More isomorphism theorems

Last lecture we learned about the 1st Isomorphism Theorem. As its name suggests, it has a few sequels, whose names are just as imaginative!

For subsets H and N of a group G we define the subset $HN \subset G$ to be:

$$HN = \{ab : \ a \in H, b \in N\}$$

▸ **2nd Isomorphism Theorem: Let H, N be subgroups a group G with N normal. Then HN is a subgroup of G, $H \cap N$ is a normal subgroup of H, and**

$$\frac{H}{H \cap N} \cong \frac{HN}{N}$$

Proof. Let us first show HN is a subgroup of G. Clearly $e \in HN$, since $e = ee$ and $e \in H$, $e \in N$. Let $ab, a'b' \in HN$ where $a, a' \in H$, $b, b' \in N$. Then $Na' = a'N$ since N is normal, and this implies $ba' = a'b''$ for some $b'' \in N$. Thus

$$(ab)(a'b') = a(ba')b' = (aa')(b''b') \in HN.$$

Next, let $ab \in HN$. Then $Na^{-1} = a^{-1}N$ implies $b^{-1}a^{-1} = a^{-1}b'$ for some $b' \in N$. Thus $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b' \in HN$. Thus HN is a subgroup of G.

The argument that $H \cap N$ is a normal subgroup of H is also straightforward, and is omitted.

To prove the isomorphism, we construct a homomorphism

$$\phi : H \longrightarrow HN/N$$

by setting, for each $a \in H$, $\phi(a) = aN$. Any coset in HN/N is of the form $abN = aN$ for $ab \in HN$, and $\phi(a) = aN$, so ϕ is onto. The map ϕ is a homomorphism:

$$\phi(aa') = aa'N = aNa'N = \phi(a)\phi(a')$$

Now let us compute $\ker(\phi)$. Suppose $a \in H$ and $\phi(a) = aN = N$. This means $a \in N$. Thus $\ker(\phi) = H \cap N$. The 1st Isomorphism Theorem then gives

$$HN/N \cong H/\ker(\phi) = H/H \cap N$$ □

▸ **Example:** Let m, n be positive integers. Let $G = \mathbb{Z}$ and $H = m\mathbb{Z}$, $N = n\mathbb{Z}$. Recall that $\gcd(m, n)$ is characterized as being the smallest positive integer contained in the set

$$m\mathbb{Z} + n\mathbb{Z} = \{am + bn : \ a, b \in \mathbb{Z}\}$$

This subset of $\mathbb{Z}$ is the subgroup "HN" in this example. We see that $m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}$. On the other hand, the subset corresponding to $H \cap N$ is given by

$$m\mathbb{Z} \cap n\mathbb{Z} = \{k \in \mathbb{Z} : k = am = an \text{ for some } a, b \in \mathbb{Z}\}$$

Noting that the smallest positive integer in this subset is the lcm of m and n, we obtain $m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m,n)\mathbb{Z}$. The 2nd Isomorphism Theorem gives us

$$\frac{m\mathbb{Z}}{\text{lcm}(m,n)\mathbb{Z}} = \frac{H}{H \cap N} \cong \frac{HN}{N} = \frac{\text{gcd}(m,n)\mathbb{Z}}{n\mathbb{Z}}$$

If k divides l, then $|k\mathbb{Z}/l\mathbb{Z}| = l/k$. (This can be shown explicitly, but is also deduced as a consequence of the 3rd Isomorphism Theorem below.) By comparing orders of groups,

$$\text{lcm}(m,n)/m = n/\text{gcd}(m,n) \quad \Longrightarrow \quad \text{lcm}(m,n)\text{gcd}(m,n) = mn.$$

▸ **Example:** Consider $G = S_4$ and the normal subgroup $N = \{e, (12)(34), (13)(24), (14)(23)\}$ of order 4 inside of S_4. Then we saw from last lecture that $S_4/N \cong S_3$ by constructing an onto homomorphism $\phi : S_4 \to S_3$, recognizing its kernel to be N, and using the 1st Isomorphism Theorem. Let us give a different method for showing $S_4/N \cong S_3$.

Take $H = S_3 \subset S_4$. This is a (non-normal) subgroup. Then we first claim

$$HN = S_4.$$

To see this, we proceed as follows. First, $S_3 = S_3 e \subset HN$, so in particular $(12), (23), (31) \in HN$. Next, as $(12), (23), (31) \in S_3$ and $(12)(34), (23)(14), (31)(24) \in N$ we get the products

$$(12)\left((12)(34)\right) = (34), \qquad (23)\left((23)(14)\right) = (14), \qquad (31)\left((31)(24)\right) = (24)$$

also as elements of HN. Thus every transposition $(12), (13), (14), (23), (24), (34)$ in S_4 is in HN. Every permutation in S_4 is a product of transpositions, and therefore $S_4 = HN$. Next note that $H \cap N = \{e\}$ by direct inspection. Finally, the 2nd Isomorphism Theorem gives

$$S_4/N = HN/N \cong S_3/S_3 \cap N = S_3/\{e\} \cong S_3$$

▸ **3rd Isomorphism Theorem: Let G be a group and suppose $H, N \subset G$ are normal subgroups with $N \subset H$. Then we have an isomorphism**

$$\frac{G}{H} \cong \frac{G/N}{H/N}$$

Proof. We define a map $\phi : G/N \to G/H$ by setting $\phi(aN) = aH$. Note that if $aN = bN$, then $ab^{-1} \in N \subset H$, and thus $aH = bH$. Thus $\phi(aN) = aH = bH = \phi(bN)$, and ϕ is well-defined. Furthermore, ϕ is an onto homomorphism, by repeating earlier arguments.

The kernel of ϕ is given by $aN \in G/N$ such that $\phi(aN) = aH = H$, which is true if and only if $a \in H$. Thus $\ker(\phi) = \{aN : a \in H\} = H/N$. The 1st Isomorphism Theorem gives

$$G/H \cong \frac{G/N}{\ker(\phi)} = \frac{G/N}{H/N}$$ □

▸ **Example:** Consider $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Note that $mn\mathbb{Z} \subset n\mathbb{Z}$ for any integer m. With $G = \mathbb{Z}$, $H = n\mathbb{Z}$ and $N = mn\mathbb{Z}$ we obtain from the 3rd Isomorphism Theorem:

$$\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}/mn\mathbb{Z}}{n\mathbb{Z}/mn\mathbb{Z}}$$

Taking orders, we get $n = |\mathbb{Z}_n| = |\mathbb{Z}/mn\mathbb{Z}|/|n\mathbb{Z}/mn\mathbb{Z}| = mn/|n\mathbb{Z}/mn\mathbb{Z}|$. In particular,

$$|n\mathbb{Z}/mn\mathbb{Z}| = m.$$

Setting $l = mn$ and $k = n$ we deduce in general that if k divides l then $|k\mathbb{Z}/l\mathbb{Z}| = l/k$.