Homework 8

1. Determine which of the following subsets are ideals. For example, $\mathbb{Q} \subset \mathbb{R}$ means: is $\mathbb{Q}$ an ideal in the ring $\mathbb{R}$? Justify your answer in each case.

   (a) $2\mathbb{Z} \subset \mathbb{Z}$

   (b) $\mathbb{Q} \subset \mathbb{R}$

   (c) $\{a + b\sqrt{-5} : a + b \equiv 0 \pmod 6\} \subset \mathbb{Z}[\sqrt{-5}]$

   (d) $\mathbb{R} \subset \mathbb{R}[x]$

   (e) $\{f(x, y) \in \mathbb{R}[x, y] : \ f(4, 5) = 0\}$

2. Consider the ideal in the ring $\mathbb{Z}[\sqrt{-5}]$ given by

$$I = (3, 1 + \sqrt{-5}) = \{3x + (1 + \sqrt{-5})y : \ x, y \in \mathbb{Z}[\sqrt{-5}]\}$$

   Suppose I is principal and generated by some $a+b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$, i.e. $I = (a+b\sqrt{-5})$.

   (a) Since $3 \in I$, we have $3 = (a + b\sqrt{-5})(c + d\sqrt{-5})$ for some $c, d \in \mathbb{Z}$. Find all possibilities for $a + b\sqrt{-5}$ based on this relation.

   (b) Do the same as in (a) but using $1 + \sqrt{-5} \in I$ instead of $3 \in I$.

   (c) Using your results in (a) and (b) show that I cannot be a principal ideal.

3. Let R be a commutative ring. Consider the ring of polynomials $R[x]$. For an element $a \in R$ define the *evaluation* at a to be the map

$$\phi_a : R[x] \longrightarrow R, \qquad \phi_a(f(x)) = f(a)$$

   given by plugging a into a polynomial $f(x) \in R[x]$ to get an element $f(a) \in R$.

   (a) Show that ϕ_a is a homomorphism of rings.

   (b) Apply the 1st Isomorphism Theorem of rings to this homomorphism.

4. For two ideals $I, J \subset R$ in a commutative ring R, define the sum

$$I + J = \{a + b \ : \ a \in I, b \in J\}.$$

   (a) Show that $I + J$ is an ideal in R.

   (b) Show that $I \subset I + J$ and $J \subset I + J$.

   Given $a_1, \ldots, a_n \in R$ one often writes $(a_1, \ldots, a_n) \subset R$ for the "ideal generated by $a_1, \ldots, a_n$". It is by definition the sum of principal ideals $(a_1) + (a_2) + \ldots + (a_n)$.

5. (a) Given ideals $I, J \subset \mathbb{R}[x, y]$ show that the varieties satisfy $V(I \cap J) = V(I) \cup V(J)$.

   (b) Let $I = (x^2 - y - \frac{1}{2})$ and $J = (x^2 + y^2 - 1)$. Draw the varieties $V(I + J)$ and $V(I \cap J)$.

6. How many prime ideals does the ring $\mathbb{Z}_{30}$ contain?