Homework 6

1. Determine whether each map defined is a homomorphism or not. If it is a homomorphism, prove it, and also determine whether it is an isomorphism.

   (a) The maps $\phi_1, \phi_2 : (\mathbb{R}^\times, \times) \to \mathrm{GL}_2(\mathbb{R})$ given by the following, for each $a \in \mathbb{R}^\times$:

   $$\phi_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \qquad \phi_2(a) = \begin{pmatrix} 2a & 0 \\ 0 & a \end{pmatrix}$$

   (b) The map $\phi : (\mathbb{Z}_n, +) \to (\mathbb{C}^\times, \times)$ given by $\phi(k \pmod{n}) = e^{2\pi\sqrt{-1}k/n}$.

   (c) The map $\phi : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$ given by sending $z = a + bi \in \mathbb{C}^\times$ to:

   $$\phi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

2. A while back on HW 1 you showed that the set $\mathbb{Z}$ equipped with the binary operation $a * b = a + b + 1$ defines a group. Show that this group $(\mathbb{Z}, *)$ is isomorphic to the usual group of integers with addition, $(\mathbb{Z}, +)$.

3. Let $\phi : G \to G'$ be a group homomorphism. Define

   $$\mathrm{im}(\phi) = \{a' \in G' : a' = \phi(a) \text{ for some } a \in G\} \subset G'$$

   This set is called the *image* of ϕ. Show that $\mathrm{im}(\phi)$ is a subgroup of G'.

4. Let G and G' be isomorphic groups. Prove that if G is cyclic, then G' is cyclic.

5. Show that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$.

6. (a) Find two non-isomorphic groups of order 4.

   (b) Show that A_4 and $\mathbb{Z}_2 \times S_3$, although both of order 12, are not isomorphic.

   (c) Show that $S_3 \times \mathbb{Z}_4$ and S_4, although both of order 24, are not isomorphic.