Homomorphisms and Isomorphisms

When are two groups "the same"? We have implicitly answered this question in our earlier investigations. Namely, we have seen several different incarnations of a certain non-abelian group of order 6. First, we abstractly defined a group

$$G = \{e, r, b, g, y, o\}$$

in Lecture 1, by writing out its Cayley table. We saw later that the group formed by the symmetries of an equilateral triangle, and also the symmetric group

$$S_3 = \{e, (12), (23), (31), (123), (132)\}$$

have the same Cayley table when the elements are matched up appropriately. The Cayley tables of G and S_3 are shown below, and we see that they are essentially the same.

	e	r	b	g	y	o
e	e	r	b	g	y	o
r	r	e	o	y	g	b
b	b	y	e	o	r	g
g	g	o	y	e	b	r
y	y	b	g	r	o	e
o	o	g	r	b	e	y

	e	(12)	(23)	(31)	(132)	(123)
e	e	(12)	(23)	(31)	(132)	(123)
(12)	(12)	e	(123)	(132)	(31)	(23)
(23)	(23)	(132)	e	(123)	(12)	(31)
(31)	(31)	(123)	(132)	e	(23)	(12)
(132)	(132)	(23)	(31)	(12)	(123)	e
(123)	(123)	(31)	(12)	(23)	e	(132)

Let us formalize what we mean by *the same*. What we really have is a map $\phi : G \to S_3$, that is, an assignment to each element of G some element of S_3. This assignment is a 1-1 and onto map, which means that every element of S_3 is mapped to by something in G, and that no two elements of G are mapped to the same element of S_3. In our example this map is:

$$\phi(e) = e, \quad \phi(r) = (12), \quad \phi(b) = (23), \quad \phi(g) = (31), \quad \phi(y) = (132), \quad \phi(o) = (132)$$

Furthermore, this map preserves the group structures, or equivalently, it maps the Cayley table of G in the correct way to the Cayley table of S_3. What this really means is that for any $a_1, a_2 \in G$ the product $a_1 a_2 \in G$ is mapped to the product $\phi(a_1)\phi(a_2) \in S_3$. For example,

$$(123) = \phi(o) = \phi(rb) = \phi(r)\phi(b) = (12)(23) = (123)$$

We formalize the kind of map that is appearing here into a definition.

▸ **A *homomorphism* from a group G to a group G' is a map $\phi : G \to G'$ such that**

$$\phi(ab) = \phi(a)\phi(b) \qquad \textbf{for all } a, b \in G.$$

A homomorphism ϕ is an *isomorphism* if the map ϕ is 1-1 and onto.

Therefore, two groups are the "same" if there is an isomorphism $\phi : G \to G'$.

The only difference between a homomorphism and an isomorphism is that the map ϕ does not need to be 1-1 and onto for a homomorphism. Every isomorphism is of course a homomorphism, and so for the moment we will develop some properties of this more general notion.

Examples

1. Let G be any group, and $\{e\}$ the trivial group. Then there is a homomorphism $\phi : G \to \{e\}$ defined by $\phi(a) = e$ for all $a \in G$. Then for any $a, b \in G$ we have $\phi(ab) = e = ee = \phi(a)\phi(b)$, and so ϕ is a homomorphism.

2. Consider the groups $\mathrm{GL}_2(\mathbb{R})$ and $(\mathbb{R}^\times, \times)$. Define a map $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ by setting $\phi(A) = \det(A)$ for each matrix $A \in \mathrm{GL}_2(\mathbb{R})$. Then

$$\phi(A)\phi(B) = \det(A)\det(B) = \det(AB) = \phi(AB)$$

Thus ϕ is a homomorphism.

3. Define a map $\phi : (\mathbb{R}, +) \to (\mathbb{R}^\times, \times)$ by $\phi(x) = 2^x$ for each $x \in \mathbb{R}$. Then

$$\phi(x + y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y)$$

for all $x, y \in \mathbb{R}$. Therefore ϕ is a homomorphism of groups.

4. Let G be a subgroup and $H \subset G$ a subgroup. Define $\phi : H \to G$ by $\phi(a) = a$ for all $a \in H$. In other words, ϕ is the "inclusion map" which simply sends H to itself viewed as a subset of G. Clearly ϕ is a homomorphism. For example, we have a homomorphism $(\mathbb{Z}, +) \to (\mathbb{Q}, +)$, and a homomorphism $(\mathbb{Q}^\times, \times) \to (\mathbb{R}^\times, \times)$.

5. Consider the symmetric group S_n, and the additive group $\mathbb{Z}_2$. Define $\phi : S_n \to \mathbb{Z}_2$ by:

$$\phi(\sigma) = \begin{cases} 0 \pmod 2 & \text{if } \sigma \text{ is even} \\ 1 \pmod 2 & \text{if } \sigma \text{ is odd} \end{cases}$$

Then ϕ is a homomorphism, thanks to our understanding of how even and odd permutations compose. For example, if σ is even and σ' is odd, we know $\sigma\sigma'$ is odd, and thus

$$\phi(\sigma\sigma') = 1 = 0 + 1 = \phi(\sigma) + \sigma(\sigma')$$

The following gives another source of many group homomorphisms, and is related to our construction of quotient groups from last lecture.

▶ **Let G be a group and $N \subset G$ a normal subgroup. Define $\phi : G \to G/N$ by $\phi(a) = aN$ for all $a \in G$. Then ϕ is a group homomorphism which is onto.**

To see this, we simply compute $\phi(a)\phi(b) = (aN)(bN) = abN = \phi(ab)$. This shows that ϕ is a homomorphism. The homomorphism ϕ is onto because given a coset $aN \in G/N$ the element $a \in G$ satisfies $\phi(a) = aN$.

As an example, we have onto homomorphisms $\mathbb{Z} \to \mathbb{Z}_n$ for every positive integer n, defined by sending an integer k to its congruence class $k \pmod{n}$.

▶ **Let $\phi : G \to G'$ be a homomorphism of groups. Then ϕ maps the identity $e \in G$ to the identity $e' \in G'$. Further, for all $a \in G$ we have $\phi(a^{-1}) = \phi(a)^{-1}$.**

To prove the first part, we compute

$$\phi(e) = \phi(ee) = \phi(e)\phi(e)$$

Multiply both sides by $\phi(e)^{-1}$ to obtain $e' = \phi(e)$. Next, for $a \in G$ we compute

$$\phi(a)\phi(a^{-1}) = \phi(aa^{-1}) = \phi(e) = e'$$

and similarly $\phi(a^{-1})\phi(a) = e'$. Thus the inverse of $\phi(a) \in G'$ is given by $\phi(a^{-1})$.