Symmetric groups

In this lecture we continue our study of the symmetric groups S_n introduced last lecture, and we then introduce the notion of *even* and *odd* permutations.

▶ **The order of the group S_n is given by $|S_n| = n!$.**

This is a basic counting exercise. Each permutation σ is a way of rearranging the elements $\{1, \dots, n\}$. There are n possibilities for $\sigma(1) \in \{1, \dots, n\}$. Once the value of $\sigma(1)$ is fixed, there are $n-1$ possibilities left for $\sigma(2)$, since it can be anything except for $\sigma(1)$. Continuing in this fashion, we see that the number of possibilities for the permutation $\sigma \in S_n$ is equal to $n(n-1)(n-2)\cdots 2 \cdot 1 = n!$.

For example, we have seen that $|S_1| = 1 = 1!$, $|S_2| = 2 = 2!$ and $|S_3| = 6 = 3!$. Recall that S_3 is essentially the symmetries of an equilateral triangle, and there are 6 such symmetries.

A *cycle* is a permutation $\sigma \in S_n$ such that there exists an $i \in \{1, \dots, n\}$ with the property that for each $j \in \{1, \dots, n\}$ either $\sigma(j) = j$ or $j = \sigma^k(i)$ for some $k \in \mathbb{Z}$. In other words, every element of $\{1, \dots, n\}$ is either fixed by σ or can be obtained from i by successively applying σ. Our notation for such a cycle, which we introduced last lecture, is as follows:

$$(i \quad \sigma(i) \quad \sigma^2(i) \quad \cdots \quad \sigma^{l-1}(i))$$

Here l is the smallest positive integer such that $\sigma^l(i) = i$.

▶ **Every $\sigma \in S_n$ can be written as a product of disjoint cycles.**

Two cycles are *disjoint* if they share no common elements in their cycle notations. For example, (13) and (246) are disjoint cycles, while (12) and (254) are not. To prove the statement, define a relation on the set $\{1, \dots, n\}$ as follows: $i \sim j$ if and only if $\sigma^k(i) = j$ for some $k \in \mathbb{Z}$. You can check that this defines an equivalence relation, and hence partitions the set $\{1, \dots, n\}$, and each equivalence class forms a cycle with respect to σ.

$$\begin{tikzcd} 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[ll, bend right] & 1 \arrow[r] & 6 \arrow[l, bend right] & 3 \end{tikzcd}$$

The content of the above is that any permutation $\sigma \in S_n$ partitions the set $\{1, 2, \dots, n\}$ into subsets each of which σ permutes in a cyclic fashion. For example, $\sigma = (245)(16) \in S_6$ is depicted above. Note that σ fixes "3" and this does not appear in cycle notation.

▶ **Disjoint cycles in S_n commute.**

The *length* of a cycle is the number of elements appearing in it: (146) is a cycle of length 3, and (4263) is length 4. A *transposition* is a cycle of length 2, such as (12).

▶ **For $n \geqslant 2$, every $\sigma \in S_n$ is a product of transpositions.**

We first note that an arbitrary cycle $(a_1 a_2 \cdots a_k)$ can be written

$$(a_1 a_2 \cdots a_{k-1} a_k) = (a_3 a_2)(a_4 a_3) \cdots (a_k a_{k-1})(a_1 a_k)$$

For example, $(245) = (45)(24)$. The general case follows from the case of cycles, because any $\sigma \in S_n$ is a product of cycles.

The way in which a cycle is written as a product of transpositions is not unique. As an example, we have $(12)(13) = (132) = (23)(12)$ in the group S_3. In fact, another formula for writing an arbitrary cycle as a product of transpositions is as follows:

$$(a_1 a_2 \cdots a_{k-1} a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_3)(a_1 a_2)$$

Even and odd permutations

Let $f = f(x_1, \ldots, x_n)$ be a polynomial in n variables. Given a permutation $\sigma \in S_n$ define a new polynomial $\sigma(f)$ to be $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Define the special polynomial

$$\Delta_n = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$$

to be the product ofl all factors $(x_i - x_j)$ where i, j range over elements of $\{1, \ldots, n\}$ with $i < j$. Then we may apply any permutation σ to this to obtain a new polynomial $\sigma(\Delta_n)$. It turns out that this new polynomial is always either Δ_n or $-\Delta_n$. This lets us divide permutations into two types, as follows:

▶ **If $\sigma(\Delta_n) = \Delta_n$ is called *even*. If $\sigma(\Delta_n) = -\Delta_n$ then σ is called *odd*.**

For example, consider $\sigma = (12) \in S_3$. Note $\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. Then

$$\begin{aligned} \sigma(\Delta_3) &= \quad (x_{\sigma(1)} - x_{\sigma(2)})(x_{\sigma(1)} - x_{\sigma(3)})(x_{\sigma(2)} - x_{\sigma(3)}) \\ &= \quad (x_2 - x_1)(x_1 - x_3)(x_2 - x_3) \\ &= -(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) \\ &= -\Delta_3 \end{aligned}$$

Thus $\sigma = (12)$ is odd. In the next lecture we will show that the *even* permutations form a subgroup of S_n called the *alternating group* A_n.