

## More isomorphisms, and kernels

In this lecture we continue our study of homomorphisms and isomorphisms and also introduce the important notion of the *kernel* of an homomorphism.

Recall that an isomorphism is a 1-1 and onto homomorphism  $\phi : G \rightarrow G'$ . The homomorphism property says that  $\phi(ab) = \phi(a)\phi(b)$  for all  $a, b \in G$ . Define a relation on groups: write  $G \cong G'$  if there is an isomorphism  $\phi : G \rightarrow G'$ .

► **The relation  $G \cong G'$  is an equivalence relation on the collection of all groups.**

In proving this we will introduce a number of important properties of isomorphisms.

First, given any group  $G$ , consider the map  $\phi : G \rightarrow G$  given by  $\phi(a) = a$ , the “identity” homomorphism. This is clearly an isomorphism, so  $G \cong G$ , i.e. the relation is reflexive. The identity homomorphism is sometimes written  $\text{id}_G = \phi : G \rightarrow G$ .

Next, suppose  $G \cong G'$ , i.e. there is an isomorphism  $\phi : G \rightarrow G'$ . Consider the inverse map of  $\phi$ , denoted  $\phi' : G' \rightarrow G$ . This is defined as follows. Let  $a' \in G'$ . We would like to describe  $\phi'(a')$ . Because  $\phi$  is onto, there is an  $a \in G$  with  $\phi(a) = a'$ , and this  $a$  is unique as  $\phi$  is 1-1. Then we define  $\phi'(a')$  by  $\phi'(a') = a$ . We see that  $\phi'$  is characterized by the relation  $\phi'(\phi(a)) = a$  for all  $a \in G$ . You can check that  $\phi'$  is 1-1 and onto, and also  $\phi(\phi'(a')) = a'$  for all  $a' \in G'$ . Next, given  $a', b' \in G'$ , let  $a, b \in G$  such that  $\phi(a) = a'$ ,  $\phi(b) = b'$ . Then

$$\phi'(a'b') = \phi'(\phi(a)\phi(b)) = \phi'(\phi(ab)) = ab = \phi'(a')\phi'(b')$$

Therefore  $\phi' : G' \rightarrow G$  is an isomorphism, and so  $G' \cong G$ . The map  $\phi'$  is called the *inverse* isomorphism of  $\phi$ . The inverse isomorphism is sometimes written  $\phi^{-1} : G' \rightarrow G$ . This establishes symmetry of the relation  $\cong$ .

In general, if  $\phi : G \rightarrow G'$  and  $\phi' : G' \rightarrow G''$  are homomorphisms, the composition map  $\phi' \circ \phi : G \rightarrow G''$  is also a homomorphism. Indeed, if  $a, b \in G$  then we compute

$$(\phi' \circ \phi)(ab) = \phi'(\phi(ab)) = \phi'(\phi(a)\phi(b)) = \phi'(\phi(a))\phi'(\phi(b)) = (\phi' \circ \phi)(a)(\phi' \circ \phi)(b)$$

If  $G \cong G'$  and  $G' \cong G''$  by isomorphisms  $\phi : G \rightarrow G'$  and  $\phi' : G' \rightarrow G''$  then  $\phi' \circ \phi$  is an isomorphism, as you may check. This establishes transitivity. Thus  $\cong$  is an equivalence relation.

We say two groups  $G$  and  $G'$  are *isomorphic* if  $G \cong G'$ . A useful criterion for  $\phi : G \rightarrow G'$  to be an isomorphism is as follows.

► **Let  $\phi : G \rightarrow G'$  be a homomorphism. Then  $\phi$  is an isomorphism if and only if there exists a homomorphism  $\phi' : G' \rightarrow G$  such that**

$$\phi' \circ \phi = \text{id}_G \quad \text{and} \quad \phi \circ \phi' = \text{id}_{G'}$$

Indeed if  $\phi$  is an isomorphism, then the inverse isomorphism considered above provides such a map  $\phi'$ . Conversely, if  $\phi$  is a homomorphism and such a  $\phi'$  in the statement exists, let us check that  $\phi$  is 1-1 and onto. First, let  $a' \in G'$ . Then  $a' = (\phi \circ \phi')(a') = \phi(\phi'(a'))$ . Thus  $\phi$  is onto. Second, suppose  $a, b \in G$  and  $\phi(a) = \phi(b)$ . Then applying  $\phi'$  we get  $a = \phi'(\phi(a)) = \phi'(\phi(b)) = b$  and thus  $\phi$  is 1-1. We conclude that  $\phi$  is an isomorphism.

In fact in the above statement,  $\phi'$  is always uniquely determined as being the inverse isomorphism of  $\phi$  that we considered earlier.

Most of the properties of groups that we are interested in are preserved under isomorphisms.

► **Let  $G$  and  $G'$  be isomorphic groups. Then:**

- (i) **If  $G$  is abelian, then  $G'$  is abelian.**
- (ii) **If  $G$  is cyclic, then  $G'$  is cyclic.**
- (iii) **If  $G$  is finite, then so is  $G'$  and  $|G| = |G'|$ .**

The verification is left as an exercise.

Let  $\phi: G \rightarrow G'$  be a homomorphism. We define the *kernel* of  $\phi$  to be

$$\ker(\phi) = \{a \in G : \phi(a) = e'\} \subset G$$

► **Let  $\phi: G \rightarrow G'$  be a homomorphism. Then  $\ker(\phi)$  is a normal subgroup of  $G$ .**

To see this, first note  $\phi(e) = e'$ , so  $e \in \ker(\phi)$ . Next, let  $a, b \in \ker(\phi)$ . This means  $\phi(a) = e'$  and  $\phi(b) = e'$  where  $e' \in G'$  is the identity. Then

$$\phi(ab) = \phi(a)\phi(b) = e'e' = e'$$

and so  $ab \in \ker(\phi)$ . Similarly,  $\phi(a^{-1}) = \phi(a)^{-1} = (e')^{-1} = e'$  and so  $a^{-1} \in \ker(\phi)$ . Therefore  $\ker(\phi)$  is a subgroup of  $G$ . Next, let  $a \in \ker(\phi)$ , and let  $g \in G$  be any element. Then

$$\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g^{-1}) = \phi(g)e'\phi(g)^{-1} = e'$$

This shows  $g \cdot \ker(\phi) \cdot g^{-1} \subset \ker(\phi)$ . Therefore  $\ker(\phi)$  is normal.

The following is a sort of converse to the above result:

► **Let  $G$  be a group and  $N \subset G$  a normal subgroup. Then there is a homomorphism  $\phi$  from  $G$  to some other group such that  $\ker(\phi) = N$ .**

In fact we can take the homomorphism  $\phi$  to be one that we have seen before:

$$\phi: G \rightarrow G/N$$

defined by  $\phi(a) = aN$  for each  $a \in G$ . The kernel of this homomorphism is the set of  $a \in G$  such that  $\phi(a) = N$ , i.e.  $aN = N$ . But these are exactly the  $a$  that are in the coset  $N$ . Thus  $\ker(\phi) = N$ . This proves what we claimed.

Finally, the following shows that kernels are useful for understanding isomorphisms.

► **An onto homomorphism  $\phi : G \rightarrow G'$  is an isomorphism if and only if  $\ker(\phi) = \{e\}$ .**

First, suppose  $\phi$  is an isomorphism. If  $a \in \ker(\phi)$  then  $\phi(a) = e'$ . But  $\phi(e) = e'$  also, so  $\phi(e) = \phi(a)$  implies  $e = a$ , as  $\phi$  is 1-1. This shows  $\ker(\phi) = \{e\}$ .

Conversely, suppose  $\ker(\phi) = \{e\}$ . We aim to show  $\phi$  is 1-1. Suppose  $\phi(a) = \phi(b)$ . Then

$$e' = \phi(a)\phi(b)^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1})$$

and in particular  $ab^{-1} \in \ker(\phi)$ . Since  $\ker(\phi) = \{e\}$  we must have  $ab^{-1} = e$ , or equivalently  $a = b$ . Thus  $\phi$  is 1-1. By assumption  $\phi$  is onto, and thus it is an isomorphism.