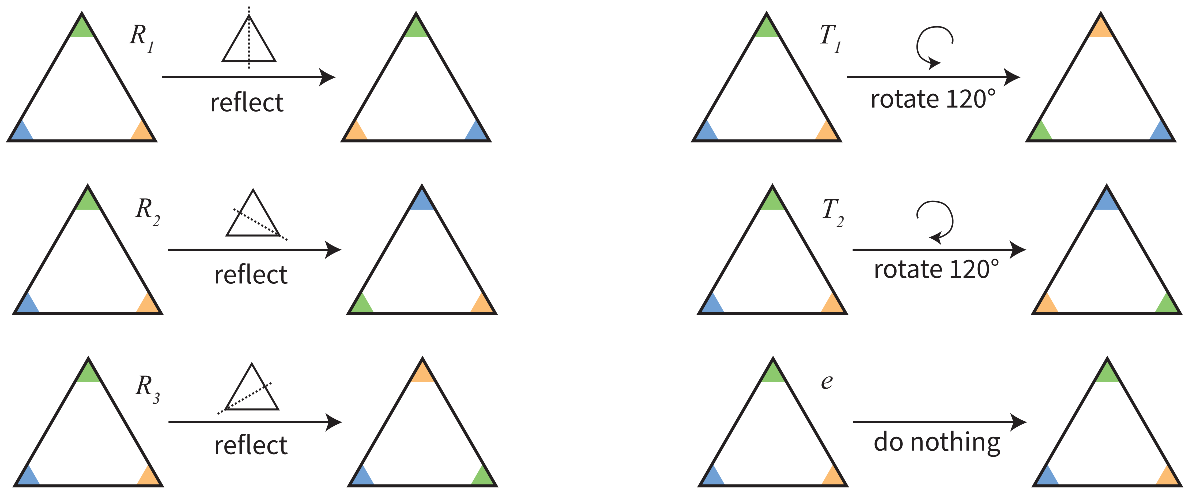
Symmetries

A *symmetry* of an object is a rearrangement of the object such that its geometric properties are preserved. If the object is a subset X of points in space, then a symmetry is a 1-1 map $f : X \to X$ that preserves distances and angles. For example, consider an equilateral triangle $X \subset \mathbb{R}^2$ in 2-dimensional space. A symmetry maps the triangle to itself in a way which sends vertices to vertices, and edges to edges. All symmetries of the triangle are as follows:



Let us give each such symmetry the name as indicated in the figure. Thus R_1 is "reflection across the vertical line" and so on. We use the notation

$$R_1 \circ T_1$$

to mean "first T_1 then R_1." In this case this means "rotate 120° counter-clockwise, then reflect across the vertical line." (The order is just as for functions, or maps.) We see that

$$R_1 \circ T_1 = R_3$$

We also have $R_1 \circ R_1 = e$, $R_1 \circ R_2 = T_2$, $T_1 \circ T_2 = e$. Eventually we obtain a Cayley table:

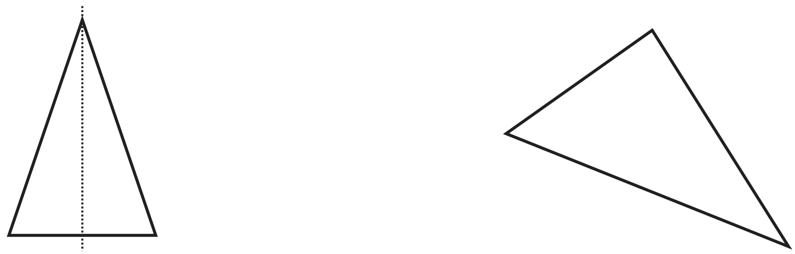
	e	R_1	R_2	R_3	T_1	T_2
e	e	R_1	R_2	R_3	T_1	T_2
R_1	R_1	e	T_2	T_1	R_3	R_2
R_2	R_2	T_1	e	T_2	R_1	R_3
R_3	R_3	T_2	T_1	e	R_2	R_1
T_1	T_1	R_2	R_3	R_1	T_2	e
T_3	T_2	R_3	R_1	R_2	e	T_1

This should look familiar: it is exactly the Cayley table for the non-abelian group of order 6 that we discussed in Lecture 1 (after renaming symbols).

▸ **The symmetries of an object forms a group.**

This can be seen explicitly in the example above, but the general case is also straightforward. The key is to think of symmetries as 1-1 maps from a set X to itself which satisfy certain properties (the maps preserve distances and angles, for example).

Let us consider other objects. Take an isosceles triangle in the plane. Then apart from the identity symmetry e, there is only one symmetry R, which is a reflection across the line indicated in the figure below on the left. Further, $R \circ R = e$. Thus the group of symmetries of an isosceles triangle is $\{e, R\}$ with $R^2 = e$. This is a cylic group of order 2.



On the other hand, if we take an arbitrary triangle with 3 different angles, so that no two sides are the same length, then there are no symmetries apart from the identity. See the figure above on the right. The group of symmetries of such a triangle is $\{e\}$, a trivial group.

Permutations

Let X be a finite non-empty set. Suppose $|X| = n$. We think of X as an abstract collection of n points, with no geometric information regarding the points. In this case the "symmetries of X" are simply the 1-1 maps $\sigma : X \to X$, i.e. the ways of rearranging the points in the set. These are called *permutations* of the set X. We will not care about the names of the elements of X, so we may as well let $X = \{1, \ldots, n\}$.

▸ **The group of permutations on $\{1, \ldots, n\}$ is called S_n, the n^{th} symmetric group.**

An element of S_n, which is a map from $\{1, \ldots, n\}$ to itself, may be represented as:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

We also write $i \mapsto j$ to mean σ sends i to j, i.e. $\sigma(i) = j$. For example, let $\sigma \in S_6$ map $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 5$, $4 \mapsto 6$, $5 \mapsto 1$, $6 \mapsto 4$. Then we write this permutation as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$$

Let us consider another permutation $\tau \in S_6$, given as follows:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 5 & 6 \end{pmatrix}$$

Let us compute $\tau \circ \sigma$. Note that the notation means "do σ, then τ." We find:

$$(\tau \circ \sigma)(1) = \tau(\sigma(1)) = \tau(2) = 1 \qquad (\tau \circ \sigma)(2) = \tau(\sigma(2)) = \tau(3) = 4$$
$$(\tau \circ \sigma)(3) = \tau(\sigma(3)) = \tau(5) = 5 \qquad (\tau \circ \sigma)(4) = \tau(\sigma(4)) = \tau(6) = 6$$
$$(\tau \circ \sigma)(5) = \tau(\sigma(5)) = \tau(1) = 3 \qquad (\tau \circ \sigma)(6) = \tau(\sigma(6)) = \tau(4) = 2$$

Writing the result in our notation introduced above we then have:

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 3 & 2 \end{pmatrix}$$

Another more compact notation, called cycle notation, is described as follows. We illustrate this notation in the current example, and also comment on the general case in the next lecture. Starting with the element "1" we can successively apply the permutation σ until we get back to "1":

$$1 \overset{\sigma}{\longmapsto} 2 \overset{\sigma}{\longmapsto} 3 \overset{\sigma}{\longmapsto} 5 \overset{\sigma}{\longmapsto} 1$$

This forms a *cycle* and is denoted (1235). The notation is meant to be cyclic, and can be written also as (2351), (3512) or (5123). To describe the rest of the permutation σ we must also know what it does to "4" and "6". Note that

$$4 \overset{\sigma}{\longmapsto} 6 \overset{\sigma}{\longmapsto} 4$$

and this forms another cycle $(46) = (64)$. We then write σ in cycle notation as follows:

$$\sigma = (1235)(46) = (46)(1235)$$

We can also write τ in cycle notation, as $\tau = (1342)$. Note that "5" and "6" are fixed by τ and they do not appear in the notation, by convention. We then compute

$$\tau \circ \sigma = (1342)(1235)(46) = (246)(35)$$

by reading cycles from right to left, and this is the same answer we obtained above.

▸ **The n^{th} symmetric group S_n is non-abelian if $n \geqslant 3$.**

Note $S_1 = \{e\}$ is the trivial group, and $S_2 = \{e, (12)\}$ is cyclic of order 2. For $n = 3$ we have

$$S_3 = \{e, (12), (23), (31), (123), (132)\}$$

This is a non-abelian group of order 6. In fact if you write out its Cayley table you may notice it looks just like the symmetries of an equilateral triangle. This is no accident, for S_3 is essentially the same group: any symmetry of the triangle is determined by how the 3 vertices are permuted. To see that S_n is non-abelian for $n \geqslant 4$, note that we can view $S_3 \subset S_n$ as the subgroup which fixes the last $n - 3$ elements.