Homework 6

1. Determine whether each map defined is a homomorphism or not. If it is a homomorphism, prove it, and also determine whether it is an isomorphism.

   (a) The maps $\phi_1, \phi_2, \phi_3 : (\mathbb{R}^\times, \times) \to \mathrm{GL}_2(\mathbb{R})$ given by the following, for each $a \in \mathbb{R}^\times$:

   $$\phi_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \qquad \phi_2(a) = \begin{pmatrix} 2a & 0 \\ 0 & a \end{pmatrix}, \qquad \phi_3(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

   (b) The map $\phi : (\mathbb{Z}_n, +) \to (\mathbb{C}^\times, \times)$ given by $\phi(k \pmod n) = e^{2\pi i k/n}$. (Here $i = \sqrt{-1}$.)

   (c) The map $\phi : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$ given by sending $z = a + bi \in \mathbb{C}^\times$ to:

   $$\phi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(a) Let $a, b \in \mathbb{R}^\times$. $\phi_1(ab) = \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = \phi_1(a)\phi_1(b)$

so ϕ_1 is a homomorphism. ϕ_1 is <u>not</u> onto (e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \phi_1(a)$ for any $a \in \mathbb{R}^\times$) so it is not an isomorphism.

$\phi_2(ab) = \begin{pmatrix} 2ab & 0 \\ 0 & ab \end{pmatrix} \neq \begin{pmatrix} 4ab & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 2b & 0 \\ 0 & b \end{pmatrix} = \phi_2(a)\phi_2(b)$

$\Rightarrow \phi_2$ <u>not</u> a homom.

$\phi_3(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (ϕ_3 should map identity to identity) $\Rightarrow \phi_3$ <u>not</u> hom.

(b) The map ϕ is a homomorphism: for $k, l \in \mathbb{Z}_n$

$\phi(k+l) = e^{2\pi i (k+l)/n} = e^{2\pi i k/n} e^{2\pi i l/n} = \phi(k)\phi(l)$.

Not an isomorphism: $\mathbb{C}^\times$ is infinite, $\mathbb{Z}_n$ finite.

(c) It's a homomorphism: let $a + ib = z$, $c + id = z' \in \mathbb{C}^\times$.

Then $\phi(zz') = \phi((ac - bd) + i(ad + bc)) = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix}$

$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \phi(z)\phi(z')$. <u>Not</u> isom. (not onto: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \phi(z)$ for any z)

2. A while back on HW 1 you showed that the set $\mathbb{Z}$ equipped with the binary operation $a * b = a + b + 1$ defines a group. Show that this group $(\mathbb{Z}, *)$ is isomorphic to the usual group of integers with addition, $(\mathbb{Z}, +)$.

Goal: find an isomorphism $\phi : (\mathbb{Z}, *) \to (\mathbb{Z}, +)$.

Recall in $(\mathbb{Z}, *)$ the identity is -1 and inverse of a is given by $-a-2$.

The isomorphism ϕ must send identity to identity so we are forced to have $\phi(-1) = 0$.

Next, $(\mathbb{Z}, *)$ is generated by 0:

$$0 * 0 = 0 + 0 + 1 = 1, \quad 0 * 0 * 0 = 2, \text{ etc}$$

and
$$0^{-1} = -0-2 = -2, \quad 0^{-2} = -3, \text{ etc.}$$

So we should probably map 0 to a generator of $(\mathbb{Z}, +)$, say 1 (although -1 is also fine): $\phi(0) = 1$.

This determines ϕ because:

$$\phi(1) = \phi(0 * 0) = \phi(0) + \phi(0) = 1 + 1 = 2$$
$$\phi(2) = \phi(0 * 1) = \phi(0) + \phi(1) = 1 + 2 = 3, \text{ etc.}$$

We get the formula $\phi(k) = k + 1$.

This is an isomorphism (its inverse is $\phi^{-1}(k) = k - 1$).

3. Let $\phi : G \to G'$ be a group homomorphism. Define

$$\mathrm{im}(\phi) = \{a' \in G' : a' = \phi(a) \text{ for some } a \in G\} \subset G'$$

This set is called the *image* of ϕ. Sometimes people write $\mathrm{im}(\phi) = \phi(G)$.

(a) Show that $\mathrm{im}(\phi)$ is a subgroup of G'.

(b) Prove or provide a counterexample: $\mathrm{im}(\phi)$ is a normal subgroup of G'.

(a) Let $a' = \phi(a)$, $b' = \phi(b) \in \mathrm{im}(\phi)$

Then $a'b' = \phi(a)\phi(b) = \phi(ab) \in \mathrm{im}(\phi)$

Also, $e' = \phi(e) \in \mathrm{im}(\phi)$

and for $a' = \phi(a) \in \mathrm{im}(\phi)$ we have $(a')^{-1} = \phi(a)^{-1} = \phi(a^{-1}) \in \mathrm{im}(\phi)$

Thus $\mathrm{im}(\phi)$ is a subgroup of G'.

(b) $\mathrm{im}(\phi)$ need not be a normal subgroup of G'.

Take any group G with a non-normal subgroup $H \subseteq G$.

Define $\phi : H \to G$ by inclusion: $\phi(a) = a$ for $a \in H$.

Then $\mathrm{im}(\phi) = H \subseteq G$ is not normal.

4. Let G and G' be isomorphic groups. Prove the following statements.
   (a) If G is abelian, then G' is abelian.
   (b) If G is cyclic, then G' is cyclic.
   (c) Show that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$.

(a) Let $\phi : G \to G'$ and $\phi' : G' \to G$ be isomorphisms.

Let $a', b' \in G'$. Then

$$\phi'(a'b') = \phi'(a')\phi'(b') \overset{G \text{ abelian}}{=} \phi'(b')\phi'(a') = \phi'(b'a')$$

$\phi'(a'b') = \phi'(b'a') \Rightarrow a'b' = b'a'$ (since ϕ' is 1-1).

Thus G' is abelian.

(b) Let a generate G, i.e. $G = \langle a \rangle = \{ a^k : k \in \mathbb{Z} \}$.

Then $\phi(a) \in G'$. ϕ is onto so for any $a' \in G'$, there is $b \in G$ such that $\phi(b) = a'$. But $b \in G \Rightarrow b = a^k$ for some $k \in \mathbb{Z}$.

So $a' = \phi(a^k) = \phi(a)^k$. Thus $G' = \{ \phi(a)^k : k \in \mathbb{Z} \}$, i.e. G' is cyclic.

(c) Suppose $\phi : \mathbb{Q} \to \mathbb{Z}$ were an isomorphism.

Then $1 \in \mathbb{Z}$ is $1 = \phi(r)$ for some $r \in \mathbb{Q}$.

But then $\phi(r) = \phi(\frac{r}{2} + \frac{r}{2}) = \phi(\frac{r}{2}) + \phi(\frac{r}{2}) = 2\phi(\frac{r}{2}) \Rightarrow \phi(\frac{r}{2}) = \frac{1}{2} \notin \mathbb{Z}$,

contradiction.

5. (a) Find two non-isomorphic groups of order 4.
   (b) Show that A_4 and $\mathbb{Z}_2 \times S_3$, although both of order 12, are not isomorphic.
   (c) Show that $S_3 \times \mathbb{Z}_4$ and S_4, although both of order 24, are not isomorphic.

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ are not isomorphic
($\mathbb{Z}_2 \times \mathbb{Z}_2$ has no element of order 4)

(b) A_4 has no elements of order 6
(in fact we showed it has no subgroup of order 6)
On the other hand, $\mathbb{Z}_2 \times S_3$ has elements of order 6, e.g. $(1, (123))$.
Thus $A_4 \not\cong \mathbb{Z}_2 \times S_3$.

(c) $S_3 \times \mathbb{Z}_4$ has elements of order 12, e.g. $((123), 1)$.

However S_4 has no such elements
(possible orders in S_4 are $1: e$, $2: (12), (12)(34)$, $3: (123)$, $4: (1234)$)

Thus $S_3 \times \mathbb{Z}_4 \not\cong S_4$.