Complex numbers

The use of complex numbers historically emerged as a way of dealing with the problem that some simple polynomial equations, such as

$$x^2 + 1 = 0,$$

have no solutions in the real numbers. It then became apparent over time that complex numbers are much more than just an invention of this necessity.

Formally, a complex number is a pair $(a, b) \in \mathbb{R} \times \mathbb{R}$, which we often write as $a + ib$:

$$\mathbb{C} = \{a + ib : \ a, b \in \mathbb{R}\}$$

The symbol i satisfies $i^2 = -1$, and we write $i = \sqrt{-1}$. The operations $+$ and $\times$ on $\mathbb{C}$ are:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$$

As usual we often omit the symbol $\times$. These operations satisfy the usual distributive law. Following earlier notation we write $\mathbb{C}^\times$ for the non-zero complex numbers.

▸ **$(\mathbb{C}, +)$ and $(\mathbb{C}^\times, \times)$ are abelian groups.**

The claim for $(\mathbb{C}, +)$ is straightforward. For $(\mathbb{C}^\times, \times)$, associativity, identity ($e = 1$), and commutativity are clear. To show inverses exist, consider $z = a + bi \in \mathbb{C}^\times$. We write

$$z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \left(\frac{a}{a^2+b^2}\right) + \left(-\frac{b}{a^2+b^2}\right) i$$

This is an element of $\mathbb{C}^\times$, and is an inverse to z. A priori "$1/z$" (and the symbols in red) does not make sense, because we only defined multiplication above: but the expression on the right of the equation can be taken as definition for $1/z$ and justifies the notation.

▸ **The order of $i \in \mathbb{C}^\times$ is equal to 4.**

This is immediate from the computations $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$.

One of the most important functions of complex numbers is the exponential function. This can be defined in several ways. One way is to use power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

If we plug $z = i\theta$ into this formula, where θ is a real number, we get:

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos(\theta) + i\sin(\theta) \end{aligned}$$

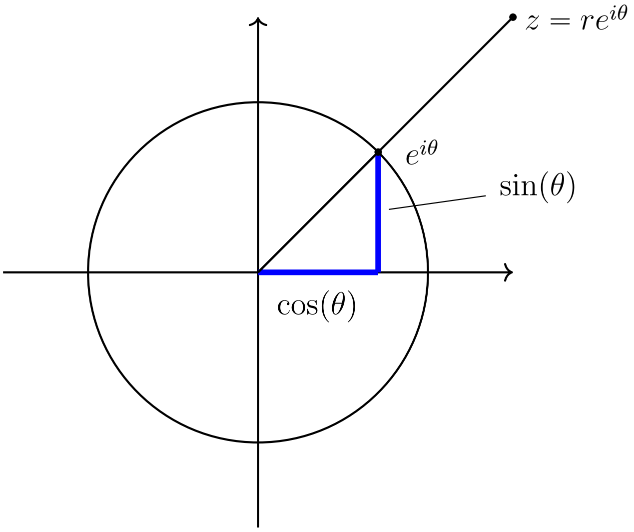
The remarkable formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ is usually called *Euler's Formula*.

The *norm* of $z = a + bi \in \mathbb{C}$ is the non-negative real number defined by $|z| = \sqrt{a^2 + b^2}$. The *conjugate* of $z = a + bi$ is the complex number defined by $\overline{z} = a - bi$. We have

$$|z|^2 = a^2 + b^2 = z\overline{z},$$

which gives the expression $1/z = \overline{z}/|z|^2$, and expands to our previous expression for $1/z$. Note that $|e^{i\theta}| = \sqrt{\cos(\theta)^2 + \sin(\theta)^2} = 1$. A general property is that $|zz'| = |z||z'|$ for $z, z' \in \mathbb{C}$.

Every complex number z can be written in *polar form* $z = re^{i\theta}$ where r is a non-negative real number, and θ is some real number. Note $|z| = |re^{i\theta}| = |r||e^{i\theta}| = r$. The number $r = |z|$ is sometimes also called the *amplitude*, and θ the *phase* of z.



The *complex plane* is the usual xy-plane, where the point (x, y) represents the complex number $z = x + yi$. Above we show how in polar form, $z = re^{i\theta}$ is determined by locating the position of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ on the unit circle (it forms angle θ with the x-axis) and then scales according to its amplitude r.

The exponential law $e^{z+z'} = e^z e^{z'}$ holds for complex numbers $z, z' \in \mathbb{C}$. This can be proved in any number of ways. When $z = i\theta$ and $z' = i\theta'$ for real numbers θ, θ' this law is equivalent to the trigonometric identities for angle addition:

$$\begin{aligned} e^{i\theta}e^{i\theta'} &= (\cos(\theta) + i\sin(\theta))(\cos(\theta') + i\sin(\theta')) \\ &= (\cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta')) + i(\cos(\theta)\sin(\theta') + \sin(\theta)\cos(\theta')) \\ &= \cos(\theta + \theta') + i\sin(\theta + \theta') \\ &= e^{i(\theta+\theta')} \end{aligned}$$

The exponential law can be phrased in terms of group theory as follows:

▶ **Exponentiation gives an onto homomorphism $(\mathbb{C}, +) \to (\mathbb{C}^\times, \times)$.**

The geometric way of multiplying $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$ now emerges: you multiply $e^{i\theta}$ and $e^{i\theta'}$ to get $e^{i\theta+i\theta'}$ (so the phases add) and then scale this by rr', the new amplitude.

We now give a group-theoretic interpretation of polar decomposition multiplication.

▶ **Let $U(1) = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$. Then with the operation of multiplication, this is a subgroup of $(\mathbb{C}^\times, \times)$, and is in particular an abelian group.**

This is straightforward to prove. The group $U(1)$ is called the *circle group*. Geometrically, it is the group of rotational symmetries of the complex plane which fix the origin. It is one of the most important and fundamental examples of a group. We have:

▶ **Exponentiation gives an onto homomorphism $(\mathbb{R}, +) \to (U(1), \times)$.**

For the next statement we let $\mathbb{R}^\times_{>0}$ be the subgroup of $(\mathbb{R}^\times, \times)$ containing the positive reals.

▶ **The map $\phi : \mathbb{R}^\times_{>0} \times U(1) \to \mathbb{C}^\times$ given by $\phi(r, e^{i\theta}) = re^{i\theta}$ is an isomorphism of groups.**

In other words, polar decomposition of complex numbers helps us understand how the group $\mathbb{C}^\times$ is really a product of two simpler groups we already understand.

To prove the statement, first we see that ϕ is 1-1 and onto: it is 1-1 because if $re^{i\theta} = r'e^{i\theta'}$ then $r = r'$ and $e^{i\theta} = e^{i\theta'}$ (here it is important that $r, r' \neq 0$). It is onto because every non-zero complex number has a polar representation. Finally, ϕ is a homomorphism because

$$\phi(r, e^{i\theta})\phi(r', e^{i\theta'}) = (re^{i\theta})(r'e^{i\theta'}) = rr'e^{i\theta}e^{i\theta'} = \phi(rr', e^{i\theta}e^{i\theta'}) = \phi((r, e^{i\theta})(r', e^{i\theta'}))$$

It follows that ϕ is an isomorphism as claimed.