Homework 8

1. Determine which of the following are rings, with the usual operations of addition and multiplication. If it is a ring you do not need to write out all the axioms, but if it is not a ring, explain why. If the set is a ring, is it a field? If not, explain why.

   (a) $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$

   (b) $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$, so that $i^2 = -1$

   (c) The set of following matrices *with entries in the ring* $\mathbb{Z}_2$:

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

   (d) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Addition and multiplication is defined pointwise. That is, for two such functions f, g, we have $f + g$ and fg defined by $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for $x \in \mathbb{R}$.

(a) It's a ring, in fact a subring of $\mathbb{C}$:

$0 = 0 + 0\sqrt{2}$, $1 = 1 + 0\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

$a+b\sqrt{2},\ c+d\sqrt{2} \in \mathbb{Q}(\sqrt{2}) \Rightarrow (a+b\sqrt{2}) - (c+d\sqrt{2}) = (a-c) + (b-d)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

and $(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd) + (ad+bc)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

(Here $a, b, c, d \in \mathbb{Q}$).

It's also a field: $(a+b\sqrt{2})^{-1} = \left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

Note: $a^2 - 2b^2 \neq 0$ if $(a,b) \neq (0,0)$ since a, b are rational.

(b) This is also a subring of $\mathbb{C}$.

Not a field: take $2 \in \mathbb{Z}[i]$. Then $2^{-1} = \frac{1}{2} \notin \mathbb{Z}[i]$.

(c) $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ $\quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \text{"}0\text{"} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{"}1\text{"}$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Closed under all operations. It's a ring.

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus all nonzero elements have mult. inverses.

$\Rightarrow$ it's a field.

(d) It's a ring.

Note "0" = constant function $f(x) \equiv 0$, "1" = constant function $f(x) \equiv 1$

It's not a field.

For example consider $f(x) = x$.

Suppose $g(x)$ has $f(x)g(x) = 1$.

Then necessarily $g(x) = \frac{1}{x}$ for all $x \neq 0$.

But cannot extend g to a continuous function at $x = 0$.

2. For each ring R given, determine the set of units $R^\times$.

    (a) The integers $\mathbb{Z}$

    (b) The ring $M_2(\mathbb{Z})$, the 2×2 matrices with integer entries

    (c) Each of the rings in problem #1

(a) $ab = 1$ in $\mathbb{Z} \Rightarrow a, b = \pm 1$.

Thus $\mathbb{Z}^\times = \{1, -1\}$.

(b) $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ has also $A^{-1} \in M_2(\mathbb{Z})$.

Then $\det(A^{-1}) = \det(A)^{-1} = \frac{1}{ad-bc} \in \mathbb{Z}$.

Thus $ad - bc = \pm 1$.

Conversely, if $ad-bc = \pm 1$ then $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in M_2(\mathbb{Z})$.

Thus $M_2(\mathbb{Z})^\times = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1 \right\}$.

(1a) $\mathbb{Q}(\sqrt{2})$ is a field so $\mathbb{Q}(\sqrt{2})^\times = \mathbb{Q}(\sqrt{2}) - \{0\}$.

(1b) Suppose $(a+bi)(c+di) = 1$ where $a, b, c, d \in \mathbb{Z}$.

Take norms squared: $(a^2+b^2)(c^2+d^2) = 1$.

$\Rightarrow (a, b) = (1,0), (0,1), (-1,0)$ or $(0,-1)$.

Thus $1, -1, i, -i$ are the units:

$$\mathbb{Z}[i]^\times = \{1, -1, i, -i\}$$

(1c) This is a field, and the units are all nonzero elements:

$$\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

(1d) The inverse in this ring of a function $f(x)$ should be $\frac{1}{f(x)}$. This is a well-defined continuous function if and only if $f(x)$ is never zero.

Thus $\underline{R^\times = \{ f(x) : f(x) \neq 0 \text{ for all } x \in \mathbb{R} \}}$