# Homework 5

1. (a) For each of (a)–(e) in #1 of Homework 4 that you completed last week, determine which subgroups are normal subgroups.

   (b) For those of (a)–(e) that are normal subgroups, determine the quotient group by writing its Cayley table. Note that the entries of the table will be cosets.
   (If you do this right, it is not so much writing. The quotient groups are small.)

2. Let $G_1$ and $G_2$ be groups, and define the set $G_1 \times G_2 = \{(a_1, a_2) : a_1 \in G_1, a_2 \in G_2\}$.

   (a) Define a binary operation on $G_1 \times G_2$ and show it makes $G_1 \times G_2$ into a group.

   (b) Show $H_2 = \{(e_1, a_2) : a_2 \in G_2\}$ is a subgroup of $G_1 \times G_2$.[1] Is it normal?

   (c) Explain how the quotient group $(G_1 \times G_2)/H_2$ is related to $G_1$.

3. Let $G$ be a group and $N \subset G$ a normal subgroup.

   (a) If $G$ is abelian, show that $G/N$ is abelian. Is the converse true?
   Prove it or give a counterexample.

   (b) Is it true that if $N$ and $G/N$ are cyclic, then $G$ is cyclic?
   Prove it or give a counterexample.

4. Consider the group $G \subset \mathrm{GL}_2(\mathbb{R})$ consisting of upper triangular matrices:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \ a, b, c \in \mathbb{R}, \ \ ac \neq 0 \right\}$$

   Let $H \subset G$ be the subset of $G$ consisting of those matrices in $G$ with $a = c = 1$.

   (a) Show $H$ is an abelian subgroup of $G$.
   (b) Show $H$ is normal in $G$.
   (c) Show $G/H$ is abelian.

[1] Here $e_1$ is the identity in $G_1$.