in a clockwise fashion (and label the others arbitrarily) then $\phi(R) = (1357)(2468)$. Note this permutation is *even*. A similar computation shows $\phi(a)$ is even for F, B, U, D, L. But every element in G can be written as a composition of these moves, so we have shown:

▸ **The homomorphism $\phi : G \to S_{20}$ has $\mathbf{im}(\phi) \subset A_{20}$.**

In other words, for any symmetry $a \in G$, the permutation $\phi(a)$ is even. Now the proof of why the configuration drawn previously is impossible is easy! Just note that it swaps exactly 2 subcubes, so the picture corresponds to a transposition in S_{20}, which is odd, and this contradicts what we have just found.

We can also prove that the following configuration is impossible:



Note that this configuration does not swap any subcubes; it only changes the "orientation" of exactly one subcube. It is in fact in the kernel of the homomorphism ϕ, so that homomorphism will not help us here.

We can define another homomorphism as follows. Notice that each edge piece (not a corner) has 2 stickers. Collecting all such pairs of stickers, there are 24. Labelling these $1, \ldots, 24$ and seeing how each $a \in G$ permutes these, we obtain a homomorphism $\psi : G \to S_{24}$. Again, it is not hard to show that $\psi(R)$ is even, and the same is true for F, B, U, D, L, and thus for all $a \in G$. On the other hand, the above drawn configuration corresponds to a transposition which swaps exactly 2 of these edge stickers, which is impossible!