# Homework 9

1. Let $R = \{a + bx : a, b \in \mathbb{Z}_3\}$ be the set of expressions $a + bx$ where $a, b$ are elements of $\mathbb{Z}_3$ and $x$ is a symbol (similar to $i = \sqrt{-1}$ in $\mathbb{C}$) which satisfies $x^2 = -1 \equiv 2 \pmod 3$. The addition and multiplication in $R$ is similar to that in the complex numbers.
   (a) How many elements does the ring $R$ have? List them.
   (b) Show that this ring is a field.

2. Show each number is algebraic over $\mathbb{Q}$ by finding its minimal polynomial.
   (a) $\sqrt{5} - 1$
   (b) $\sqrt[3]{3 + i\sqrt{2}}$
   (c) $\sqrt{3} + \sqrt{5}$

3. Consider $\mathbb{Z}_2[x]/(x^3 + x + 1)$. Show this is a field with 8 elements. Write the multiplication table for the non-zero elements of this field.

4. (a) Let $\alpha = \sqrt[4]{5}$ and $\beta = \sqrt[4]{5} \cdot i$. Find the minimal polynomials of $\alpha$ and $\beta$.
   (b) Find the degrees of the extensions $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ over $\mathbb{Q}$.
   (c) Show that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic fields.
   (d) Show that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are not the same field.

5. Find a basis for each field extension, and compute the degree of the extension.
   (a) $\mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$
   (b) $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}(\sqrt{27})$
   (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$

6. (a) Decide whether each given number is algebraic over $\mathbb{Q}$, and explain your answer. For the ones which are algebraic, find their minimal polynomials.

$$\alpha_1 = \sqrt{3}, \qquad \alpha_2 = \sqrt{1 + \sqrt{5^2 - 4^2}}, \qquad \alpha_3 = \pi + 1,$$

$$\alpha_4 = \sqrt[3]{5 + \sqrt{2}},, \qquad \alpha_5 = \sqrt{\pi}, \qquad \alpha_6 = \sqrt{5}$$

   (b) Compute the extension degree $[\mathbb{Q}(\alpha_i) : \mathbb{Q}]$ for each $\alpha_i$ appearing above.
   (c) Determine which are algebraic:

$$\alpha_1\alpha_2, \qquad \alpha_2/\alpha_4, \qquad \alpha_3 - \alpha_5^2, \qquad \alpha_5^{100}, \qquad \sqrt[101]{\alpha_6}$$

   (d) Show that $\mathbb{Q}(\alpha_1 + \alpha_6) = \mathbb{Q}(\alpha_1, \alpha_6)$.