Normal subgroups and quotient groups

In this lecture we discuss an important class of subgroups called *normal subgroups*. Given a normal subgroup N in a group G, we then construct the *quotient group* G/N. The construction is a generalization of our construction of the groups $(\mathbb{Z}_n, +)$.

Fix a group G and a subgroup H. If we have a Cayley table for G, then it is easy to find the right and left cosets of H in G. Let us illustrate this with an example we have encountered before. Consider the group S_3, and choose the subgroup $H = \{e, (12)\}$. To find the right cosets of H in the Cayley table, first find H in the left column. It's shaded gray below.

	e	(12)	(23)	(31)	(132)	(123)
e	e	(12)	(23)	(31)	(132)	(123)
(12)	(12)	e	(123)	(132)	(31)	(23)
(23)	(23)	(132)	e	(123)	(12)	(31)
(31)	(31)	(123)	(132)	e	(23)	(12)
(132)	(132)	(23)	(31)	(12)	(123)	e
(123)	(123)	(31)	(12)	(23)	e	(132)

Each column, restricted to the rows determined by H, gives a right coset. In our example above, each such column has 2 elements. Equal cosets are given the same color. We see in our example we have 3 distinct right cosets, compatible with Lagrange's Theorem.

The description of left cosets from the Cayley table is similar, except the roles of column and row are switched. We illustrate this for our example:

	e	(12)	(23)	(31)	(132)	(123)
e	e	(12)	(23)	(31)	(132)	(123)
(12)	(12)	e	(123)	(132)	(31)	(23)
(23)	(23)	(132)	e	(123)	(12)	(31)
(31)	(31)	(123)	(132)	e	(23)	(12)
(132)	(132)	(23)	(31)	(12)	(123)	e
(123)	(123)	(31)	(12)	(23)	e	(132)

We see in this example that the right cosets and left cosets are not the same.

For another example, consider the subgroup $H = \{e, (123), (132)\}$ in S_3. The same procedure above carried out for H shows that in this case, the right cosets and left cosets are actually the same. More precisely, $aH = Ha$ for each group element a. This is illustrated in the tables below, where right and left cosets are shown.

	e	(12)	(23)	(31)	(132)	(123)
e	e	(12)	(23)	(31)	(132)	(123)
(12)	(12)	e	(123)	(132)	(31)	(23)
(23)	(23)	(132)	e	(123)	(12)	(31)
(31)	(31)	(123)	(132)	e	(23)	(12)
(132)	(132)	(23)	(31)	(12)	(123)	e
(123)	(123)	(31)	(12)	(23)	e	(132)

	e	(12)	(23)	(31)	(132)	(123)
e	e	(12)	(23)	(31)	(132)	(123)
(12)	(12)	e	(123)	(132)	(31)	(23)
(23)	(23)	(132)	e	(123)	(12)	(31)
(31)	(31)	(123)	(132)	e	(23)	(12)
(132)	(132)	(23)	(31)	(12)	(123)	e
(123)	(123)	(31)	(12)	(23)	e	(132)

This latter kind of subgroup is called a *normal subgroup* and is defined as follows.

▶ **A subgroup N in a group G is *normal* if for all $a \in G$ we have $aN = Na$.**

The first example $H = \{e, (12)\}$ in S_3 from above is not normal, while the second example $H = \{e, (123), (132)\} \subset S_3$ is normal. Note that if G is abelian, every subgroup is normal. The following gives alternative characterizations for a subgroup to be normal.

▶ **Let G be a group and $N \subset G$ a subgroup. The following are equivalent:**

(i) **N is a normal subgroup.**

(ii) **$aNa^{-1} \subset N$ for all $a \in G$.**

(iii) **$aNa^{-1} = N$ for all $a \in G$.**

Proof. (i)⇒(ii): Assume N is normal, i.e. $aN = Na$ for all $a \in G$. Let $ana^{-1} \in aNa^{-1}$ where $n \in N$. Since $aN = Na$, we have $an \in Na$, i.e. $an = n'a$ for some $n' \in N$. Then $ana^{-1} = n'aa^{-1} = n' \in N$. Thus $aNa^{-1} \subset N$.

(ii)⇒(iii): Assume $aNa^{-1} \subset N$ for all $a \in G$. Let $n \in N$. Then (ii) with a^{-1} in place of a says $a^{-1}Na \subset N$. (Note (ii) holds for *all* $a \in G$.) Thus $a^{-1}na = n'$ for some $n' \in N$. Then $n = an'a^{-1} \in aNa^{-1}$, so $N \subset aNa^{-1}$. It follows that $N = aNa^{-1}$.

(iii)⇒(i): Assume $aNa^{-1} = N$ for all $a \in G$. Let $an \in aN$ where $n \in N$. Our assumption implies $ana^{-1} = n'$ for some n'. Then $an = n'a$. Thus $aN \subset Na$. Similar reasoning shows $Na \subset aN$, and thus $aN = Na$. □

Now fix a normal subgroup N within any group G. Define G/N to be:

$$G/N = \{\text{right cosets of } N \text{ in } G\} = \{\text{left cosets of } N \text{ in } G\}$$

We define a binary operation on G/N as follows: for $aN, bN \in G/N$ we set

$$(aN)(bN) = abN$$

Of course we could have also written $(Na)(Nb) = Nab$, and the fact that it doesn't matter if we use right or left cosets is thanks to the normality of N.

▶ **Let G be a group and $N \subset G$ a normal subgroup. Then the set of cosets G/N endowed with the above binary operation satisfies the axioms of a group.**

To see this we first check that the binary operation is well-defined. More precisely, suppose $aN = a'N$ and $bN = b'N$. Then $a = a'n_1$ and $b = b'n_2$ for some $n_1, n_2 \in N$. Since $b'N = Nb'$ we have $n_1 b' = b' n_3$ for some $n_3 \in N$. We compute

$$ab = (a'n_1)(b'n_2) = a'(n_1 b')n_2 = a'(b'n_3)n_2 = a'b'(n_3 n_2)$$

Thus $ab \equiv a'b' \pmod N$, or in other words $abN = a'b'N$.

Next, it is easy to see that $N = eN = Ne$ is an identity. Associativity follows because $(aNbN)cN = (abN)cN = abcN = aN(bcN) = aN(bNcN)$. Finally, the inverse of aN is given by the coset $a^{-1}N$. Thus G/N is a group.

We call the group G/N from the above construction the *quotient group* of G by N. The construction crucially depends on N being a normal subgroup, and does not work for subgroups that are not normal.

▶ **The order of the group G/N is equal to $[G : N]$.**

This is just from the definitions. Note that if G is finite, Lagrange's Theorem implies

$$|G/N| = [G : N] = |G|/|N|$$

Examples

1. For the normal subgroup $N = \{e, (123), (132)\} \subset S_3$ we have $|S_3/N| = |S_3|/|N| = 6/3 = 2$. Thus the quotient group is of order 2. Note $S_3/N = \{N, (12)N\}$.

2. Let $G = (\mathbb{Z}, +)$ and consider the subgroup $N = n\mathbb{Z} \subset G$ for any positive integer n. Then N is a normal subgroup, as G is abelian, and G/N is exactly the group $(\mathbb{Z}_n, +)$.

3. Consider the alternating group A_n as a subgroup of S_n. Using the properties of even and odd permutations, we have that A_n is a normal subgroup. Then $|S_n/A_n| = 2$ and $S_n/A_n = \{E, O\}$ where $E = \{\text{even permutations}\}$, $O = \{\text{odd permutations}\}$.