Homework 7

1. Consider the following group, which you may think of as $\mathrm{GL}_2(\mathbb{Z}_2)$:

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, b, c, d \in \mathbb{Z}_2, \;\; ad - bc \equiv 1 \pmod 2 \right\}$$

   The operation is matrix multiplication, although we now use addition and multiplication in $\mathbb{Z}_2$ instead of $\mathbb{R}$. You may verify that this is a group, although you do not need to prove it on your homework. Show that G is isomorphic to the group S_3.

2. In class we applied the construction of Cayley's Theorem to the quaterion group Q_8 to produce a 1-1 homomomorphism $\phi : Q_8 \to S_8$. Apply the same construction to the following groups, to get homomorphisms into symmetric groups:

   (a) $(\mathbb{Z}_3, +)$
   (b) $(\mathbb{Z}_8^\times, \times)$
   (c) S_3

3. Consider the quaternion group $Q_8 = \{1, -1, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}$.

   (a) List all subgroups of Q_8.
   (b) Show that every subgroup of Q_8 is normal.
   (c) Construct an onto homomorphism $Q_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2$. From your homomorphism, what does the 1st Isomorphism Theorem imply?

4. Use the last result stated in Lecture 18 to show the following: a group of order p^2, where p is a prime, must have a normal subgroup of order p.

5. Determine which of the following are rings, with the usual operations of addition and multiplication. If it is a ring you do not need to write out all the axioms, but if it is not a ring, explain why. If the set is a ring, is it a field? If not, explain why.

   (a) $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : \; a, b \in \mathbb{Q}\}$
   (b) $\mathbb{Z}[i] = \{a + bi : \; a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$, so that $i^2 = -1$
   (c) The set of following matrices *with entries in the ring* $\mathbb{Z}_2$:

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

   (d) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Addition and multiplication is defined pointwise. That is, for two such functions f, g, we have $f + g$ and fg defined by $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for $x \in \mathbb{R}$.

6. For each ring R given, determine the set of units $R^\times$.

   (a) The integers $\mathbb{Z}$
   (b) The ring $\mathrm{M}_2(\mathbb{Z})$, the 2×2 matrices with integer entries
   (c) Each of the rings in problem #1